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A Remainder Estimate of Stationary Phase Method for Oscillatory Integrals over a Space of Large Dimension and Its Application to Feynman Path Integrals

By

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Abstract

This is an introduction to stationary phase method for oscillatory integrals over a space of large dimension. In particular, an estimate of the remainder term of stationary phase method is explained. As an application, such estimate is used to give rigorous mathematical meaning to Feynman path integral if the potential is smooth and of $O(|x|^2)$ at the infinity[1]. We do not discuss Feynman path integral thus obtained is the propagator[2].

§ 1. Feynman Path Integrals.

In quantum mechanics, state of a particle in Euclidean space $\mathbf{R}^d$ is described by an element φ in Hilbert space $L^2(\mathbf{R}^d)$ with unit norm (cf. for example [2] or [19]). φ is represented by a function $\varphi(x)$, called wave function, with the property

$$\|\varphi\|^2 = \int_{\mathbf{R}^d} |\varphi(x)|^2 dx = 1.$$

The integral of $|\varphi(x)|^2$ over a domain Q in $\mathbf{R}^d$

$$\int_Q |\varphi(x)|^2 dx$$

gives the probability for the particle to be found in Q.

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[1]The case with vector potential is treated in [12].
[2]That is given in [8].

If a particle moves, state φ_t of particle changes as time t changes. Motion of the particle is one parameter family $\{\varphi_t\}_{t\in\mathbf{R}}$ parameterized by time t. The family φ_t is represented by a function $\varphi(t,x)$ of $(t,x)\in\mathbf{R}^{1+d}$, with

$$\int_{\mathbf{R}^d}|\varphi(t,x)|^2dx=1.$$

Assume that the motion occurs under the influence of given force with potential $V(t,x)$. Then there is a mapping $U(t,s)\colon L^2(\mathbf{R}^d)\ni\varphi_s\mapsto\varphi_t\in L^2(\mathbf{R}^d)$ and $U(t,s)$ is a unitary operator called the evolution operator. Since $U(t,s)$ is a linear operator, it is represented, at least formally, by an integral transformation:

$$\varphi(t,x)=\int_{\mathbf{R}^d}k(t,x;s,y)\varphi(s,y)dy. \tag{1.1}$$

The function $k(t,x;s,y)$ is called the propagator.

Quantization is the process to determine the evolution operator $U(t,s)$ or equivalently the propagator from the potential $V(t,x)$.

There exist two ways of quantization. One is Schrödinger's method and the other is Feynman's method. Schrödinger's method is to obtain $U(t,x)$ and Feynman's method is to obtain propagator.

§ 1.1. Schrödinger's Quantization – Schrödinger Equation.

In classical mechanics the motion of a particle is described by a curve $(p(t),q(t))$ in the phase space $T^*(\mathbf{R}^\mathbf{d})=\mathbf{R}^{2d}$. $q(t)$ is the position of the particle at time t and $p(t)$ is the momentum. $(p(t),q(t))$ is the solution of Hamilton's equation. cf. for example [20]:

$$\frac{d}{dt}q(t)=\frac{\partial}{\partial p}H(t,p,q),$$
$$\frac{d}{dt}p(t)=-\frac{\partial}{\partial q}H(t,p,q),$$

here $H(t,p,q)$ is Hamilton's function

$$H(t,p,q)=\frac{1}{2}p^2+V(t,q), \tag{1.2}$$

if physical unit system is suitably chosen.

To make notations simpler we always assume that $d=1$ in the following. ∂_x denotes partial differentiation by x. i.e. $\partial_x=\dfrac{\partial}{\partial x}$.

Now we summarize Schrödinger's quantization (cf. for example [19] and [2]).

Replace q and p in $H(t,p,q)$ by x and by partial differential operator $\frac{\hbar}{i}\partial_x$ respectively. Here $i=\sqrt{-1}$ and $\hbar$ is a very small positive constant, which plays an important role in quantum mechanics[3]. Then we obtain the partial differential operator, Hamiltonian operator,

$$H(t)=-\frac{1}{2}(\hbar\partial_x)^2+V(t,x).$$

[3] $\hbar=h/2\pi$, here h is Planck constant.

Then Schrödinger's quantization is the following rule:

$$\frac{d}{dt}\varphi_t = \frac{-i}{\hbar}H(t)\varphi_t.$$

This means that the wave function $\varphi(t,x)$ is the solution of the partial differential equation, the Schrödinger equation,

$$-\frac{\hbar}{i}\partial_t\varphi(t,x) = \frac{1}{2}(\frac{\hbar}{i}\partial_x)^2\varphi(t,x) + V(t,x)\varphi(t,x). \tag{1.3}$$

If initial condition $\varphi(s,x)$ is given, $\varphi(t,x)$ is determined uniquely. This correspondence is the evolution operator $U(t,s)$.

§ 1.2. Feynman's Quantization – Feynman Path Integral.

Feynman's quantization introduced by [3] is a method to construct propagator $k(t,x;s,y)$ using Lagrangian of classical mechanics $L(t,\dot{x},x) = \frac{1}{2}\dot{x}^2 - V(t,x)$. Here $V(t.x)$ is the potential field and x is the position of the particle and $\dot{x}$ is the velocity. $L(t,\dot{x},x)$ is a function on $T(\mathbf{R}^d)$.

