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Bounds of minimal dilatation for pseudo-Anosovs and the magic 3-manifold

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1 Minimal dilatation of pseudo-Anosovs

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures, and let $\mathrm{Mod}(\Sigma)$ be the mapping class group. Mapping classes $\phi \in \mathrm{Mod}(\Sigma)$ are classified into 3 types, periodic, reducible, pseudo-Anosov. There exist two numerical invariants of pseudo-Anosov mapping classes. One is the entropy $\mathrm{ent}(\phi)$ which is the logarithm of the dilatation $\lambda(\phi) > 1$. The other is the volume $\mathrm{vol}(\phi)$ which is the hyperbolic volume of the mapping torus of ϕ

$$\mathbb{T}(\phi) = \Sigma \times [0,1]/\sim,$$

where $\sim$ identifies $(x,1)$ with $(f(x),0)$ for any representative $f \in \phi$.

We denote by $\delta_{g,n}$, the minimal dilatation for pseudo-Anosov elements $\phi \in \mathrm{Mod}(\Sigma_{g,n})$. We set $\delta_g = \delta_{g,0}$. A natural question arises.

Question 1.1. *What is the value of $\delta_{g,n}$? Find a pseudo-Anosov element of* $\mathrm{Mod}(\Sigma_{g,n})$ *whose dilatation is equal to $\delta_{g,n}$.*

The above question is hard in general. For instance, in the case of closed surfaces, it is open to determine the values δ_g for $g \geq 3$. On the other hand, one understands the asymptotic behavior of the minimal entropy $\log \delta_g$. Penner proved that $\log \delta_g \asymp \frac{1}{g}$ [12]. The following question posed by McMullen.

Question 1.2 ([11]). *Does $\lim\limits_{g\to\infty} g \log \delta_g$ exist? What is its value?*

Note that $\lim\limits_{g\to\infty} g \log \delta_g$ exists if and only if $\lim\limits_{g\to\infty} |\chi(\Sigma_g)| \log \delta_g$ exists, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

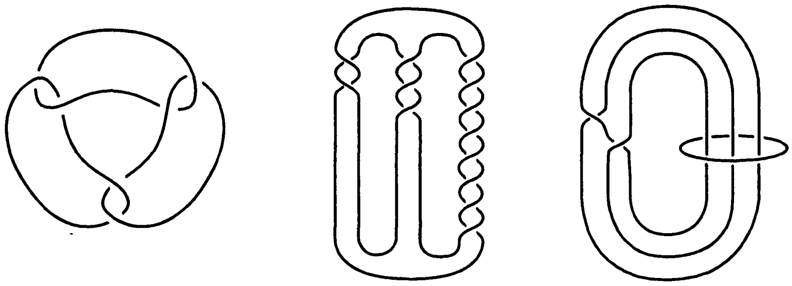

Figure 1: (left) 3 chain link $\mathcal{C}_3$. (center) $(-2,3,8)$-pretzel link or Whitehead sister link. (right) link 6^2_2. (N equals the exterior of $\mathcal{C}_3$. $N(\frac{-3}{2})$ is homeomorphic to the $(-2,3,8)$-pretzel link exterior. $N(\frac{-1}{2})$ is homeomorphic to the 6^2_2 link exterior.)

Related questions on the minimal dilatation are ones for orientable pseudo-Anosovs. A pseudo-Anosov mapping class ϕ is said to be *orientable* if the invariant (un)stable foliation for a pseudo-Anosov homeomorphism $\Phi \in \phi$ is orientable. We denote by δ_g^+, the minimal dilatation for orientable pseudo-Anosov elements of $\mathrm{Mod}(\Sigma_g)$ for a closed surface Σ_g of genus g.

In this paper, we report our results in [7, 8] on the minimal dilatation by investigating the so called *magic manifold* N which is the exterior of the 3 chain link $\mathcal{C}_3$, see Figure 1(left). In Section 2, we describe a motivation for the study of pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of N. In Section 3, we state our results.

We would like to note that this paper only contains some results in [7, 8] and does not contain their proofs. The readers who are interested in the details should consult (the introduction of) [7, 8].

2 Why is the magic manifold an intriguing example?

Gordon and Wu named the exterior of the link $\mathcal{C}_3$ the *magic manifold* N, see [3]. The reason why this manifold is called "magic" is that many important examples for the study of the exceptional Dehn fillings can be obtained from the Dehn fillings of a single manifold N. The magic manifold is fibered and it has the smallest known volume among orientable hyperbolic 3-manifolds having 3 cusps. Many manifolds having at most 2 cusps with small volume are obtained from N by Dehn fillings, see [10].

2.1 Entropy versus volume

Both invariants entropy $\mathrm{ent}(\phi)$ and volume $\mathrm{vol}(\phi)$ know some complexity of pseudo-Anosovs ϕ. A natural question is how these are related.

