数理解析研究所講究録
第 1720 巻 2010 年 1-27

HARMONIC MAPS OF SURFACES AND INTEGRABLE SYSTEM APPROACH (A SURVEY)
(曲面の調和写像と可積分系的アプローチ (サーベイ))

大阪市立大学・大学院理学研究科　大仁田　義裕 (Yoshihiro Ohnita)
Department of Mathematics & OCAMI,
Graduate School of Science,
Osaka City University

INTRODUCTION

The purpose of this survey lecture is to provide an exposition on the theory of harmonic maps of surfaces, especially integrable system approach to harmonic map theory of surfaces into symmetric spaces. For the recent progress in this area, see, e. g. [OCAMI2008],

The harmonic map theory of surfaces into symmetric spaces investigates the construction, the classification and the moduli spaces of solutions to the harmonic map equations. The content of this article consists of the following topics:

(1) Harmonic map equation of Riemann surfaces into Lie groups and symmetric spaces.
(2) Extended solutions of the harmonic map equation.
(3) Loop groups and infinite dimensional Grassmannian.
(4) Loop group actions and DPW representation formulas.
(5) Uniton transforms and harmonic maps of finite uniton number.
(6) Harmonic maps of finite type and harmonic maps of tori.

This article is based on the author's lectures at the RIMS meeting "The Progress and View of Harmonic Map Theory", organized by Professor Hiroshi Iriyeh (Tokyo Denki University), RIMS, Kyoto Univ., 2 (Wed)-4 (Thu) June, 2010. The author would like to thank Hiroshi Iriyeh for his excellent organization and his kind invitation to a keynote lecture at the meeting.

1. HARMONIC MAP EQUATIONS

1.1. **Harmonic maps of Riemann manifolds.** Let (M^m, g_M) be an m-dimensional Riemanninan manifold and (N^n, g_N) be an n-dimensional Riemanninan manifold. Let $\varphi : M^m \to N^n$ be a smooth map.

2010 *Mathematics Subject Classification.* Primary 53C43, Secondary 58E20, 37K25.
Key words and phrases. harmonic map, Riemann surface, integrable system.
Partially supported by JSPS Grant-in-Aid for Scientific Research (A) No. 19204006 and and the Priority Research of Osaka City University "Mathematics of knots and wide-angle evolutions to scientific objects ".

Definition 1.1. The *energy functional* for smooth maps φ is defined by

$$E(\varphi) := \frac{1}{2}\int_M ||d\varphi||^2 dv_g.$$

Definition 1.2. φ is a *harmonic map*

$\underset{\text{def}}{\Longleftrightarrow}$

For any compact supported C^∞-variation $\{\varphi_t\}$ of φ,

$$\frac{d}{dt}E(\varphi_t)|_{t=0} = 0\,.$$

Example. (1) Constant maps.

(2) Geodesics = 1-dimensional harmonic maps $(\dim(M) = 1)$.

(3) Minimal surfaces (surfaces satisfying the equations of *soup films*) = conformal harmonic maps.

(4) The Gauss map of constant mean curvature surfaces (surfaces satisfying the equations of *soup bubbles*) is a harmonic map into a 2-dimensional unit sphere,

(5) Besides so many various examples of harmonic maps are known (cf. J. Eells and L. Lemaire, Two Reports on Harmonic Maps, [6]).

Generally the harmonic map theory has different aspects in the cases $\dim(M) = 1$, $\dim(M) = 2$ and $\dim(M) \geq 3$, respectively.

Let $\varphi : M \longrightarrow N$ be a smooth map.

$$\begin{tikzcd}
\varphi^{-1}TN \arrow[r] \arrow[d, "\nabla^{\varphi}=\varphi^{-1}\nabla"'] & (TN, \nabla^N) \arrow[d] \\
(M, g_M) \arrow[r, "\varphi"] & (N, g_N)
\end{tikzcd}$$

The *second fundamental form* of a smooth map φ is defined by

$$\beta(X, Y) := \nabla^{\varphi}_X d\varphi(Y) - d\varphi(\nabla^M_X Y) \quad (\forall X, Y \in C^\infty(TM)).$$

The *tension field* of the map φ is defined by

$$\tau(\varphi) := (g_M)^{ij}\beta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \in C^\infty(\varphi^{-1}TN).$$

Definition 1.3. *Harmonic Map Equation* (HME) :

$$\tau(\varphi) = 0\,.$$

Let $\varphi : M \longrightarrow N$ be a smooth map. Suppose that N is equipped with a semi-Riemannian metric g_N, or more generally a torsion-free affine connection ∇^N.

Then

φ is a harmonic map

$\iff$

$$\tau(\varphi) = g_M^{ij}\left(\frac{\partial^2 \varphi^a}{\partial x^i \partial x^j} + (\Gamma_N)^a_{bc}\frac{\partial \varphi^b}{\partial x^i}\frac{\partial \varphi^c}{\partial x^j} - (\Gamma_M)^k_{ij}\frac{\partial \varphi^a}{\partial x^k}\right)\frac{\partial}{\partial u^a} = 0 .$$

Here $(g_M)_{ij}$, g_M^{ij}, $(\Gamma_M)^k_{ij}$ denotes the components of g_M and its Levi-Civita connection, and $(\Gamma_N)^a_{bc}$ denote the the components of the Levi-Civita connection of g_N, or a torsion-free affine connection equipped on N.

1.2. **Harmonic maps of Riemann surfaces.**

Fact. In the case when M is 2-dimensional, the energy functional and harmonicity of smooth maps are invariant under conformal deformations of a Riemannian metric of M (**conformal invariance!**).

Suppose that M is an oriented 2-dimensional smooth manifold. Let

$$[g] := \{\rho g \mid \rho \text{ is a positive smooth function on } M\}$$

be a conformal class of a Riemannian metric g of M.

As a domain manifold of harmonic maps, we consider a Riemann surface (i.e. a 1-dimensional complex manifold) $(M, [g]) = (M, J)$ rather than an oriented 2-dimensional Riemannian manifold (M, g).

Lemma 1.1. $\varphi : (M, [g]) = (M, J) \to (N, \nabla^N)$ *is a harmonic map*
$\iff$

$$\nabla^{\varphi}_{\frac{\partial}{\partial \bar{z}}} d\varphi\left(\frac{\partial}{\partial z}\right) = 0 .$$

Here $\{z, \bar{z}\}$ *denotes a local complex coordinate system of the Riemann surface* (M, J).

This harmonic map equations means that $d\varphi\left(\frac{\partial}{\partial z}\right)$ is a local holomorphic section of $\varphi^{-1}(TN)^{\mathbf{C}}$ with the holomorphic vector bundle structure defined by the $\bar{\partial}$-operator $\nabla^{\varphi}_{\frac{\partial}{\partial \bar{z}}}$

1.3. **Famous theorems on harmonic maps.** The first result is a classical result due to the direct method of variations as follows:

Theorem 1.1. *Let M and N be two compact Roemannnian manifolds. Suppose that* $\dim(M) = 1$, *that is,* $M = S^1$ *(a circle). Then any homotopy class of continuous map from M to N contains a harmonic map of minimum energy. Hence each element of the fundamental group* $\pi_1(N)$ *of N can be represented by a closed geodesic of minimum energy.*

The second one is the Eells and Sampson's theorem shown by nonlinear heat equation method (**a breakthrough**!).

Theorem 1.2 (Eells-Sampson, 1964). *Let M and N be two compact Roemannnian manifolds. Suppose that the sectional curvatures of N are non-positive. Then Then any homotopy class of continuous map from M to N contains a harmonic map of minimum energy.*

Remark. The homotopy class of a continuous map of degree ± 1 from a torus T^2 (a compact Riemann surface of genus 1) to a unit 2-sphere S^2 does not contain any harmonic map (cf. [6]).

Thirdly, we mention Sacks-Uhlenbeck's reults [28]. Let M be a compact Riemann surface and N be a compact Riemannain manifold. For each $\alpha \geq 1$, the *α-energy functional* for smooth maps $\varphi : M \to N$ is define as follows:

$$E_\alpha(\varphi) := \int_M (1 + \|d\varphi\|^2)^\alpha dv_M$$

Here dv_M is a volume form of a Riemannian metric of M. If $\alpha = 1$, then E_α is equivalent to the usual energy functional E. It is known that if $\alpha > 1$, then E_α satisfies the Palais-Smale Condition (C).

The first result of Sacks-Uhlenbeck is the Removability theorem for an isolated singularity of harmonic maps :

Theorem 1.3 (Sacks-Uhlenbeck). *Let N be a compact Riemannian manifold. Suppose that a harmonic map $\varphi : D \setminus \{p\} \to N$ defined outside a point p in a domain D of the Gauss plane $\mathbf{C}$. If φ has finite energy, then φ extends to a smooth harmonic map from M to N. In particular, any harmonic map $\varphi : \mathbf{C} \to N$ with finite energy from the complex plane $\mathbf{C}$ to N extends to a harmonic map from a Riemann sphere $S^2 = \mathbf{C} \cup \{\infty\}$ to N.*

The second result is on convergence, degeneration and bubbling of harmonic maps:

Theorem 1.4 (Sacks-Uhlenbeck). *Let M be a compact Riemann surface and N be a compact Riemannain manifold. Suppose that $\alpha(i) \geq 1$, $\alpha(i) \to 1\, (i \to \infty)$, $\varphi_{\alpha(i)} : M \setminus \{p\} \to N$ is a sequence of critical maps of $E_{\alpha(i)}$ and $E(\varphi_{\alpha(i)}) \leq$ C (positive constant). Then there exist a subsequence $\{\alpha(j)\} \subset \{\alpha(i)\}$, a finite set $\{p_1, \cdots, p_\ell\} \subset M$, a harmonic map $\varphi_\infty : M \to N$, non-constant harmonic maps $\tilde{\varphi}^{(k)} : S^2 \longrightarrow N\, (k = 1, \cdots, \ell)$ s such that*

(1) *$\varphi_{\alpha(j)} \to \varphi_\infty\, (j \to \infty)$ C^1-converges on any compact subset of $M \setminus \{p_1, \cdots, p_\ell\}$.*
(2) *$e(\varphi_{\alpha(j)}) \to e(\varphi_\infty) + \sum_{k=1}^{\ell} m_k\, \delta(p_k)$ converges as measures. In particular, $E(\varphi_\infty) \to E(\varphi_\infty) + \sum_{k=1}^{\ell} E(\tilde{\varphi}^{(k)}) \leq \overline{\lim}_{j\to\infty} E(\varphi_{\alpha(j)}) \leq$ C and $E(\tilde{\varphi}^{(k)}) \leq m_k$.*

In my lecture at the RIMS meeting I mentioned about Micallef and Moore [17] on sphere theorem for compact Riemannian manifolds with positive isotropic sectional curvature as one of most successful applications of the Sack-Uhlenbeck's theory. There has been many other important applications and progress of the Sack-Uhlenbeck's theory: the construction of "Bubble tree", the compactification of the moduli space of harmonic maps, J-holomorphic curves and the Gromov-Witten theory, etc.

2. Harmonic maps into symmetric spaces

2.1. **Symmetric Spaces.** Symmetric spaces form a class of smooth manifolds of particularly high symmetry. Here we give a brief explanation on: What is a symmetric space ? We refer [12], [15] as the excellent textbooks.

We give attention to the following two conditions on a smooth manifold N, which are equivalent each other :

(1) N is a semi-Riemannian manifold (or more generally a smooth manifold with a torison-free affine connection) such that the geodesic symmetry at each point of N extends to an isometry (affine transformation) of N.
(2) N is a homogeneous space
$$N = G/K,$$
where G is a Lie group with an involutive automorphism σ and K is a closed subgroup of G such that $G_\sigma^0 \subset K \subset G_\sigma$. Here G_σ denotes the subgroup of G consisting of all elements fixed by σ and G_σ^0 its identity component.

