数理解析研究所講究録
第1719巻 2010年 21-25

Phase field model of mode III-type crack evolution and its energy estimation

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Keywords: crack evolution, phase field model, numerical simulation

1 Model for Crack Evolution

Many methods are proposed for numerical computations on crack evolution problems, because of the difficulty of the singularity arising from the crack tip, change of computational boundary arising from the new crack surface, lack of the explicit method to determine the direction of expand the crack. We show that the mode III (anti-plane shear mode) crack growth on a plate is described as reaction-diffusion system that is consisted by the of the anti-plane displacement and the phase field that describes the crack[2]. This system is derived from the energy description introduced by Francfort and Marigo[1], We show that this reaction-diffusion system make the computation of the crack problem easy.

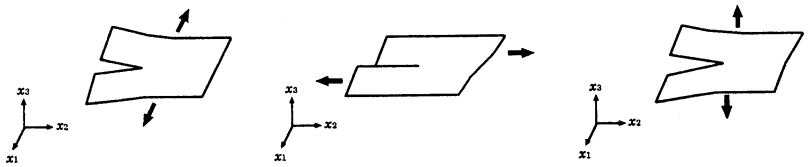

Figure 1: 3 modes of the crack evolution on a plate (a) mode I, (b) mode II, (c) mode III.

We derive equation for the crack evolution of the plate, that is expanded by the small anti-plane displacement (Figure 1 (c)). Let Γ be a bounded two dimensional domain with a piecewise smooth boundary Γ_N, and let Γ_D be a nonempty open portion of Γ_N which consists of a finite number of connected

components. We define $\Gamma_N := \Gamma \backslash \Gamma_D$. For $t > 0$, we consider the equations:

$$
\begin{cases}
\alpha_1 \dfrac{\partial u}{\partial t} = \operatorname{div}\left((1-z)^2 \nabla u\right) & x \in \Omega,\ t > 0 \\
\alpha_2 \dfrac{\partial z}{\partial t} = \left(\varepsilon \Delta z - \dfrac{\gamma^2}{\varepsilon} z + |\nabla u|^2 (1-z)\right)_+ & x \in \Omega,\ t > 0 \\
u(x,t) = g(x,t) & x \in \Gamma_D,\ t > 0 \\
\dfrac{\partial u}{\partial n} = 0 \quad x \in \Gamma_N,\ t > 0, \quad \dfrac{\partial z}{\partial n} = 0 & x \in \Gamma,\ t > 0 \\
u(x,0) = u_0(x), \quad z(x,0) = z_0(x) & x \in \Omega
\end{cases}
\tag{1}
$$

where $u(x,t)$ represents the small anti-plane displacement at the position $x \in \overline{\Omega}$ and time $t \geq 0$, and $g(x,t)$ is a given anti-plane displacement on the boundary Γ_D. The variable $z(x,t)$ satisfies $0 \leq z(x,t) \leq 1$ in Ω and represents the crack shape, as $z \approx 0$ in the region without crack and $z \approx 1$ near the crack. The minimum length scale of z is given as $O(\varepsilon)$ with a small regularization parameter $\varepsilon > 0$ for the numerical stability. The function $z(x,t)$ is called the phase field for the crack shape. For stable numerical simulations, we also introduce small time relaxation parameters $\alpha_1 \geq 0$ and $\alpha_2 > 0$.

The first equation of (1) expresses the force balance in the uncracked region ($z \approx 0$), and the second equation expresses the crack evolution due to the modulus of the stress $|\nabla u|$. The material constant $\gamma > 0$ is called the fracture toughness, which prescribes the critical value of the energy release rate in the Griffith's criterion. It is harder for the crack to grow, if the value of γ is larger.

A crack once generated can be no longer repaired. We put $(\ )_+$ to the right hand side of the second equation, where $(a)_+ = \max(a, 0)$. It guarantees the non-repair condition for the crack: $\frac{\partial z}{\partial t} \geq 0$.

This model has advantages for numerical simulation of crack evolution as followings: i) automatic path selection of the crack that means the possibility of calculation of the outbreak of new crack and sub-crack, ii) possibility to use the ordinal method (for example, FDM, FVM, and FEM) for numerical simulation because of the PDE model on fixed domain, iii) numerical stability introduced by the regularization parameter $\varepsilon > 0$, iv) potential to adopt the spacial profile or hysteresis of the fracture toughness γ.

2 Energy estimation

Griffith focused on the energy balance of the material that includes the crack, because he thinks that the rupture strength is mainly dominated by the stress concentration at the tip of the crack. He introduced the relation between the energy for making new crack surface and that for releasing by the crack evolution under the quasi-static condition Griffith's fracture criterion).

Starting from the internal energy proposed by Francfort-Marigo [1], we treat the following energy description:

$$\begin{cases} \mathcal{E}(u,z) &:= \mathcal{E}_1(u,z) + \mathcal{E}_2(z) \\ \mathcal{E}_1(u,z) &:= \frac{\mu}{2}\int_\Omega (1-z)^2 |\nabla u|^2 \, dx - \int_\Omega fu \, dx \\ \mathcal{E}_2(z) &:= \frac{1}{2}\int_\Omega \gamma(x)\left(\epsilon|\nabla z|^2 + \frac{1}{\epsilon}z^2\right) dx \end{cases} \tag{2}$$

where $\mathcal{E}_1(u,z)$ is the regularized elastic energy of the plate, and $\mathcal{E}_2(z)$ is the surface energy of the crack. $\gamma(x) > 0$ is fracture toughness on $x \in \Omega$. In [1], they proposed this energy and investigated precisely. Bourdin et al. and Buliga made some numerical simulations of crack evolution that minimize the energy similar to (2).