Let $[a,b]$ be a time interval. A motion of a particle during this period of time is a curve, or a path, γ: $[a,b] \ni t \mapsto \gamma(t) \in \mathbf{R}^d$. To any path γ we define its action $S(\gamma)$ by

$$S(\gamma) = \int_a^b L(t,\dot{\gamma}(t),\gamma(t))dt.$$

$S(\gamma)$ changes as γ changes, in other words, $S(\gamma)$ is a functional ofγ. Let x,y be arbitrary points of $\mathbf{R}^d$. Let Ω be the set of all paths γ: $[a,b] \to \mathbf{R}^d$ such that

$$\gamma(a) = y,\ \gamma(b) = x.$$

Although Ω contains a huge number of paths, Hamilton's least action principle of classical mechanics (cf. for example, [20]) states that the only path γ_0 that is realized under Newton's law of motion is the solution of the variational problem,

$$\delta S(\gamma_0) = 0, \quad \gamma_0(a) = y, \quad \gamma_0(b) = x.$$

We call such path as the classical path.

Feynman's quantization is the following formal formula.

$$k(b,x;a,y) = \frac{1}{N}\sum_{\gamma\in\Omega}\exp\Big(\frac{i}{\hbar}S(\gamma)\Big). \tag{1.4}$$

Here $k(b,x;a,y)$ is the integral kernel of (1.1), $S(\gamma)$ is the action of path γ, summation $\sum_{\gamma\in\Omega}$ is summation over all paths in Ω and N is a normalizing factor.

Since Ω is a continuum, it is better to replace $\sum_{\gamma\in\Omega}$ by symbol of integration over Ω, i.e.

$$k(b,x;a,y) = \int_\Omega \exp\Big(\frac{i}{\hbar}S(\gamma)\Big)\mathcal{D}[\gamma]. \tag{1.5}$$

The right-hand side is an integration over the path space Ω. This is called Feynman path integral.

More generally, one can discuss integration of the form

$$\int_\Omega F(\gamma)\exp\Big(\frac{i}{\hbar}S(\gamma)\Big)\mathcal{D}[\gamma]$$

for functional $F(\gamma)$ of γ. This integration is also called Feynman path integral.

§ 2. Feynman's Original Formulation of the Path Integral.

The formula (1.4) or (1.5) is quite formal. Feynman gave more solid formulation in [3] and we follow him. We assume that $d = 1$ for simplicity.

Let $[a,b]$ be an interval of time. Let Δ be an arbitrary division of $[a,b]$

$$\Delta : a = T_0 < T_1 < \cdots < T_J < T_{J+1} = b. \tag{2.1}$$

We set $\tau_j = T_j - T_{j-1}$ $(j = 1,2,\ldots,J+1)$ and $|\Delta| = \max_{1\leq j\leq J+1}\{\tau_j\}$.

For $j = 1,2,\ldots,J$, choose an arbitrary point $x_j \in \mathbf{R}$. We set $x_0 = y$, $x_{J+1} = x$. We have thus $J+2$ points $\{(T_j,x_j)\}$ in time-space $\mathbf{R}\times\mathbf{R}$. Consider classical path γ_1 starting from (T_0,X_0) and ending at (T_1,x_1). If such a classical path is not unique, then we choose the one for which the action is the smallest. Similarly we consider classical path γ_2 starting from (T_1,x_1) and ending at (T_2,x_2). Continuing this process we obtain classical path $\gamma_j,(j = 1,2,\ldots,J+1)$ starting (T_{j-1},x_{j-1}) and arriving at (T_{j+1},x_{j+1}) in time-space. Finally we connect all of these $J+1$ classical paths and obtain a path connecting (T_0,x_0) and (T_{J+1},x_{J+1}) in time-space. We name this long path $\gamma_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)$ because it depends on the division Δ and points $(x_0,x_1,\ldots,x_{J+1})$. Although this is a continuous curve, it is not, in general, a smooth one. It may have edge at (T_j,x_j) ,$j = 1,2,\ldots,J$. We call such a path a piecewise classical path. Some time we use γ_Δ as an abbreviation of $\gamma_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)$.

The action $S(\gamma_\Delta)$ of $\gamma_\Delta(x_{J+1},x_J,\ldots,x_0)$ is a function of $(x_{J+1},x_J,\ldots,x_1,x_0)$ if Δ is fixed.

$$S(\gamma_\Delta)(x_{J+1},x_J,\ldots,x_0) = \int_a^b L(t,\dot\gamma_\Delta(t),\gamma_\Delta(t))dt = \sum_{j=1}^{J+1}\int_{T_{j-1}}^{T_j} L(t,\dot\gamma_j(t),\gamma_j(t))dt. \tag{2.2}$$

Similarly if a functional $F(\gamma)$ of γ is given, $F(\gamma_\Delta)$ is a function of $(x_{J+1},x_J,\ldots,x_1,x_0)$. For the sake of brevity we often write $F(\gamma_\Delta)$ by F_Δ and $S(\gamma_\Delta)$ by S_Δ.