N is called a *symmetric space* if N satisfies such a condition. A symmetric space is locally characterized by the curvature condition $\nabla R = 0$.

Examples of symmetric spaces.

(1) Euclidean space $\mathbf{E}^n$, standard sphere $S^n(c)$, real hyperbolic space form $H^n(c)$.
(2) Projective spaces $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathbf{O}P^2 = F_4/Spin(9)$. Grassmann manifolds of k-planes $\mathrm{Gr}_k(\mathbf{R}^n)$, $\mathrm{Gr}_k(\mathbf{C}^n)$, $\mathrm{Gr}_k(\mathbf{H}^n)$, etc.
(3) Lie groups G, S^1, $SO(3)$, $SU(2)$, $SO(n)$, $SU(n)$, $U(n)$, G_2, etc. Homogeneous spaces $G^{\mathbf{C}}/G$, etc.

Riemannian symmetric spaces were created and classified first by Elie Cartan. There is a duality between Riemannian symmetric spaces of compact type (nonnegatively curved!) and Riemannian symmetric spaces of noncompact type (nonpositively curved!) such as S^n and H^n. All simply connected irreducible Riemannian symmetric spaces are classified into 9 types of group manifolds (4 classical types and 5 exceptional types) and 19 types of non-group manifolds (7 classical types and 12 exceptional types).

Non-symmetric homogeneous spaces related to symmetric spaces are also important in geometry of symmetric spaces. For instance, Hopf fibrations, genralized flag manifolds, twistor spaces, etc.

2.2. **Harmonic map equations of Riemann surfaces into Lie groups.** Let M be a Riemann surface and G be a compact Lie group equipped with biinvariant Riemannian metric g_M. Let $\theta = \theta_G$ denote the left-invariant Maurer-Cartan form of G and it is fundamental that $\theta = \theta_G$ satisfies the Maurer-Cartan equations
$$d\theta_G + \frac{1}{2}[\theta_G \wedge \theta_G] = 0. \tag{2.1}$$
Here $[\beta_1 \wedge \beta_2](X, Y) := [\beta_1(X), \beta_2(Y)] - [\beta_1(Y), \beta_2(X)]$.

Let $\varphi : M \to G$ be a smooth map. Set

$$\alpha := \varphi^*\theta = \varphi^{-1}d\varphi = \alpha' + \alpha'',$$

where α' and α'' denote the $(1,0)$-part and the $(0,1)$-part of α, respectively. Then α is a 1-form on M with values in $\mathfrak{g}$ and by (2.1) α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

The harmonic map equation for the map φ is written as

$$\bar{\partial}\alpha' + \frac{1}{2}[\alpha' \wedge \alpha''] = 0. \tag{2.2}$$

By using (2.2) we can show that (2.2) is equivalent to the equation

$$d * \alpha = -\sqrt{-1}\bar{\partial}\alpha' + \sqrt{-1}\partial\alpha'' = \sqrt{-1}(-\bar{\partial}\alpha' + \partial\alpha'') = 0. \tag{2.3}$$

2.3. **Zero curvature formalism of harmonic map equation.** For each $\lambda \in S^1$ or $\lambda \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, we define

$$\alpha_\lambda := \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha'',$$

which α_λ is a 1-form on M with values in $\mathfrak{g}$ for $\lambda \in S^1$ and $\mathfrak{g}^{\mathbf{C}}$ for $\lambda \in \mathbf{C}^*$.

Theorem 2.1 ([23], [35], [36], [32]). *The system of the Maurer-Cartan equation* (2.2) *and the harmonic map equation* (2.3) *is equivalent to the system of the Maurer-Cartan equations*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad (\forall \lambda \in S^1 \text{ or } \mathbf{C}^*) \tag{2.4}$$

This equation is also called the "Uhlenbeck equation".

2.4. **Lax equation formalism of harmonic map equation.** The equation (2.4) is equivalent to the Lax equation

$$\begin{aligned} &\frac{\partial L}{\partial \bar{z}} = [K, L], \\ &L := \frac{\partial}{\partial z} + (1 - \lambda^{-1})A_z, \quad K := -(1 - \lambda)A_{\bar{z}}. \end{aligned} \tag{2.5}$$

Here λ is the *spectral parameter* and set

$$A_z := \frac{1}{2}\alpha'\left(\frac{\partial}{\partial z}\right), \quad A_{\bar{z}} := \frac{1}{2}\alpha''\left(\frac{\partial}{\partial \bar{z}}\right). \tag{2.6}$$

2.5. **Gauge-theoretic formulation of harmonic map equation.** The harmonic map equation from a Riemann surface M to a Lie group G can be formulated as the Yang-Mills-Higgs equation over a Riemann surface in the following way. Let $P = M \times G$ be a trivial principal bundle with structure group G over a Riemann surface M. Let $\mathcal{A}_P$ denote the affine space of all smooth connections on P and $\Omega^1(\mathfrak{g}_P)$ denote the vector space of all smooth 1-forms with values in the adjoint bundle $\mathfrak{g}_P$. Let $A \in \mathcal{A}_P$ be a connection on P defined by $d_A = d+\frac{1}{2}\alpha$ and $\phi \in \Omega^1(\mathfrak{g}_P)$ the Higgs field defined by $\phi = \frac{1}{2}\alpha$. Then the harmonic map equation is described as the Yang-Mills-Higgs equation

$$\begin{cases} F(A) + \dfrac{1}{2}[\phi \wedge \phi] = 0, \\ d_A\phi = d_A * \phi = 0. \end{cases} \tag{2.7}$$

On the other hand, the slightly different Yang-Mills-Higgs equation over a Riemann surface M

$$\begin{cases} F(A) - \dfrac{1}{2}[\phi \wedge \phi] = 0, \\ d_A\phi = d_A * \phi = 0 \end{cases} \tag{2.8}$$

locally corresponds to the harmonic map equation into noncompact symmetric space $G^{\mathbf{C}}/G$ and the moduli space of its solutions is called the *Hitchin System.* See also [18], [19].

2.6. **Extended solutions of the harmonic map equation.** A solution

$$\Phi_\lambda : M \to G \quad (\lambda \in S^1)$$

or

$$\Phi_\lambda : M \to G^{\mathbf{C}} \quad (\lambda \in \mathbf{C}^*)$$

to the linear partial differential equations

$$\Phi^*\theta = \Phi^{-1}d\Phi = \alpha_\lambda \quad (\forall\, \lambda \in S^1 \text{ or } \mathbf{C}^*) \tag{2.9}$$

or equivalently locally

$$\Phi^{-1}\frac{\partial}{\partial z}\Phi = (1-\lambda^{-1})A_z,\ \Phi^{-1}\frac{\partial}{\partial \bar{z}}\Phi = (1-\lambda)A_{\bar{z}},\ (\forall\, \lambda \in S^1 \text{ or } \mathbf{C}^*) \tag{2.10}$$

is called an *extended solution* of a harmonic map φ (Uhlenbeck [32]). Here we set $\alpha' = 2A_z dz$ and $\alpha'' = 2A_{\bar{z}} d\bar{z}$.

If M is simply connected, there exists uniquely an extended solution Φ for any initial condition $\Phi_\lambda(z_0) = \gamma(\lambda)$ ($\forall\, \lambda \in S^1$ or $\mathbf{C}^*$). Here γ can be considered as a loop in a Lie group.

2.7. **Extended solutions and loop groups.** ([26])

The (free) loop group of G is defined by

$$\Lambda G := \{\gamma : S^1 \to G \mid C^\infty\}.$$

The based loop group of G is defined by

$$\Omega G := \{\gamma : S^1 \to G \mid C^\infty,\ \gamma(1) = e\}.$$

The extended solution of a harmonic map $\Phi_\lambda = \sum_{i=-\infty}^{\infty} \lambda^i T_i$ with $\Phi_1 = e$ can be considered as a map into the based loop group

$$\Phi : M \ni z \longmapsto \Phi(z) \in \Omega G.$$

Assume that G is a compact Lie group. It is known that ΩG has the infinite dimensional complex Kähler manifold structure and if $H^3(G, \mathbf{Z}) \cong H^2(\Omega G, \mathbf{Z}) \cong \mathbf{Z}$, then it is Einstein-Kähler. The Kähler form (and thus a symplectic form) is given by

$$\begin{aligned} \omega_{\Omega G}(\xi, \eta) &:= \int_0^1 \langle \xi'(t), \eta(t) \rangle dt \\ &= \langle \xi'(t), \eta(t) \rangle_{L^2} = \langle J_{\Omega G}(\xi(t)), \eta(t) \rangle_{L^2_{1/2}} \end{aligned} \tag{2.11}$$

for each $\xi, \eta \in \Omega\mathfrak{g}$.

Proposition 2.1. *An extended solution $\Phi_\lambda : M \to G\,(\lambda \in S^1)$ of a harmonic map with $\Phi_1 = e$ is nothing but a holomorphic map $\Phi : M \to \Omega G$ whose differential $d\Phi$ satisfying the condition*

$$\Phi^{-1} d\Phi\left(\frac{\partial}{\partial z}\right) \in (1 - \lambda^{-1})\mathfrak{g}^{\mathbf{C}}.$$

2.8. **Correspondence between harmonic maps and extended solutions.** Assume that M is a simply connected Riemann surface, that is, is conformal to Riemann sphere S^2, Gauss plane $\mathbf{C}$, unit open disk $B^2(1)$. Then from the above argument we see that there is a bijective correspondence between the quotient space of all extended solutions modulo left translations by loops $\gamma : S^1 \to G$

$$\begin{aligned} &\Omega G \setminus \{\, \Phi : M \longrightarrow \Omega G \mid \text{ extended solutions } \} \\ &\cong \{\, \Phi : M \longrightarrow \Omega G \mid \text{ extended solutions}, \Phi(z_0) = e \,\} \end{aligned}$$

and the quotient space of harmonic maps modulo left translations by elements of G

$$\begin{aligned} &G \setminus \{\varphi : M \longrightarrow G \mid \text{ harmonic maps}\} \\ &\cong \{\, \varphi : M \longrightarrow G \mid \text{ harmonic maps}, \varphi(z_0) = e \,\}. \end{aligned}$$

Remark. The extended solutions for harmonic maps of a Riemann surface M into a symmetric space G/K can also be formulated (cf. [9], [7]). The *Cartan immersion* of a symmetric space G/K into G is fitting and useful in the formulation. It is known that every compact Lie group and every compact symmetric spaces can be immersed into a unitary group and a complex Grassmann manifold as a totally geodesic submanifold. Note that a composition $\iota \circ \varphi$ of a harmonic map φ and a totally geodesic immersion ι is also a harmonic map.

3. Infinite dimensional Grassmannian and loop groups

The harmonic map theory in symmetric spaces is built up in the framework of geometry of loop groups and infinite dimensional Grassmannian due to Pressley-Segal [25], Segal-Wilson[27].

Suppose that $G = U(n)$ (for the simplicity). Define

$$H^{(n)} := L^2(S^1, \mathbf{C}^n),$$
$$H_+^{(n)} := \{f \in L^2(S^1, \mathbf{C}^n) \mid f(\lambda) = \sum_{i \geq 0} \lambda^i c_i\},$$
$$H_-^{(n)} := \{f \in L^2(S^1, \mathbf{C}^n) \mid f(\lambda) = \sum_{i < 0} \lambda^i c_i\},$$
$$H^{(n)} = H_+^{(n)} \oplus H_-^{(n)}.$$

Define an infinite dimensional complex Grassmannian $\mathrm{Gr}(H^{(n)})$ by

$\mathrm{Gr}(H^{(n)}) := \{W \mid$ a closed vector subspace of $H^{(n)}$ satisfying the conditions (1), (2)$\}$

(1) $\mathrm{pr}_+ : W \to H_+^{(n)}$ is a Fredholm linear operator,

(2) $\mathrm{pr}_- : W \to H_-^{(n)}$ is a Hilbert-Schmidt linear operator.