Theorem 2.1 ([6]). *There exists a constant $B = B(\Sigma)$ depending only on the topology of Σ such that the inequality,*

$$B\,\mathrm{vol}(\phi) \leq \mathrm{ent}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ. Furthermore, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \Sigma) > 1$ depending only on ε and the topology of Σ such that the inequality

$$\mathrm{ent}(\phi) \leq C\,\mathrm{vol}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ whose mapping torus $\mathbb{T}(\phi)$ has no closed geodesics of length $< \varepsilon$.

The first part of Theorem 2.1 says that if the entropy is small, the volume can not be large.

For a non-negative integer c, we set

$$\begin{aligned} \lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \mathrm{Mod}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}, \\ \mathrm{vol}(\Sigma; c) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathrm{Mod}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}. \end{aligned}$$

A variation on the questions of the minimal dilatations is to determine $\lambda(\Sigma; c)$ and to find a mapping class realizing the minimum. In [6], the authors and S. Kojima obtain experimental results concerning the minimal dilatation. In the case the mapping class group $\mathrm{Mod}(D_n)$ of an n-punctured disk D_n, they observe that for many pairs (n, c), there exists a pseudo-Anosov element simultaneously reaching both minima $\lambda(D_n; c)$ and $\mathrm{vol}(D_n; c)$. Experiments tell us that in case $c = 3$, the mapping tori reaching both minima are homeomorphic to N. Moreover when $c = 2$, it is observed that there exists a mapping class ϕ

realizing both $\lambda(D_n;2)$ and $\mathrm{vol}(D_n;2)$ and its mapping torus $\mathbb{T}(\phi)$ is homeomorphic to a Dehn filling of N along one cusp. This is a motivation for us for focusing on N.

2.2 Small dilatation pseudo-Anosovs

After we have finished our papers [6, 7], we leaned the small dilatation pseudo-Anosovs, introduced by Farb, Leininger and Margalit.

For any number $P > 1$, define the set of pseudo-Anosov homeomorphisms

$$\Psi_P = \{\text{pseudo-Anosov } \Phi : \Sigma \to \Sigma \mid \chi(\Sigma) < 0,\ |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.$$

Elements $\Phi \in \Psi_P$ are called *small dilatation pseudo-Anosov homeomorphisms.* If one takes P sufficiently large (e.g. $P \geq 2+\sqrt{3}$), then Ψ_P contains a pseudo-Anosov homeomorphism $\Phi_g : \Sigma_g \to \Sigma_g$ for each $g \geq 2$. By a result in [5], Ψ_P also contains pseudo-Anosov homeomorphism $\Phi_n : D_n \to D_n$ for each $n \geq 3$. Let $\Sigma^\circ \subset \Sigma$ be the surface obtained by removing the singularities of the (un)stable foliation for Φ and $\Phi|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ$ denotes the restriction. Observe that $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$. The set

$$\Psi_P^\circ = \{\Phi|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ \mid (\Phi : \Sigma \to \Sigma) \in \Psi_P\}$$

is infinite. Let $\mathcal{T}(\Psi_P^\circ)$ be the set of homeomorphism classes of mapping tori by elements of Ψ_P°.

Theorem 2.2 ([2]). *The set $\mathcal{T}(\Psi_P^\circ)$ is finite. Namely, for each $P > 1$, there exist finite many complete, non compact hyperbolic 3-manifolds $M_1, M_2, \cdots, M_r$ fibering over S^1 so that the following holds. Any pseudo-Anosov $\Phi \in \Psi_P$ occurs as the monodromy of a Dehn filling of one of the M_k. In particular, there exists a constant $V = V(P)$ such that* $\mathrm{vol}(\Phi) \leq V$ *holds for any $\Phi \in \Psi_P$.*

It is not known that how large the set of manifolds $\{M_1, \cdots, M_r\}$ is. By Theorem 2.2, one sees that the following set $\mathcal{V}$ is finite.

$$\mathcal{V} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid n \geq 3,\ \Phi \text{ is pseudo-Anosov on } \Sigma = D_n,\ \lambda(\Phi) = \delta(D_n)\},$$

where $\delta(D_n)$ denotes the minimal dilatation for pseudo-Anosov elements of $\mathrm{Mod}(D_n)$ on D_n.

In [7], we show that for each $n \geq 9$ (resp. $n = 3, 4, 5, 7, 8$), there exists a pseudo-Anosov homeomorphism $\Phi_n : D_n \to D_n$ with the smallest known entropy (resp. the smallest entropy) which occurs as the monodromy on an n-punctured disk fiber for the Dehn filling of N. A pseudo-Anosov homeomorphism $\Phi_6 : D_6 \to D_6$ with the smallest entropy occurs as the monodromy on a 6-punctured disk fiber for N. In particular, $N \in \mathcal{V}$. See also work of Venzke [13]. This result suggests that one may have a chance to find pseudo-Anosov homeomorphisms with small dilatation on other surfaces which arise as the monodromies on fibers for Dehn fillings of N. This is another motivation for us.