We set the total energy $\mathcal{E}$ as a free energy of Ginzburg-Landau Theory, and derive the temporal evolution equations of the displacement and the phase field.

Though the detailed derivation of (1) is written in [3], the reaction-diffusion equation on two scalar variable (1) is given when we set $f = 0, \gamma(x) \equiv \gamma > 0, \mu = 1, \varepsilon \equiv \epsilon\gamma$.

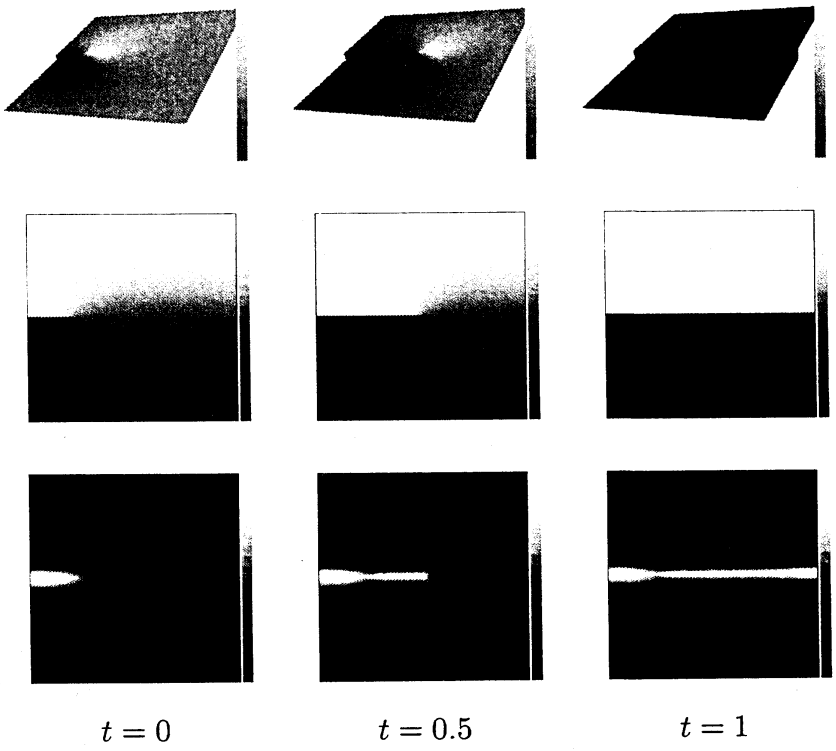

Figure 2: Birdsview of u (top), u (middle) and z(bottom).

We studied that crack evolution phenomena can be developed by the numerical simulations of (1) with $g(x,t)$ ([2],[3]). In this model, quasi-static energy relaxation that leads the crack evolution is assumed, then, we confirm that this assumption is kept in the numerical simulation.

Set the initial crack at $t = 0$, we make a numerical simulation fixed the boundary condition as $g(x,t) = g(x)$. In the following simulations, we put $\varepsilon = 10^{-3}, \alpha_1 = 0, \alpha_2 = 10^{-3}, \gamma = 0.5$ in (1), and set the computational domain as $\Omega = (-1,1) \times (-1,1)$, with $\Gamma_D = \{(x_1,x_2)|x_1 \in (-1,1), x_2 = \pm 1\}$. The boundary condition for u is given as $g(x,t) = 5x_2$ ($x \in \Gamma_D, t \geq 0$).

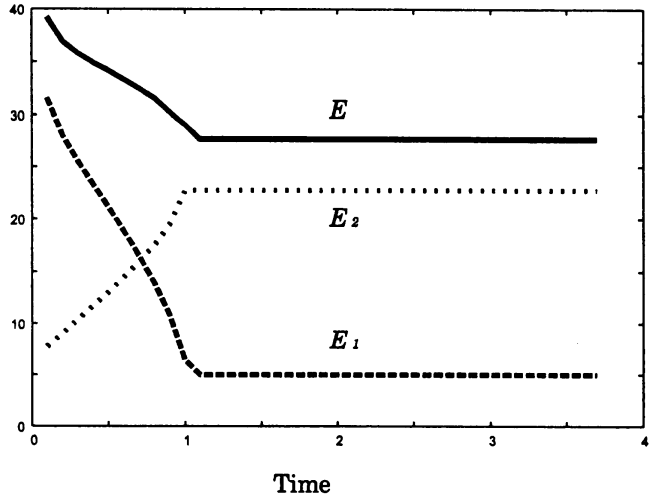

Figure 3: Temporal evolution of $\mathcal{E}$ (solid line), $\mathcal{E}_1$ (dashed line), $\mathcal{E}_2$ (dotted line).

The temporal evolution of u (Figure 2) shows the crack evolution, however, velocity of the crack expansion becomes slower by time. From the results of numerical simulation, we calculate the energy of system (2). Figure 3 shows the temporal evolution of energy that the elastic energy $\mathcal{E}_1$ is decaying, surface energy $\mathcal{E}_2$ is growing, and total energy is decaying slowly as the crack growth till the material is fractured ($t \sim 1$). We confirm that these numerical results follow our model and describe the crack evolution phenomena, The physical characteristics of the material can be estimated by calculating the stress intensity factor from these numerical results.

References

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