Moreover, we define an infinite dimensional submanifold of the infinite dimensional Grassmannian $\mathrm{Gr}(H^{(n)})$ as follows:

$$\mathrm{Gr}_\infty^{(n)} := \{W \in \mathrm{Gr}(H^{(n)}) \mid W \text{ satisfying the conditions (3), (4)}\}$$

(3) $\lambda W \subset W$.

(4) $\mathrm{pr}_+(W^\perp)$, $\mathrm{pr}_-(W)$ consists of C^∞-functions.

Then there is a diffeomorphism (after a suitable completion) between

$$\Omega G \ni \gamma \longmapsto \gamma H_+ \in \mathrm{Gr}_\infty^{(n)}.$$

$\mathrm{Gr}_\infty^{(n)}$ is called the *infinite dimensional Grassmannian model* of ΩG.

The two fundamental splitting theorems for loops are obtained from theory of infinite dimensional Grassmannian models.

Let T denote the maximal torus of G, that is, the subgroup of all diagonal matrices of $U(n)$. Define the complex (free) loop group of $G^{\mathbf{C}}$ by

$$\Lambda G^{\mathbf{C}} := \{\gamma : S^1 \to G^{\mathbf{C}} \mid C^\infty\}$$

and its subgroups by

$$\Lambda^+ G^{\mathbf{C}} := \{\gamma \in \Lambda G^{\mathbf{C}} \mid \gamma \text{ extends continuously to holomorphic } D_0 \longrightarrow G^{\mathbf{C}}\},$$
$$\Lambda^- G^{\mathbf{C}} := \{\gamma \in \Lambda G^{\mathbf{C}} \mid \gamma \text{ extends continuously to holomorphic } D_\infty \longrightarrow G^{\mathbf{C}}\},$$
$$\Lambda_1^- G^{\mathbf{C}} := \{\gamma \in \Lambda^- G^{\mathbf{C}} \mid \gamma(\infty) = e\},$$
$$\check{T} := \{\delta : S^1 \longrightarrow T \subset G \text{ continuous group homomorphism}\},$$

Here

$$D_0 := \{\lambda \in \mathbf{C} \cup \{\infty\} \mid |\lambda| < 1\},$$
$$D_\infty := \{\lambda \in \mathbf{C} \cup \{\infty\} \mid |\lambda| > 1\}.$$

The following splitting theorem is called the *polar decomposition* or *Iwasawa decomposition* of the complex loop group $\Lambda G^{\mathbf{C}}$:

Theorem 3.1 ([25]). *Any $\gamma \in \Lambda G^{\mathbf{C}}$ can be uniquely decomposed into*

$$\gamma = \gamma_u \gamma_+ ,$$

where $\gamma_u \in \Omega G$, $\gamma_+ \in \Lambda^+ G^{\mathbf{C}}$. The multiplication map

$$\Omega G \times \Lambda^+ G^{\mathbf{C}} \ni (\gamma_u, \gamma_+) \longmapsto \gamma_u \gamma_+ \in \Lambda G^{\mathbf{C}}$$

is a diffeomorphism (after a suitable completion).

This theorem was shown by proving

$$\Omega G \cong Gr_{\infty}^{(n)} \cong \Lambda G^{\mathbf{C}} / \Lambda^+ G^{\mathbf{C}} .$$

The next splitting theorem is called the *Birkhoff decomposition* of the complex loop group $\Lambda G^{\mathbf{C}}$:

Theorem 3.2 ([25]). *Any $\gamma \in \Lambda G^{\mathbf{C}}$ can be decomposed into*

$$\gamma = \gamma_- \delta \gamma_+ ,$$

where $\gamma_- \in \Lambda^- G^{\mathbf{C}}$, $\delta \in \check{T}$, $\gamma_+ \in \Lambda^+ G^{\mathbf{C}}$. Moreover, $\Lambda^- G^{\mathbf{C}} \cdot \Lambda^+ G^{\mathbf{C}}$ is a dense open subset ("Bigg Cell") of the identity component of $\Lambda G^{\mathbf{C}}$ and the multiplication map

$$\Lambda_1^- G^{\mathbf{C}} \times \Lambda^+ G^{\mathbf{C}} \ni (\gamma_-, \gamma_+) \longmapsto \gamma_- \gamma_+ \in \Lambda^- G^{\mathbf{C}} \cdot \Lambda^+ G^{\mathbf{C}} \subset \Lambda G^{\mathbf{C}}$$

is a diffeomorphism (after a suitable completion).

The Birkhoff splitting theorem for loops describes the Morse theoretic stratification of ΩG for the energy functional of loops ([24]). The complement of the Big Cell can be characterized by zeros of a canonical global holomorphic section σ of the dual determinant line bundle Det^* of $\mathrm{Gr}(H^{(n)})$ (cf. [27]).

Moreover we introduce another setting of loop groups and it is necessary to define loop group actions on extended solutions of harmonic maps ([32], [1], [8]).

Choose a real number ε with $0 < \varepsilon < 1$. Take two circles on a Riemann sphere $\mathbf{C} \cup \{\infty\}$ as follows :

$$C_\varepsilon := \{\lambda \in \mathbf{C} \mid |\lambda| = \varepsilon\},$$
$$C_{\varepsilon^{-1}} := \{\lambda \in \mathbf{C} \mid |\lambda| = \varepsilon^{-1}\}.$$

Regarding C_ε as a circle with center O we denote by I_ε its interior. Regarding $C_{\varepsilon^{-1}}$ as a circle with center ∞, we denote by $I_{\varepsilon^{-1}}$ its interior.

$$I_\varepsilon := \{\lambda \in \mathbf{C} \mid |\lambda| < \varepsilon\},$$
$$I_{\varepsilon^{-1}} := \{\lambda \in \mathbf{C} \mid |\lambda| > \varepsilon^{-1}\}.$$

Set $I := I_\varepsilon \sqcup I_{\varepsilon^{-1}}$. We denote the complementary subset of $\mathbf{C} \cup \{\infty\}$ to the closure $\bar{I}$ of I by

$$E := (\mathbf{C} \cup \{\infty\}) \setminus \bar{I}.$$

At this setting we define different groups of loops in $G^{\mathbf{C}}$.

$\Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}} := \{g : C_\varepsilon \sqcup C_{\varepsilon^{-1}} \longrightarrow G^{\mathbf{C}},\ \text{smooth map}\}$,
$\Lambda^{E,\varepsilon}G^{\mathbf{C}} := \{g \in \Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}} \mid g \text{ extends continuously to holomorphic } g^E : E \to G^{\mathbf{C}}\}$,
$\Lambda_1^{E,\varepsilon}G^{\mathbf{C}} := \{g \in \Lambda^{E,\varepsilon}G^{\mathbf{C}} \mid g^E(1) = e\}$,
$\Lambda^{I,\varepsilon}G^{\mathbf{C}} := \{g \in \Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}} \mid g \text{ extends continuously to holomorphic } g^I : I \to G^{\mathbf{C}}\}$.

In our case we define the *reality condition* on $g \in \Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}$ as follows :

$$g(\lambda)^{-1} = g(\bar{\lambda}^{-1})^* \quad (\forall \lambda \in C_\varepsilon \sqcup C_{\varepsilon^{-1}}).$$

$$\begin{aligned}
\Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}} &:= \{g \in \Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}} \mid g \text{ satisfies the reality condition}\},\\
\Lambda_{\mathbf{R}}^{E,\varepsilon}G^{\mathbf{C}} &:= \Lambda^{E,\varepsilon}G^{\mathbf{C}} \cap \Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}},\\
\Lambda_{\mathbf{R},1}^{E,\varepsilon}G^{\mathbf{C}} &:= \Lambda_1^{E,\varepsilon}G^{\mathbf{C}} \cap \Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}},\\
\Lambda_{\mathbf{R}}^{I,\varepsilon}G^{\mathbf{C}} &:= \Lambda^{I,\varepsilon}G^{\mathbf{C}} \cap \Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}.
\end{aligned}$$

We describe the splitting theorem for these loop groups. This formulation was inspired by Uhlenbeck [32]. The latter half of the statement is essential and was proved by Ian McIntosh [16]. His proof is an ingenious combination of the Iwasawa decomposition and the Birkhoff decomposition.

Theorem 3.3 ([32], [1], [8], [16]). $\Lambda^{E,\varepsilon}G^{\mathbf{C}} \cdot \Lambda^{I,\varepsilon}G^{\mathbf{C}}$ *is a dense open subset of the identity component of* $\Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}$*, and the multiplication map*

$$\Lambda^{E,\varepsilon}G^{\mathbf{C}} \times \Lambda^{I,\varepsilon}G^{\mathbf{C}} \ni (\gamma_E, \gamma_I) \longmapsto \gamma_E\gamma_I \in \Lambda^{E,\varepsilon}G^{\mathbf{C}} \cdot \Lambda^{I,\varepsilon}G^{\mathbf{C}} \subset \Lambda^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}$$

is a diffeomorphism (after a suitable completion). Moreover, the restriction of this multiplication map to real elements induces a diffeomorphism onto $\Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}$*:*

$$\Lambda_{\mathbf{R},1}^{E,\varepsilon}G^{\mathbf{C}} \times \Lambda_{\mathbf{R}}^{I,\varepsilon}G^{\mathbf{C}} \longrightarrow \Lambda_{\mathbf{R}}^{E,\varepsilon}G^{\mathbf{C}} \cdot \Lambda_{\mathbf{R}}^{I,\varepsilon}G^{\mathbf{C}} = \Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}}G^{\mathbf{C}}.$$

For each nonnegative integer $k \geq 0$ or $k = \infty$, we define certain subsets of $\Omega G^{\mathbf{C}}$ and ΩG as follows:

$$\begin{aligned}
\mathcal{X}_k :=& \{\delta : \mathbf{C}^* \longrightarrow G^{\mathbf{C}} \mid \delta \text{ is holomorphic on } \mathbf{C}^*,\ \delta(1) = e,\\
&\qquad \delta(\lambda) = \sum_{i=-k}^{k} \lambda^i A_i,\ \delta(\lambda)^{-1} = \sum_{i=-k}^{k} \lambda^i B_i, \},\\
\mathcal{X}_{k,\mathbf{R}} :=& \{\delta \in \mathcal{X}_k \mid \delta \text{ satisfies the reality condition},\\
&\qquad \text{i.e. } \delta(\lambda)^{-1} = \delta(\bar{\lambda}^{-1})^* \ (\forall \lambda \in \mathbf{C}^*)\}.
\end{aligned}$$

Here notice that $\mathcal{X}_0 \subset \mathcal{X}_1 \cdots \subset \mathcal{X}_k \subset \mathcal{X}_{k+1} \subset \cdots \subset \mathcal{X}_\infty \subset \Omega G^{\mathbf{C}}$, $\mathcal{X}_\infty$ is a subgroup of $\Omega G^{\mathbf{C}}$ and $\mathcal{X}_{0,\mathbf{R}} \subset \mathcal{X}_{1,\mathbf{R}} \cdots \subset \mathcal{X}_{k,\mathbf{R}} \subset \mathcal{X}_{k+1,\mathbf{R}} \subset \cdots \subset \mathcal{X}_{\infty,\mathbf{R}} \subset \Omega G$, $\mathcal{X}_{\infty,\mathbf{R}}$ is a subgroup of ΩG.