3 Results

Let us introduce the following polynomial

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \text{ for } k > 0, \ -k < \ell < k,$$

having a unique real root $\lambda_{(k,\ell)}$ greater than 1 [8]. For the rational number r, let $N(r)$ be the Dehn filling of N along the slope r.

Theorem 3.1. *Let $r \in \{\frac{-3}{2}, \frac{-1}{2}, 2\}$. For each $g \geq 3$, there exists a monodromy $\Phi_g = \Phi_g(r)$ on a closed fiber of genus g for a Dehn filling of $N(r)$, where the filling is on the boundary slope of a fiber of $N(r)$, such that*

$$\lim_{g \to \infty} g \log \lambda(\Phi_g) = \log(\tfrac{3+\sqrt{5}}{2}).$$

In particular

$$\lim_{g \to \infty} \sup g \log \delta_g \leq \log(\tfrac{3+\sqrt{5}}{2}).$$

Remark 3.2. *Independently, Hironaka has obtained Theorem 3.1 in case $r = \frac{-1}{2}$ [4]. Independently, Aaber and Dunfield have obtained Theorem 3.1 in case $r = \frac{-3}{2}$ [1]. They have obtained similar results on the dilatation to those given in [8].*

By using monodromies on closed fibers coming from $N(\frac{-3}{2})$, we find an upper bound of δ_g for $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$ and $g \geq 5$.

Theorem 3.3. **(1)** $\delta_g \leq \lambda_{(g+2,1)}$ *if* $g \equiv 0, 1, 5, 6 \pmod{10}$ *and* $g \geq 5$.

(2) $\delta_g \leq \lambda_{(g+2,2)}$ *if* $g \equiv 7, 9 \pmod{10}$ *and* $g \geq 7$.

For more details of an upper bound of δ_g for other g (e.g. $g \equiv 2, 4 \pmod{10}$), see [8]. The bound in Theorem 3.3 improves the one by Hironaka [4].

We turn to the study on the minimal dilatations δ_g^+ for orientable pseudo-Anosovs. The minima δ_g^+ were determined for $g = 2$ by Zhirov [14], for $3 \leq g \leq 5$ by Lanneau-Thiffeault [9], and for $g = 8$ by Lanneau-Thiffeault and Hironaka [9, 4]. Those values are given by $\delta_2^+ = \lambda_{(2,1)}$, $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$, $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$, $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$ and $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$.

We recall the lower bounds of δ_6^+ and δ_7^+ and the question on δ_g^+ for g even by Lanneau-Thiffeault.

Theorem 3.4 ([9]).

(1) $\delta_6^+ \geq \lambda_{(6,1)} \approx 1.17628$.

(2) $\delta_7^+ \geq \lambda_{(9,2)} \approx 1.11548$.

Question 3.5 ([9]). *For g even, is δ_g^+ equal to $\lambda_{(g,1)}$?*

We give an upper bound of δ_g^+ in case $g \equiv 1, 5, 7, 9 \pmod{10}$ using orientable pseudo-Anosov monodromies coming from $N(\frac{-3}{2})$.

Theorem 3.6. **(1)** $\delta_g^+ \leq \lambda_{(g+2,2)}$ *if* $g \equiv 7, 9 \pmod{10}$ *and* $g \geq 7$.

(2) $\delta_g^+ \leq \lambda_{(g+2,4)}$ *if* $g \equiv 1, 5 \pmod{10}$ *and* $g \geq 5$.

The bound in Theorem 3.6 improves the one by Hironaka [4]. Theorem 3.6(1) together with Theorem 3.4(2) gives:

Corollary 3.7. $\delta_7^+ = \lambda_{(9,2)}$.

Independently, Corollary 3.7 was established by Aaber and Dunfiled [1].

The following tells us that the sequence $\{\delta_g^+\}_{g \geq 2}$ is not monotone decreasing.

Proposition 3.8. *If Question 3.5 is true, then $\delta_g^+ < \delta_{g+1}^+$ whenever $g \equiv 1, 5, 7, 9 \pmod{10}$ and $g \geq 7$. In particular the inequality $\delta_7^+ < \delta_8^+$ holds.*

Our pseudo-Anosov homeomorphisms providing the upper bound of δ_g in Theorem 3.3(1) are not orientable. This together with the inequality $\lambda_{(7,1)} < \lambda_{(6,1)} = \delta_5^+$ implies:

Corollary 3.9. $\delta_5 < \delta_5^+$.

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