4. Loop group actions and representation formulas for harmonic maps

In this section we explain two fundamental and important structures of harmonic map from Riemann surfaces to Lie groups and symmetric spaces The first one is a structure of *infinite dimensional group actions* on all such harmonic maps. The second one is a structure of *Weierstrass type representation formulas*, which represents locally all such harmonic maps in terms of infinite dimensional holomorphic potentials.

4.1. **S^1-action on harmonic maps.** The group $S^1 = \{\zeta \in \mathbf{C}^* \mid |\zeta| = 1\}$ acts on the based loop group ΩG by

$$(\zeta^\natural \gamma)(\lambda) := \gamma(\zeta^{-1}\lambda)\gamma^{-1}(\zeta^{-1}) \quad (\zeta \in S^1, \gamma \in \Omega G).$$

The S^1-action on extended solutions (and thus harmonic maps) is defined as follows (C.-L. Terng): For each $\zeta \in S^1$ and each extended solution $\Phi_\lambda : M \to G\,(\lambda \in S^1)$, we define

$$(\zeta^\natural \Phi)_\lambda := \Phi_{\zeta^{-1}\lambda}\Phi_{\zeta^{-1}}^{-1}.$$

Then the map $(\zeta^\natural \Phi)_\lambda : M \to G\,(\lambda \in S^1)$ is a new extended solution.

Moreover the semigroup $\mathbf{C}^*_{\leq 1}$ and the complex group $\mathbf{C}^*$ also acts on extended solutions of harmonic maps ([8], [33]).

4.2. **Loop group action $\sharp$.** In this subsection we assume the setting of the Birkhoff-Uhlenbeck decomposition in Section 3.

There is a natural injection

$$\Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \ni h \longmapsto h^E|_{S^1} \in \Omega G,$$

where h^E denotes the continuous extension of $h \in \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}}$ to a holomorphic map $h^E : I \to G^{\mathbf{C}}$. We regard this injection as

$$\Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G.$$

Now, by using the Birkhoff-Uhlenbeck Decomposition Theorem 3.3, we define the group action $\sharp$ of the infinite dimensional group $\Lambda^{\varepsilon,\varepsilon^{-1}}_{\mathbf{R}} G^{\mathbf{C}}$ on $\Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G$ as follows : For each $g \in \Lambda^{\varepsilon,\varepsilon^{-1}}_{\mathbf{R}} G^{\mathbf{C}}$ and each $h \in \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G$,

$$g^\sharp h := gh(gh)_I^{-1} = (gh)_E \in \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G.$$

Theorem 4.1 ([32], [1], [8]). *Each $g \in \Lambda^{\varepsilon,\varepsilon^{-1}}_{\mathbf{R}} G^{\mathbf{C}}$ and each extended solution $\Phi : M \to \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}}$, $g^\sharp \Phi : M \to \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G$ is a new extended solution.*

This group action $\sharp$ is called the *Birkhoff-Uhlenbeck group action* (cf. [1], [8]).

4.3. **Loop group action $\natural$.** By the Iwasawa Decomposition Theorem 3.1, the natural group action $\natural$ of the infinite dimensional group $\Lambda G^{\mathbf{C}}$ on ΩG is defined as follows : For each $\gamma \in \Lambda G^{\mathbf{C}}$ and each $\delta \in \Omega G$,

$$\gamma^{\natural}\delta := \gamma\delta(\gamma\delta)_{+}^{-1} = (\gamma\delta)_{u} \in \Omega G .$$

Theorem 4.2 ([8]). *For each $\gamma \in \Lambda G^{\mathbf{C}}$ and each extended solution $\Phi : M \to \Omega G$, $\gamma^{\natural}\Phi : M \to \Omega G$ is a new extended solution.*

This group action $\natural$ is called the *natural group action* (cf. [8]).

4.4. **Relationship between the Birkhoff-Uhlenbeck action $\sharp$ and the natural action $\natural$.** We easily see that the group actions $\sharp$ and $\natural$ of the subgroups $\Lambda_{\mathbf{R},1}^{E,\varepsilon} G^{\mathbf{C}}$ and ΩG are simply left translations of extended solutions by loops. Thus we should compare the group actions $\sharp$ and $\natural$ of $\Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}$ and $\Lambda^{+} G^{\mathbf{C}}$.

For any $\varepsilon > 0$, the group $\Lambda^{+} G^{\mathbf{C}}$ can be embedded into the group $\Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}$ by the following injective group homomorphism:

$$\Lambda^{+} G^{\mathbf{C}} \ni \gamma \longmapsto \hat{\gamma} \in \Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}, \tag{4.1}$$

where for each $\lambda \in \mathbf{C} \cup \{\infty\}$,

$$\hat{\gamma} := \begin{cases} \gamma(\lambda) & (\lambda \in \mathbf{C} \cup \{\infty\},\ |\lambda| \geq \varepsilon) \\ (\gamma(\bar{\lambda}^{-1})^{-1})^{*} & (\lambda \in \mathbf{C} \cup \{\infty\},\ |\lambda| \leq \varepsilon) . \end{cases} \tag{4.2}$$

Then we obtain

Theorem 4.3 ([8]). *For each $\lambda \in \Lambda^{+} G^{\mathbf{C}}$ and $\delta \in \mathcal{X}_{k,\mathbf{R}}$ $(0 \leq k \leq \infty)$,*

$$\gamma^{\natural}\delta = \hat{\gamma}^{\sharp}\delta . \tag{4.3}$$

Corollary 4.1 ([8]). *For each $\lambda \in \Lambda^{+} G^{\mathbf{C}}$ and extended solution $\Phi : M \to \Omega G$ such that Φ_{λ} is holomorphic in $\lambda \in \mathbf{C}^{*}$ entirely, we have*

$$\gamma^{\natural}\Phi = \hat{\gamma}^{\sharp}\Phi . \tag{4.4}$$

The properties of the loop group action for harmonic maps, its Morse theoretic aspect and applications to the study on spaces of harmonic maps were discussed in [8].

4.5. **DPW formula for harmonic maps (Iwasawa decomposition).** Another important structure of harmonic maps of Riemann surfaces into Lie groups and symmetric spaces is a Weierstrass type representation formula of all such harmonic maps in terms of holomorphic functions with values in a certain infinite dimensional vector space. It is due to Dorfmeister-Pedit-Wu ([5]), the so-called *DPW formula*, and here we shall explain their representation formula for harmonic maps.

Assume that $M \subset \mathbf{C}$ is a simply connected domain of the complex plane. Fix a base point $z_0 \in M$.

Let $\varphi : M \to G$ be a harmonic map. We may assume that $\varphi(z_0) = e$ after a suitable left translation of G. Let $\Phi : M \to \Omega G$ be its extended solution with $\Phi(z_0) = e$.

We consider the equation of the holomorphicity on $g = \Phi b : M \to \Lambda G^{\mathbf{C}}$ with respect to $b : M \to \Lambda^+ G^{\mathbf{C}}$:

$$0 = \bar\partial g = \bar\partial\Phi\, b + \Phi\, \bar\partial b\,.$$

It $\bar\partial$-equation for $b : M \to \Lambda^+ G^{\mathbf{C}}$

$$\begin{aligned}\bar\partial\, b &= -\,(\Phi^{-1}\bar\partial\Phi)\, b\\ &= -\,\frac{1}{2}(1-\lambda)\alpha''\, b\end{aligned} \tag{4.5}$$

Then there exists a solution $b : M \to \Lambda^+ G^{\mathbf{C}}$ to the $\bar\partial$-equation (4.5) satisfying $b(z_0) = e$, which has the freedom of right multiplication by holomorphic maps $h : M \to \Lambda^+ G^{\mathbf{C}}$ with $h(z_0) = e$. Thus we obtain $g = \Phi\, b : M \to \Lambda G^{\mathbf{C}}$ which is a holomorphic map in the sense that $\bar\partial g = 0$ and satisfies $g(z_0) = e$. Moreover we define $\mu_\varphi := g^{-1}dg$. Then we have a formula

$$\begin{aligned}\mu_\varphi &= g^{-1}dg = g^{-1}\partial g\\ &= b^{-1}(\Phi^{-1}\,\partial\Phi)b + b^{-1}\,\partial b\\ &= -\,\lambda^{-1}\mathrm{Ad}(b|_{\lambda=0})^{-1}(\alpha') + [\text{terms of } \lambda^i\ (\geq 0)]\,.\end{aligned}$$

Define an infinite dimensional complex vector space

$$\Lambda_{-1,\infty} := \left\{ \xi \in \Lambda\mathfrak{g}^{\mathbf{C}} \mid \xi \text{ has Fourier series expansion } \xi = \sum_{i=-1}^{\infty} \lambda^i \xi_i \right\}.$$

Denote by $\Omega^{1,0}(M, \Lambda_{-1,\infty})$ the complex vector space of all smooth $(1,0)$-forms with values in $\Lambda_{-1,\infty}$ defined on M. Then we define the infinite dimensional vector space of all *holomorphic potentials* with values in $\Lambda_{-1,\infty}$ by

$$\mathcal{P} := \left\{ \mu \in \Omega^{1,0}(M, \Lambda_{-1,\infty}) \mid \bar\partial\mu = 0 \right\}.$$

Each $\mu \in \mathcal{P}$ is expressed as

$$\mu = \sum_{i=-1}^{\infty} \lambda^i \mu_i = \mu_z\, dz\,,$$

where each μ_i is a holomorphic 1-form on M with values in $\mathfrak{g}^{\mathbf{C}}$ and μ_z is a holomorphic function with values in $\Lambda_{-1,\infty}$ on M. Then we have $\mu_\varphi \in \mathcal{P}$.

We discuss the inverse construction from μ to a harmonic map. For each $\mu \in \mathcal{P}$, it holds

$$d\mu + \frac{1}{2}[\mu \wedge \mu] = \bar\partial\mu = 0$$

and thus there exists a unique smooth map $g^\mu : M \to \Lambda G^{\mathbf{C}}$ such that $g^\mu(z_0) = e$ and $(g^\mu)^{-1}dg^\mu = \mu$. In particular, $g^\mu : M \to \Lambda G^{\mathbf{C}}$ is a holomorphic map in the sense that $\bar\partial g^\mu = 0$. By Iwasawa Decomposition Theorem 3.1, there exist uniquely $\Phi^\mu : M \to \Omega G$ and $b^\mu : M \to \Lambda^+ G$ such that

$$g^\mu = \Phi^\mu \cdot b^\mu\,.$$

Note that $\Phi^{\mu}(p_0) = e$, $b^{\mu}(e) = e$. Then Φ^{μ} is an extended solution of a harmonic map. Indeed, we have a formula

$$(\Phi^{\mu})^{-1} d\Phi^{\mu} = (1-\lambda^{-1})(\mathrm{Ad}(b^{\mu}|_{\lambda=0})\mu_{-1}) + (1-\lambda)\overline{(\mathrm{Ad}(b^{\mu}|_{\lambda=0})\mu_{-1})}.$$

Via the Grassmannian model of ΩG

$$\Omega G \cong \mathrm{Gr}^{(n)} \cong \Lambda G^{\mathbf{C}}/\Lambda^{+}G^{\mathbf{C}},$$

the corresponding extended solution of a harmonic map is expressed in terms of the above holomorphic map $g : M \to \Lambda G^{\mathbf{C}}$ as

$$\Phi : M \ni x \longmapsto \Phi(x) H_{+}^{(n)} = g(x) H_{+}^{(n)} \in \mathrm{Gr}^{(n)} \cong \Lambda G^{\mathbf{C}}/\Lambda^{+}G^{\mathbf{C}}. \tag{4.6}$$

Hence the natural group action $\natural$ of $\gamma \in \Lambda^{+}G^{\mathbf{C}} \subset \Lambda G^{\mathbf{C}}$ is given by

$$\begin{aligned}(\gamma^{\natural}\Phi)(z) H_{+}^{(n)} &= (\gamma^{\natural}\Phi(z)) H_{+}^{(n)} \\ &= \gamma\Phi(z) H_{+}^{(n)} \\ &= \gamma g(z) H_{+}^{(n)} \\ &= \gamma g(z)\gamma^{-1} H_{+}^{(n)} \in \mathrm{Gr}^{(n)} \cong \Lambda G^{\mathbf{C}}/\Lambda^{+}G^{\mathbf{C}}\end{aligned}$$

for each $z \in M$. Note that an extended solution $\gamma^{\natural}\Phi : M \to \Omega G$ also satisfies $(\gamma^{\natural}\Phi)(z_0) = e$. A holomorphic map representing the extended solution $\gamma^{\natural}\Phi : M \to \Omega G$ is

$$\gamma g \gamma^{-1} : M \ni p \longmapsto \gamma g(p)\gamma^{-1} \in \Lambda G^{\mathbf{C}}$$

and the corresponding holomorphic potential is given by

$$\begin{aligned}\mu_{\gamma^{\natural}\Phi} &= (\gamma g\gamma^{-1})^{-1} d(\gamma g \gamma^{-1}) \\ &= \gamma g^{-1}\gamma^{-1}\gamma d g \gamma^{-1} \\ &= \gamma(g^{-1}dg)\gamma^{-1} \\ &= \gamma\mu_{\Phi}\gamma^{-1} \\ &= \mathrm{Ad}(\gamma)(\mu_{\Phi}).\end{aligned}$$

The holomorphic gauge transformation group

$$\mathcal{G} := \{ h : M \longrightarrow \Lambda^{+}G^{\mathbf{C}} \mid \bar{\partial} h = 0 \} \tag{4.7}$$

acts on the infinite dimensional affine space $\mathcal{P}$ of holomorphic potentials as follows : For each $h \in \mathcal{G}$ and each $\mu \in \mathcal{P}$, define

$$h \cdot \mu := (\mathrm{Ad} h)\mu - (dh) h^{-1}$$

and then $h \cdot \mu \in \mathcal{P}$. The based holomorphic transformation group is defined by

$$\mathcal{G}^{e} := \{ h \in \mathcal{G} \mid h(z_0) = e \}, \tag{4.8}$$

which is a normal subgroup of $\mathcal{G}$. Then the above construction implies that

$$\begin{aligned}\mathcal{G}^{e}\backslash\mathcal{P} &\cong \{ \Phi : M \longrightarrow \Omega G \mid \text{ extended solutions}, \Phi(z_0) = e \} \\ &\cong \{ \varphi : M \longrightarrow G \mid \text{ harmonic maps}, \varphi(z_0) = e \}.\end{aligned}$$

Let $h \in \mathcal{G}$. We set

$$g_{h\cdot\mu} := h(z_0)\, g_\mu\, h^{-1} : M \longrightarrow \Lambda G^{\mathbf{C}}. \tag{4.9}$$

Then we have $g_{h\cdot\mu}(z_0) = e$ and $g_{h\cdot\mu}^{-1} dg_{h\cdot\mu} = h \cdot \mu$. Hence we obtain the formula

$$\begin{aligned}\Phi_{h\cdot\mu} &= (g_{h\cdot\mu})_u \\ &= (h(z_0) g_\mu)_u \\ &= (h(z_0)\Phi_\mu)_u \\ &= h(z_0)^\natural \Phi_\mu .\end{aligned}$$

We mention about a notion of the *normalized meromorphic potential* of a harmonic map ([5]). Let $\varphi : M \to G$ be a harmonic map with $\varphi(z_0)$ and $\Phi : M \to \Omega G$ be its extended solution with $\Phi(z_0) = e$. In order to construct the holomorphic potential corresponding to φ and Φ, we can use the Birkhoff decomposition theorem. Set $M' := \Phi^{-1}(\text{Big Cell}) \subset M$, which is an open set of M, and $M \setminus M' = \Phi^{-1}((\text{Big Cell})^c)$ is a discrete set of M by the holomorphicity of Φ. On M', by Birkhoff decomposition theorem 3.2, we decompose Φ uniquely as

$$\Phi = h_- h_+ ,$$

where $h_- : M \to \Lambda_1^- G^{\mathbf{C}}$, $h_+ : M \to \Lambda^+ G^{\mathbf{C}}$.

$$\begin{aligned}\frac{1}{2}(1-\lambda)\,\alpha'' &= \Phi^{-1}\bar{\partial}\Phi \\ &= \mathrm{Ad}(h_+^{-1})(h_-^{-1}\bar{\partial}h_-) + h_+^{-1}\bar{\partial}h_+\end{aligned}$$

and thus

$$\frac{1}{2}(1-\lambda)\,\mathrm{Ad}(h_+)(\alpha'') = h_-^{-1}\bar{\partial}h_- + \mathrm{Ad}(h_+)(h_+^{-1}\bar{\partial}h_+)$$

Comparing the we have $h_-^{-1}\bar{\partial}h_- = 0$ and

$$\frac{1}{2}(1-\lambda)\,\alpha'' = h_+^{-1}\bar{\partial}h_+$$

On the other hand,

$$\begin{aligned}\frac{1}{2}(1-\lambda^{-1})\,\alpha' &= \Phi^{-1}\partial\Phi \\ &= \mathrm{Ad}(h_+^{-1})(h_-^{-1}\partial h_-) + h_+^{-1}\partial h_+\end{aligned}$$

$$\frac{1}{2}(1-\lambda^{-1})\,\mathrm{Ad}(h_+)(\alpha') = h_-^{-1}\partial h_- + \mathrm{Ad}(h_+)(h_+^{-1}\partial h_+).$$

Comparing with the coefficients of λ^{-1} on the both sides, we have

$$-\frac{1}{2}\lambda^{-1}\,\mathrm{Ad}(h_+|_{\lambda=0})(\alpha') = h_-^{-1}\partial h_-.$$

Hence we obtain

$$h_-^{-1}\partial h_- = -\frac{1}{2}\lambda^{-1}\,\mathrm{Ad}(h_+|_{\lambda=0})(\alpha') = \lambda^{-1}\,\eta_{-1}.$$

$\mu = h_-^{-1}\partial h_- = \lambda^{-1}\eta_{-1}$ is a holomorphic potential defined on M' corresponding to the extended solution Φ. It is possible to show that μ extends to a meromorphic 1-form on M entirely by the geometric argument of the infinite dimensional Grassmannian on the Big Cell and the dual determinant line bundle ([5]). This meromorphic 1-form $\mu = \lambda^{-1}\eta_{-1}$ on M is called the *normalized meromorphic potential.*

4.6. **DPW formula for harmonic maps (Birkhoff-Uhlenbeck decomposition).** In this subsection we assume the setting of the Birkhoff-Uhlenbeck decomposition in Section 3.

Let M be a simply connected domain of the complex plane $\mathbf{C}$ and $z_0 \in M$ be a base point. Suppose that

$$\Phi : M \longrightarrow \Lambda^{E,\varepsilon}_{\mathbf{R},1} G^{\mathbf{C}} \subset \Omega G$$

is an extended solution of a harmonic map satisfying $\Phi(z_0) = e$.

We use the following complex loop groups defined over a circle C_ε:

$$\Lambda^{\varepsilon} G^{\mathbf{C}} := \{\gamma : C_\varepsilon \longrightarrow G^{\mathbf{C}} \mid \gamma \text{ is smooth}\},$$

$$\Lambda^{I_\varepsilon} G^{\mathbf{C}} := \{\gamma \in \Lambda^{\varepsilon} G^{\mathbf{C}} \mid \gamma \text{ extends continuously to holomorphic } \gamma^I : I_\varepsilon \longrightarrow G^{\mathbf{C}}\}.$$

Then by a solution to the $\bar\partial$-problem there exists $b = (b_\varepsilon, \overline{b_\varepsilon}) : M \to \Lambda^{I,\varepsilon}_{\mathbf{R}} G^{\mathbf{C}}$ with $b(z_0) = e$ such that

$$g = \Phi\, b = (g_\varepsilon, \overline{g_\varepsilon}) : M \longrightarrow \Lambda^{\varepsilon,\varepsilon^{-1}}_{\mathbf{R}} G^{\mathbf{C}}$$

and $g_\varepsilon = \Phi\, b_\varepsilon : M \to \Lambda^{\varepsilon} G^{\mathbf{C}}$ is a holomorphic map in the sense that $\bar\partial g_\varepsilon = \bar\partial(\Phi\, b_\varepsilon) = 0$. Such a map $b_\varepsilon : M \to \Lambda^{I_\varepsilon} G^{\mathbf{C}}$ has the freedom of right multiplications by holomorphic maps $h_\varepsilon : M \to \Lambda^{I_\varepsilon} G^{\mathbf{C}}$ with $h_\varepsilon(z_0) = e$.

The holomoprhic 1-form on M with values in $\Lambda \mathfrak{g}^{\mathbf{C}}$

$$\begin{aligned}\mu^{\varepsilon}_{\Phi} :=& g_\varepsilon^{-1} dg_\varepsilon = g_\varepsilon^{-1}\partial g_\varepsilon \\ =& b_\varepsilon^{-1}\Phi^{-1}\partial\Phi\, b_\varepsilon + b_\varepsilon^{-1}\,\partial b_\varepsilon \\ =& \frac{1}{2}(1-\lambda^{-1}) b_\varepsilon^{-1}\alpha' b_\varepsilon + b_\varepsilon^{-1}\,\partial b_\varepsilon\end{aligned}$$

is holomorphic with respect to $\lambda \in I_\varepsilon \setminus \{0\} = D(0,\varepsilon)\setminus\{0\}$ and has at most a simple pole (a pole of at most order 1) at $\lambda = 0$.

Set

$$\begin{aligned}\Lambda^{\varepsilon}_{-1,\infty}\mathfrak{g}^{\mathbf{C}} := \{\xi : C_\varepsilon \longrightarrow \mathfrak{g}^{\mathbf{C}} \mid\ & \text{smooth},\\ & \xi \text{ extends continuously to holomorphic } I_\varepsilon\setminus\{0\} \longrightarrow \mathfrak{g}^{\mathbf{C}}\\ & \text{which has at most a simple pole at } 0\}\end{aligned}$$

and define

$$\mathcal{P}^{\varepsilon} := \left\{\mu \in \Omega^{1,0}(M, \Lambda^{\varepsilon}_{-1,\infty}\mathfrak{g}^{\mathbf{C}}) \mid \bar\partial\mu = 0\right\}.$$

Each $\mu \in \mathcal{P}^{\varepsilon}$ can be expressed as

$$\mu = \sum_{i=-1}^{\infty} \mu_i \lambda^i$$

on C_ε. Here each μ_i is a holomorphic 1-form on M with values in $\mathfrak{g}^{\mathbf{C}}$. Then we have $\mu_\Phi^\varepsilon \in \mathcal{P}^\varepsilon$.

Conversely, for each $\mu \in \mathcal{P}^\varepsilon$, there exists

$$g = g_\mu = (g^\varepsilon, \overline{g^\varepsilon}) = (g_\mu^\varepsilon, \overline{g_\mu^\varepsilon}) : M \longrightarrow \Lambda_{\mathbf{R}}^{\varepsilon,\varepsilon^{-1}} G^{\mathbf{C}}$$

such that

$$(g^\varepsilon)^{-1} d(g^\varepsilon) = (g^\varepsilon)^{-1} \partial(g^\varepsilon) = \mu, \quad g(z_0) = e$$

on C_ε. We take the Birkhoff-Uhlenbeck decomposition

$$g = \Phi b,$$

where $\Phi : M \longrightarrow \Lambda_{\mathbf{R},1}^{E,\varepsilon} G^{\mathbf{C}}$, $b : M \longrightarrow \Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}$. Then

$$\Phi : M \longrightarrow \Lambda_{\mathbf{R},1}^{E,\varepsilon} G^{\mathbf{C}} \subset \Omega G$$

is an extended solution of harmonic map. Indeed, we have a formula

$$\begin{aligned}\Phi^{-1} d\Phi &= \mathrm{Ad}(b)\mu - db\, b^{-1} \\ &= [\mathrm{Ad}(b)\mu]_{\Lambda_{\mathbf{R},1}^{E,\varepsilon}\mathfrak{g}^{\mathbf{C}}} \\ &= (1 - \lambda^{-1})(\mathrm{Ad}(b(0))\mu_{-1}) + (1 - \lambda)\overline{(\mathrm{Ad}(b(0))\mu_{-1})}.\end{aligned}$$

The holomorphic gauge transformation group

$$\mathcal{G}^\varepsilon := \{h : M \longrightarrow \Lambda^{I_\varepsilon} G^{\mathbf{C}} \mid \bar{\partial} h = 0\} \tag{4.10}$$

acts on the infinite dimensional affine space $\mathcal{P}^\varepsilon$ as follows : For each $h \in \mathcal{G}^\varepsilon$ and each $\mu \in \mathcal{P}^\varepsilon$, define

$$h \cdot \mu := (\mathrm{Ad}\, h)\mu - dh \cdot h^{-1}. \tag{4.11}$$

Then we have $h \cdot \mu \in \mathcal{P}^\varepsilon$. The based holomorphic gauge transformation group is a normal subgroup of $\mathcal{G}^\varepsilon$ defined by

$$\mathcal{G}^{\varepsilon,e} := \{h \in \mathcal{G} \mid h(z_0) = e\}. \tag{4.12}$$

Now we set

$$g_{h\cdot\mu}^\varepsilon := h(z_0)\, g_\mu^\varepsilon\, h^{-1} : M \longrightarrow \Lambda^\varepsilon G^{\mathbf{C}}. \tag{4.13}$$

Then we have $g_{h\cdot\mu}^\varepsilon(z_0) = e$ and

$$(g_{h\cdot\mu}^\varepsilon)^{-1}\, d(g_{h\cdot\mu}^\varepsilon). = h \cdot \mu \tag{4.14}$$

So we define

$$\begin{aligned} g_{h\cdot\mu} &:= (g_{h\cdot\mu}^\varepsilon, \overline{g_{h\cdot\mu}^\varepsilon}) : M \longrightarrow \Lambda^{\varepsilon,\varepsilon^{-1}} G^{\mathbf{C}}, \\ \tilde{h} &= (h, \bar{h}) : M \longrightarrow \Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}. \end{aligned} \tag{4.15}$$

Since $\tilde{h}(z_0) = (h(z_0), \overline{h(z_0)}) \in \Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}$, we have

$$
\begin{aligned}
g_{h\cdot\mu} &=(g^{\varepsilon}_{h\cdot\mu}, \overline{g^{\varepsilon}_{h\cdot\mu}}) \\
&=(h(z_0)\, g^{\varepsilon}_{\mu}\, h^{-1}, \overline{h(z_0)\, g^{\varepsilon}_{\mu}\, h^{-1}}) \\
&=(h(z_0)\, g^{\varepsilon}_{\mu}\, h^{-1}, \overline{h(z_0)}\,\overline{g^{\varepsilon}_{\mu}}\,\overline{h^{-1}}) \qquad (4.16)\\
&=(h(z_0), \overline{h(z_0)})(g^{\varepsilon}_{\mu}, \overline{g^{\varepsilon}_{\mu}})(h^{-1}, \overline{h^{-1}}) \\
&=\tilde{h}(z_0)\, g_{\mu}\, \tilde{h}^{-1} .
\end{aligned}
$$

Hence we obtain the formula

$$
\begin{aligned}
\Phi_{h\cdot\mu} &= (g_{h\cdot\mu})_E \\
&= (\tilde{h}(z_0)\, g_{\mu}\, \tilde{h}^{-1})_E \\
&= (\tilde{h}(z_0)\, \Phi_{\mu}\, b_{\mu}\, \tilde{h}^{-1})_E \qquad (4.17)\\
&= (\tilde{h}(z_0)\Phi_{\mu})_E \\
&= \tilde{h}(z_0)^{\sharp}\Phi_{\mu} .
\end{aligned}
$$

4.7. **Relationship of two kinds of DPW formulas for harmonic maps.**

Theorem 4.4. *The natural injective linear map over* $\mathbf{C}$

$$\mathcal{P} \ni \mu \longmapsto \mu|_{C_\varepsilon} \in \mathcal{P}^{\varepsilon}$$

induces a bijective correspondence between the moduli spaces of holomorphic potentials by the based holomorphic gauge transformation groups

$$\mathcal{G}^{e} \backslash \mathcal{P} \cong \mathcal{G}^{\varepsilon,e} \backslash \mathcal{P}^{\varepsilon} .$$

Moreover, they are equivariant with respect to the natural injective group homomorphism between the holomorphic gauge transformation groups $\mathcal{G} \to \mathcal{G}^{\varepsilon}$. *In particular, they are equivariant with respect to the loop group actions* $\natural$ *and* $\sharp$.

5. Harmonic maps of finite uniton number and classification problem of harmonic 2-spheres

5.1. **Uniton transform.** Suppose that $G = U(n)$. Set

$$\mathrm{Gr}(\mathbf{C}^n) = \{a \in G \,|\, a^2 = \mathrm{I}_n\}.$$

Each $a \in \mathrm{Gr}(\mathbf{C}^n)$ can be expressed as

$$a = \pi_W - \pi_{W^{\perp}} = \pi - \pi^{\perp}$$

in terms of the orthogonal projection

$$\pi = \pi_W : \mathbf{C}^n = W \oplus W^{\perp} \longrightarrow W$$

onto a vector subspace of $\mathbf{C}^n$

$$W := \{v \in \mathbf{C}^n \,|\, av = v\}.$$

The finite dimensional complex Grasssmanian of complex vector subspaces of $\mathbf{C}^n$ $\mathrm{Gr}(\mathbf{C}^n)$ is decomposed into connected components as

$$\mathrm{Gr}(\mathbf{C}^n) = \coprod_{k=0}^{n} \mathrm{Gr}_k(\mathbf{C}^n).$$

Here $\mathrm{Gr}_k(\mathbf{C}^n)$ is a complex Grassmann manifold of k-dimensional vector subspaces of $\mathbf{C}^n$.

A smooth map into the complex Grassmannian

$$\pi - \pi^{\perp} : M \to \mathrm{Gr}(\mathbf{C}^n) \subset U(n)$$

can be identified with a complex vector subbundle η of the trivial vector bundle $\underline{\mathbf{C}}^n = M \times \mathbf{C}^n$.

Let $\varphi : M \to U(n)$ be a harmonic map. We use the same notation as in the previous sections, such as $\alpha = \varphi^*\theta = \varphi^{-1}d\varphi$, a connection $d_A = d + \frac{1}{2}\alpha \in \mathcal{A}_P$ of the trivial principal bundle $P := M \times G$, a Higgs field $\phi = \frac{1}{2}\alpha \in \Omega^1(\mathfrak{g}_P)$, $\phi = \phi' + \phi''$.

Let $\Phi_\lambda : M \to U(n) (\lambda \in S^1)$ be an extended solution of a harmonic map φ. Using a smooth map into a complex Grassmannian $\pi - \pi^{\perp} : M \to \mathrm{Gr}(\mathbf{C}^n) \subset U(n)$, we define

$$\tilde{\Phi}_\lambda := \Phi_\lambda(\pi + \lambda\pi^{\perp}) : M \longrightarrow U(n) \quad (\lambda \in S^1)$$

and then we have

Lemma 5.1. *$\tilde{\Phi}$ is also a new extended solution if and only if a complex Grassmannian $\pi - \pi^{\perp} : M \to \mathrm{Gr}(\mathbf{C}^n) \subset U(n)$ satisfies the equations*

$$\begin{cases} \pi^{\perp}(\bar{\partial} + \phi'')\pi = 0, \\ \pi^{\perp}\phi'\pi = 0. \end{cases} \tag{5.1}$$

In this case $\tilde{\varphi} = \tilde{\Phi}_{-1} = \pi \circ \tilde{\Phi} = \varphi(\pi - \pi^{\perp})$ is a harmonic map.

The equations (5.1) is called the *uniton equation* of a harmonic map φ and we say that a harmonic map $\tilde{\varphi}$ can be obtained by making a uniton transform or by adding a uniton to a harmonic map φ.

We equip the trivial complex vector bundle $\underline{\mathbf{C}}^n = M \times \mathbf{C}^n$ over M with the holomorphic vector bundle structure d_A'' as the $\bar{\partial}$-operator. The harmonic map equation $d_A''\phi' = 0$ implies that ϕ' is a holomorphic Higgs field and thus we obtain a holomorphic Higgs vector bundle structure $(\underline{\mathbf{C}}^n, d_A'', \phi')$. The first equation of the uniton equations means the complex vector subbundle η corresponding to a smooth map $\pi - \pi^{\perp}$ into a complex Grassmannian is a holomorphic vector subbundle of $(\underline{\mathbf{C}}^n, d_A'')$. The second equation of the uniton equations means that the complex vector subbundle η is invariant under the action of a holomorphic Higgs field ϕ', namely, $\phi'(\eta) \subset \eta$.

The procedure of the *Gauss bundle* and the *harmonic sequence* of harmonic maps of Riemann surfaces into complex projective spaces and complex Grassmannians is an examples of the uniton transform (cf. [6]).

Lemma 5.2 (Valli [34]). *Let M be a compact Riemann surface. Assume that a harmonic map $\tilde{\varphi}$ is obtained by adding a harmonic map φ to a uniton η. Then the energy formula*

$$E(\psi) - E(\varphi) = -8\pi \deg(\eta), \quad \deg(\eta) := \int_M c_1(\eta) \in \mathbf{Z}. \tag{5.2}$$

holds. Here $c_1(\eta)$ denotes the first Chern class of the complex vector bundle η. The invariant inner product of Lie algebra $\mathfrak{u}(n)$ of $U(n)$ is defined as $\langle A, B\rangle := -\mathrm{tr}(AB)\,(A, B \in \mathfrak{u}(n))$.

Definition 5.1. Set $E = (\underline{\mathbf{C}}^n, d''_A)$, which is a holomorphic vector bundle. Consider a holomorphic Higgs bundle (E, ϕ'). If it holds $\mu(V) \leq \mu(E)$ for any holomorphic vector subbundle $V \subset E$ invariant by ϕ', then the holomorphic Higgs bundle E is called *semi-stable*. Here $\mu(V) := \deg(V)/\mathrm{rank}(V)$.

From Lemma 5.2 and the concept of the semi-stability of holomorphic Higgs bundle, we obtain :

Theorem 5.1 ([34], [21]). *Any harmonic map of a compact Riemann surface M into the unitary group $U(n)$ can be transformed by a finite number of uniton transforms into a harmonic map whose associated holomorphic Higgs bundle is semistable. It is not possible to decrease the energy of a harmonic map with the semistable holomorphic Higgs bundle by any uniton transform. In particular, if M is a Riemann sphere, then any harmonic map of M into $U(n)$ can be transformed by a finite number of uniton transforms into a constant map.*

5.2. **Harmonic maps of finite uniton number.** Suppose that $G = U(n)$.

Definition 5.2. If a harmonic map $\varphi : M \to U(n)$ has an extended solution $\Phi : M \to \Omega U(n)$

$$\begin{aligned} \Phi &= \sum_{i=0}^{m} T_i \lambda^i, \\ \Phi_{-1} &= \pi \circ \Phi = a\varphi \quad (\exists\, a \in U(n)), \end{aligned} \tag{5.3}$$

then φ is said to be of *finite uniton number*. Such a harmonic map $\varphi : M \to U(n)$ is called *harmonic map of finite uniton number* or a *uniton solution* to the harmonic map equation. Or equivalently, it means that a harmonic map $\varphi : M \to U(n)$ has an extended solution Φ such that

$$\Phi(M) \subset X_{m,\mathbf{R}} \subset \Omega U(n) \tag{5.4}$$

for some nonnegative integer m. We call such a minimal number m the *minimal uniton number* and then φ or Φ an *m-uniton*.

A harmonic map $\varphi : M \to U(n)$ of finite uniton number is always weakly conformal, that is, a branched minimal immersion. ([21]).

A 0-uniton solution is a constant map. A 1-uniton solution φ is a left translation $\varphi = c\,h$ by some $c \in U(n)$ of a holomorphic map from a Riemann surface M to a complex Grassmann manifold $h : M \longrightarrow \mathrm{Gr}(\mathbf{C}^n)$.

Theorem 5.2 ([32], [26]). *Assume that a Riemann surface M is compact and $\Phi : M \to \Omega U(n)$ is an extended solution satisfies the base point condition $\Phi(z_0) = \mathrm{I}_n$. Then Φ has finite Laurent expansion*

$$\Phi_\lambda = \sum_{i=-p}^{q} T_i \lambda^i \quad (\exists\, p, q \in \mathbf{Z}, p, q \geq 0) \tag{5.5}$$

with respect to $\lambda \in \mathbf{C}^*$.

Corollary 5.1. *If $\Phi : M \to \Omega U(n)$ is an extended solution on a compact Riemann surface, then $\varphi = \pi \circ \Phi : M \to U(n)$ is a harmonic maps of finite uniton number.*

Corollary 5.2. *Any harmonic map $\varphi : S^2 \to U(n)$ of a Riemann sphere into a unitary group is always a a harmonic maps of finite uniton number.*

Theorem 5.3 ([32]). *Suppose that $\varphi : M \to U(n)$ is a harmonic map of finite uniton number. Then there exists a unique extended solution $\Phi : M \to U(n)$ such that*

(1) $\Phi_{-1} = \pi \circ \Phi = a\varphi \quad (\exists\, a \in U(n))$,
(2) $\Phi_\lambda = \sum_{i=0}^{m} T_i \lambda^i \, (\forall\, \lambda \in \mathbf{C}^*)$, $T_m \not\equiv 0$,
(3) $V_0(\Phi) = \mathbf{C}^n$,

where $V_0(\Phi)$ denotes a complex vector subspace of $\mathbf{C}^n$ spanned by $\{(T_0)_z v \mid z \in M, v \in \mathbf{C}^n\}$. Moreover this number m is equal to the minimal uniton number of φ.

Such an extended solution is called the *normalized extended solution* of a harmonic map of finite uniton number.

Uhlenbeck proved the factorization theorem into unitons for harmonic maps of finite uniton number, repeating the uniton transform procedure by a uniton given by the kernel bundle of T_0 for the normalized extended solution.

Theorem 5.4 ([32]). *Suppose that $\varphi : M \to U(n)$ is a harmonic map of finite uniton number. Then for some $c \in U(n)$, φ can be decomposed into a product of a finite number of smooth maps into complex Grassmann manifolds :*

$$\varphi = c(\pi_1 - \pi_1^\perp) \cdots (\pi_m - \pi_m^\perp) .$$

(1) *Each $\varphi^{(i)} = c(\pi_1 - \pi_1^\perp) \cdots (\pi_i - \pi_i^\perp)\,(i = 1, \cdots, n)$ is a harmonic map.*
(2) *Each $\pi_i - \pi_i^\perp$ is a uniton for a harmonic map $\varphi^{(i)}$.*
(3) *$\pi_1 - \pi_1^\perp : M \to \mathrm{Gr}(\mathbf{C}^n)$ is a holomorphic map.*
(4) *$m < n$ and m is equal to the minimal uniton number of φ.*

Moreover, if M is compact, then $E(\varphi) = E(\varphi^{(m)}) > E(\varphi^{(m-1)}) > \cdots > E(\varphi^{(1)})$.

G. Segal [26] provided the different proofs of these results by the method of loop groups and infinite dimensional Grassmannian.

The loop group action $\sharp$ of $\Lambda_{\mathbf{R}}^{I,\varepsilon} G^{\mathbf{C}}$ coincides with the loop group action $\natural$ of $\Lambda^+ G^{\mathbf{C}}$ on harmonic maps of finite uniton number ([8]). This loop group action is used in order to study the topological properties (such as path-connectedness, fundamental groups) of the spaces of harmonic maps of a Riemann sphere into some compact symmetric spaces ([8]).

The factorization theorem into unitons is a fundamental principle for classification and explicit construction of a Riemann sphere into a compact symmetric space, generalizing the known results in the cases of $N = S^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathrm{Gr}_2(\mathbf{C}^n)$, $Q_n(\mathbf{C})$, etc.

Problem 5.1. For each compact symmetric space $N = G/K$, investigate the complete classification, the explicit construction and the properties of the space of all harmonic maps of a Riemann sphere into N.

6. Harmonic maps of finite type and classification problem of harmonic tori

6.1. **Harmonic maps of finite type.** Consider the based complex loop algebra

$$\Omega\mathfrak{g}^{\mathbf{C}} := \left\{ \xi : S^1 \to \mathfrak{g}^{\mathbf{C}},\ \text{smooth}\ \xi(1) = 0 \right\}.$$

Each $\xi \in \Omega\mathfrak{g}^{\mathbf{C}}$ has Fourier series expansion

$$\xi = \sum_{j \in \mathbf{Z}\setminus\{0\}} (1 - \lambda^{-j})\xi_j, \quad \xi_j \in \mathfrak{g}^{\mathbf{C}}.$$

Define the based real loop algebra

$$\Omega\mathfrak{g} := \left\{ \xi : S^1 \to \mathfrak{g},\ C^\infty\text{-級},\ \xi(1) = 0 \right\}.$$

Each $\xi \in \Omega\mathfrak{g}$ has Fourier series expansion

$$\xi = \sum_{j \in \mathbf{Z}\setminus\{0\}} (1 - \lambda^{-j})\xi_j,\ \xi_j \in \mathfrak{g}^{\mathbf{C}},\ \bar{\xi}_j = \xi_{-j}\ (j \in \mathbf{Z} \setminus \{0\}).$$

For each $d \in \mathbf{N}$, define a finite dimensional real vector space of $\Omega\mathfrak{g}$ by

$$\Omega_d := \left\{ \xi \in \Omega\mathfrak{g} \mid \xi = \sum_{0<|j|\leq d} (1 - \lambda^{-j})\xi_j \right\}.$$

Introduce a Lax equation over Ω_d. Denote by ξ a smooth function on Ω_d with values in $M = \mathbf{C} = \mathbf{R}^2$ and by $\{z = x + \sqrt{-1}y\}$ the standard complex coordinate system of $M = \mathbf{C} = \mathbf{R}^2$. The Lax equation is the partial differential equation of the first order:

$$\frac{\partial \xi}{\partial z} = [\,\xi, 2\sqrt{-1}(1 - \lambda^{-1})\xi_d\,]. \tag{6.1}$$

The Lax equation(6.1) has the following properties: Define two vector fields X_1, X_2 on Ω_d:

$$\frac{1}{2}(X_1 - \sqrt{-1}X_2)_\xi = [\,\xi, 2\sqrt{-1}(1 - \lambda^{-1})\xi_d\,] \quad (\forall \xi \in \Omega_d). \tag{6.2}$$

The following fact holds. The compactness of G is used in the proof of the second statement.

Lemma 6.1. *The two vector fields X_1 and X_2 commute, that is, the bracket product of vector fields on Ω_d satisfies $[X_1, X_2] = 0$. Moreover, X_1 and X_2 are complete vector fields on Ω_d.*

So let ϕ_1^t and ϕ_2^t denote one-parameter transformation groups (flows) generated by vector fields. For each $\xi^0 \in \Omega_d$, a function

$$\xi : M = \mathbf{C} \ni z = x + \sqrt{-1}y \longmapsto \xi(x,y) := (\phi_1^x \circ \phi_2^y)(\xi^0) = (\phi_2^y \circ \phi_1^x)(\xi^0) \in \Omega_d \quad (6.3)$$

is a solution to the Lax equation (6.1) with the initial condition $\xi(0) = \xi^0$.

On the coefficient $\xi_d = \bar{\xi}_{-d}$ of λ^{-d} the Fourier expansion in λ for th solution $\xi : \mathbf{C} \to \Omega_d$, the following lemma holds:

Lemma 6.2. *The* 1*-form on* $\mathbf{C}$ *with values in* $\mathfrak{g}$

$$\alpha_\lambda := 2\sqrt{-1}(1 - \lambda^{-1})\xi_d dz - 2\sqrt{-1}(1 - \lambda)\bar{\xi}_d d\bar{z}$$

satisfies the Maurer-Cartan equation

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for each $\lambda \in S^1$. *From this result, an extended solution* $\Phi : M = \mathbf{C} \to \Omega G$ *satisfying* $\Phi^*\theta = \Phi^{-1}d\Phi = \alpha_\lambda$ *exists. Hence we obtain a harmonic map* $\varphi = \pi \circ \Phi : \mathbf{C} \to G$.

The harmonic map obtained in this way is called a *harmonic maps of finite type* or *finite type solutions* (Burstall-Ferus-Pinkall-Pedit [3]) . Moreover, a harmonic map of finite type has the property that $\alpha'(\frac{\partial}{\partial z}) = \varphi^{-1}d\varphi(\frac{\partial}{\partial z})$ is contained in an $\mathrm{Ad}G^{\mathbf{C}}$-orbit in $\mathfrak{g}^{\mathbf{C}}$. In particular, if $\alpha'(\frac{\partial}{\partial z})$ is contained in an $\mathrm{Ad}G^{\mathbf{C}}$-orbit through a semisimple element of $\mathfrak{g}^{\mathbf{C}}$, $\varphi : M = \mathbf{C} \to G$ is called a *harmonic map of semisimple finite type*.

Here we mention about the results due to Burstall and Pedit [4] on orbits of loop group actions on harmonic maps (*dressing orbits*).

For $\xi^0 \in \Omega_d$, set $\mu = (\lambda^{d-1}z\xi^0)dz \in \mathcal{P}$. A holomorphic map $g_\mu : \mathbf{C} \to \Lambda G^{\mathbf{C}}$ with $g_\mu(0) = e$, $g_\mu^{-1}dg_\mu = \mu$ is $g_\mu(z) = \exp(\lambda^{d-1}z\xi^0)$ $(z \in \mathbf{C})$. By Iwasawa decomposition thorem, there exists uniquely $\Phi^\mu : \mathbf{C} \to \Omega G$ and $b^\mu : \mathbf{C} \to \Lambda^+ G^{\mathbf{C}}$ such that we decompose g_μ as

$$g_\mu(z) = \exp(\lambda^{d-1}z\,\xi(0)) = \Phi^\mu(z)\,b^\mu(z) \quad (\forall z \in \mathbf{C}).$$

Then $\Phi^\mu : \mathbf{C} \to \Omega G$ is an extended solution of harmonic map of finite type. Via the identification $\Omega G \cong \mathrm{Gr}_\infty^{(n)}$, we can express Φ^μ as

$$\Phi^\mu(z)\,H_+^{(n)} = \exp(\lambda^{d-1}z\,\xi(1))\,H_+^{(n)}.$$

The so obtained harmonic map $\varphi = (\Phi^\mu)_{-1} : M = \mathbf{C} \to G$ is of finite type.

A vacuum solution: Let $A \in \mathfrak{g}^{\mathbf{C}} \in$ be an arbitrary element satisfying $[A, \bar{A}] = 0$ (thus A is semisimple). Set

$$\xi^0 := \frac{1}{2}(1 - \lambda^{-1})A + \frac{1}{2}(1 - \lambda)\bar{A} \in \Omega_1$$

and

$$\mu_A := z\{(1 - \lambda^{-1})A + (1 - \lambda)\bar{A}\}dz \in \mathcal{P}.$$

Then its Iwasawa decomposition is

$$\begin{aligned} g_{\mu_A} &= \exp(z\{\frac{1}{2}(1-\lambda^{-1})A + \frac{1}{2}(1-\lambda)\bar{A}\}) \\ &= \exp(\{\frac{z}{2}(1-\lambda^{-1})A + \frac{\bar{z}}{2}(1-\lambda)\bar{A}\}) \exp(\{\frac{z}{2}(1-\lambda)\bar{A} - \frac{\bar{z}}{2}(1-\lambda)\bar{A}\}) \end{aligned} \tag{6.4}$$

and thus we obtain an extended solution

$$\Phi_A : \mathbf{C} \ni z \longrightarrow \exp(\frac{z}{2}(1-\lambda^{-1})A + \frac{\bar{z}}{2}(1-\lambda)\bar{A}) \in \Omega G \tag{6.5}$$

and the corresponding harmonic map of finite type is

$$\varphi_A = \Phi_{-1} : \mathbf{C} \ni z \longrightarrow \exp(zA + \bar{z}\bar{A}) \in G. \tag{6.6}$$

Such an extended solution or harmonic map is called a *vacuum solution.*

Burstall and Pedit [4] studied the orbit of the loop group $\Lambda_{\mathrm{R}}^{I,\varepsilon}G^{\mathbf{C}}$ (dressing orbit) of a vacuum solution and they proved

Theorem 6.1 ([4]). *Any harmonic map of semisimple finite type is contained in a $\Lambda_{\mathrm{R}}^{I,\varepsilon}G^{\mathbf{C}}$-orbit (dressing orbit) of a vacuum solution.*

6.2. **Classification problem of harmonic tori.** Suppose that $\mathbf{C}/\Gamma$ is a compact Riemann surface of genus 1 (a torus) and G (or G/K) is a compact Lie group (or a compact symmetric space). Let $\varphi : M = \mathbf{C}/\Gamma \to G$ (or G/K) be a harmonic map.

Theorem 6.2 (BFPP [3]). *Assume that φ is semisimple, that is, the function $(\varphi^*\theta)\left(\frac{\partial}{\partial z}\right)$ on M has values in a set of semisimple elements of $\mathfrak{g}^{\mathbf{C}}$. Then φ is a harmonic map of (semisimple) finite type.*

Theorem 6.3 (Burstall [2]). *Assume that $G/K = S^n$ or $G/K = \mathbf{C}P^n$. φ is an isotropic (=superminimal) harmonic map (thus a harmonic map of finite uniton number) or a harmonic map of finite type.*

In particular, in the case $G/K = S^2$, any harmonic map $\varphi : M = \mathbf{C}/\Gamma \to S^2$ is a ±holomorphic map or a harmonic map of finite type.

The cases of $G/K = \mathrm{Gr}_2(\mathbf{C}^n)$ and $G/K = \mathbf{H}P^n$ are discussed in [30], [31]

Corollary 6.1 (Pinkall-Sterling [22]). *The Gauss map $g : M \to S^2(1)$ of a constant mean curvature torus $M = \mathbf{C}/\Gamma \to \mathbf{R}^3$ immersed in 3-dimensional Euclidean space $\mathbf{R}^3$ is a harmonic map of finite type.*

Problem. Assume that N is a compact symmetric space other than S^n, $\mathbf{C}P^n$. Then is any harmonic map $\varphi : M = \mathbf{C}/\Gamma \to N$ of a torus into N a harmonic map of finite uniton number or of finite type ?

Theory of harmonic maps of finite type on compact Riemann surfaces of genus greater than 1 was discussed in [20].

7. Generalization to pluriharmonic maps

The notion of *pluriharmonic maps* is a natural generalization of harmonic maps of Riemann surfaces to higher dimensional complex manifolds, focused on the complex structure of the domain manifold of harmonic maps. Theory of pluriharmonic maps of complex manifolds into Lie groups and symmetric spaces are discussed in [21], [20], etc. Pluriharmonic maps of complex manifolds are very useful and significant even in the study of harmonic maps of Riemann surfaces.

References

[1] M. J. Bergvelt and M. A. Guest, *Actions of loop groups on harmonic maps*, Trans. Amer. Math. Soc. **326** (1991), 861–886.

[2] F. E. Burstall, *Harmonic tori in spheres and complex projective spaces*, J. Reine Angew. Math. **469** (1995), 149–177.

[3] F. E. Burstall, and D. Ferus, and F. Pedit, and U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. **138** (1993), 173–212.

[4] F. E. Burstall and F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J., **80** (1995), 353–382.

[5] J. Dorfmeister, F. Pedit and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom., **6** (1998), 633–668.

[6] J. Eells and L. Lemaire, *Two Reports of Harmonic Maps*, 1995, World Scientific Publishing Co. Pte. Ltd. ISBN: 981-02-1466-9.

[7] Y. Ohnita, *Group actions and deformations for harmonic maps into symmetric spaces*, Kodai Math. J. **17** (1994), 463–475.

[8] M. A. Guest and Y. Ohnita, *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan **45** (1993), 671–704.

[9] M. A. Guest and Y. Ohnita, *Loop group actions on harmonic maps and their applications*, in : Harmonic Maps and Integrable Systems, edited by A. P. Fordy and J. C. Wood, Aspects of Mathematics E23, Vieweg, Braunschweig/Wiesbaden, 1994, 273–292.

[10] M. A. Guest and Y. Ohnita, *Loop group actions and deformations for harmonic maps and their applications*, (Japanese) Sugaku *46* (1994), no. 3, 228–242.

[11] M. A. Guest and Y. Ohnita, *Actions of loop groups, deformations of harmonic maps, and their applications*, In : "Selected Papers on Harmonic Analysis, Groups and Invariants", Amer. Math. Soc. Translations Ser. 2, **183** (1998), (K. Nomizu, editor), 33–50.

[12] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, Inc., New York, San Diego, London, Tokyo (1978)

[13] N. J. Hitchin, *Harmonic maps from a* 2*-torus to the* 3*-sphere*, J. Differential Geom. **31** (1990), 627–710.

[14] N. J. Hitchin, *Integrable systems in Riemannian geometry*, edited by C.-L. Terng and K. Uhlenbeck, Surveys in Differential Geometry : Integrable Systems, **IV**, 1998, International Press, 21–81.

[15] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I, II.* 1963, 1969, Interscience, New York, Chichester, Brisbane, Toronto.

[16] I. McIntosh, *Global solutions of the elliptic 2D periodic Toda lattice*, Nonlinearity, **7**(1994 a), 85–108. Correction in **8** (1995), 629–630.

[17] M. Micallef and J. D. Moore, *Minimal* 2*-spheres and the topology of manifolds with positive curvature on totally isotropic two planes*, Ann. of Math. **127** (1988), 199–227.

[18] M. Mukai-Hidano and Y. Ohnita, *Geometry of the moduli spaces of harmonic maps into Lie groups via gauge theory over Riemann surfaces*, International J. Math., **12**, no.3 (2001), 339–371.

[19] M. Mukai-Hidano and Y. Ohnita, *Gauge-theoretic approach to harmonic maps and subspaces in moduli spaces*, "Integrable systems, Geometry and Topology ", AMS/IP Studies in Advanced Mathematics, Volume **36**, 2006, edited by C.-L. Terng, American Mathematical Society/International Press, 191–234.

[20] Y. Ohnita and S. Udagawa, *Harmonic maps of finite type into generalized flag manifolds and twistor fibrations*, Contemporary Mathematics 308 (2002), "Differential Geometry and Integrable Systems ", (the proceedings of the 9-th MSJ-IRI Tokyo 2000, Integrable Systems in Differential Geometry), edited by M. Guest, R. Miyaoka, Y. Ohnita, American Mathematical Society, 245–270.

[21] Y. Ohnita and G. Valli, *Pluriharmonic maps into compact Lie groups and factorization into unitons*, Proc. London Math. Soc. (3) **61** (1990), 546–570.

[22] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. **130** (1989), 407–451.

[23] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through quadratic constraints*, Comm. Math. Phys. **46** (1976), 207–221.

[24] A. N. Pressley, *The energy flow on the loop space of a compact Lie group*, J. London Math. Soc. , **26** (1982), 557–566.

[25] A. N. Pressley and G. B. Segal, *Loop Groups*, Oxford University Press, 1986.

[26] G. B. Segal, *Loop groups and harmonic maps*, Advances in Homotopy Theory, L.M.S. Lecture Notes, **139**, (1989), 153–164.

[27] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Publ. Math. IHES, **61**, (1985), 5–65.

[28] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of* 2*-spheres*, Ann. of Math. **113** (1981), 1-24.

[29] G. B. Segal, *Loop groups and harmonic maps*, Advances in Homotopy Theory, L.M.S. Lecture Notes, **139**, (1989), 153–164.

[30] S. Udagawa, *Harmonic maps from a two-torus into a complex Grassmann manifold*, International J. Math. **6**, (1995), 447–459.

[31] S. Udagawa, *Harmonic tori in quaternionic projective* 3*-spaces*, Proc. Amer. Math. Soc. **125** (1997), 275–285.

[32] K. Uhlenbeck, *Harmonic maps into Lie groups (Classical solutions of the chiral model)*, J. Differential Geom. **30** (1989), 1–50.

[33] M. Vajiac and K. Uhlenbeck, *Virasoro actions and harmonic maps (after Schwarz)*, J. Differential Geom., **80** (2008), 327–341.

[34] G. Valli, *Harmonic gauges on Riemann surfaces and stable bundles*, Ann. Inst. H. Poincaré Anal. Nonlinéare, **6** (1989), 233–245.

[35] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Soviet Phys. JETP, 47 (1978), 1017–1027.

[36] V. E. Zakharov and A. B. Shabat, *Integration of non-linear equations of mathematical physics by the inverse scattering method II*, Funct. Anal. Appl., 13 (1979), 166–174.

[OCAMI2008] Riemann Surfaces, Harmonic Maps and Visualization, Proceedings of the 16th Osaka City University International Academic Symposium 2008, OCAMI Studies Volume 3, 2009, edited by Y.Ohnita, M.Guest, R.Miyaoka, and W.Rossman, Osaka Municipal Universities Press. Published in March, 2010. ISBN978-4-901409-69-8 C3041

[大仁田 - 宮岡] 大仁田義裕・宮岡礼子,「調和写像と可積分系理論」, 裳華房, 執筆中.

Osaka City University Advanced Mathematical Institute & Department of Mathematics, Osaka City University, 558-8585, JAPAN

E-mail address: ohnita@sci.osaka-cu.ac.jp