Piecewise classical path γ_Δ approaches to any $\gamma\in\Omega$ as close as one like, if $|\Delta|$ and $\{x_j\}$ are suitably chosen. Taking this fact in mind, Feynman formulated:

$$\frac{1}{N}\sum_{\gamma\in\Omega}F(\gamma)\exp\frac{i}{\hbar}S(\gamma) = \lim_{|\Delta|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi\hbar\tau_j}\right)^{1/2}\int_{\mathbf{R}^J}F(\gamma_\Delta)\exp\left(\frac{i}{\hbar}S(\gamma_\Delta)\right)\prod_{j=1}^{J}dx_j.$$

In other words, with $\nu = \hbar^{-1}$,

$$\int_\Omega F(\gamma)\exp(i\nu S(\gamma))\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,b,a,x,y), \tag{2.3}$$

where

(2.4) $I[F_\Delta](\Delta;\nu,b,a,x,y)$

$$= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\mathbf{R}^J}F(\gamma_\Delta)(x_{J+1},x_J,\ldots,x_1,x_0)\exp\left(i\nu S(\gamma_\Delta)(x_{J+1},x_J,\ldots,x_1,x_0)\right)\prod_{j=1}^{J}dx_j.$$

We shall name $I[F_\Delta](\Delta; \nu, b, a, x, y)$ time slicing approximation of path integral.

Does the right hand side of (2.3) give a finite number ? The following questions should be answered.

Q1 Does $I[F_\Delta](\Delta; \nu, b, a, x, y)$ exist for fixed $|\Delta| > 0$?

Q2 Does the limit $\lim_{|\Delta|\to 0} I[F_\Delta](\Delta; \nu, b, a, x, y)$ exist ?

We will answer these questions under certain assumptions for $V(t,x)$ which will be given later in §5.

§ 3. Oscillatory Integrals.

First we discuss question Q1.

Once the division Δ is fixed, $I[F_\Delta](\Delta; \nu, b, a, x, y)$ is a special case of the following type of integrals:

(3.1)
$$\int_{\mathbf{R}^n} a(x,y) e^{i\nu\phi(x,y)} dy,$$

where $\phi(x,y)$ is a real valued function of $(x,y) \in \mathbf{R}^m \times \mathbf{R}^n$ and $a(x,y)$ is a function of (x,y).

Among others, $a(x,y) = 1$ is the most important case. In this case the integral (3.1) does not converge absolutely. How can one give definite meaning to it ?

Heuristic explanation is the following. The value $\phi(x,y)$ changes and hence $e^{i\nu\phi(x,y)}$ oscillates as y changes from one place to another in $\mathbf{R}^n$ and they cancel each other. As a result the integral (3.1) give finite value. So integral of the type (3.1) is called an oscillatory integral (with parameter x). $\phi(x,y)$ is called phase function and $a(x,y)$ is called amplitude function.

If parameter ν goes to ∞, then $e^{i\nu\phi(x,y)}$ oscillates very rapidly and hence as a result of cancellation main contribution to (3.1) comes from the critical,in other words, stationary points of $\phi(x,y)$ with respect to y, i.e.,we expect good approximation formula: cf. [16]

(3.2)
$$I(x) \propto \sum_p a(x, y_p) e^{i\nu\phi(x,y_p)} + 0(\nu^{-1}).$$

where, $\{y_p\}$ are the solution to

$$\frac{\partial}{\partial y}\phi(x, y_p) = 0.$$

Approximate evaluation formula (3.2) is the stationary phase method.

The precise meaning of oscillatory integral (3.1) is the following. Consider arbitrary family of smooth functions $\{\omega_\epsilon(y)\}_{\epsilon>0}$ with the following properties:

1. For any y
$$\lim_{\epsilon\to 0} \omega_\epsilon(y) = 1.$$
2. For any multi index α
$$\lim_{\epsilon\to 0} \left(\frac{\partial}{\partial y}\right)^\alpha \omega_\epsilon(y) = 0.$$

3. If ϵ is fixed, for any multi-index α and for any positive integer N there exists a positive constant C_ϵ such that
$$|\left(\frac{\partial}{\partial y}\right)^\alpha \omega_\epsilon(y)| \le C_\epsilon(1+|y|)^{-N}.$$

Definition 3.1. Let
$$I_\epsilon(x) = \int_{\mathbf{R}^n} \omega_\epsilon(y)a(x,y)e^{iv\phi(x,y)}dy.$$
If
$$\lim_{\epsilon\to 0} = I(x)$$
exists and does not depend on choice of family of functions $\{\omega_\epsilon\}$, $I(x)$ is called oscillatory integral (3.1). And we write
$$\int_{\mathbf{R}^n} a(x,y)e^{iv\phi(x,y)}dy = I(x).$$

Now we give a sufficient condition for oscillatory integral (3.1) to exist.

Assume $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ and the following conditions.

A1 Phase function $\phi(x,y) \in C^\infty(\mathbf{R}^m \times \mathbf{R}^n)$ is real valued. For any multi-indices α,β with $|\alpha| + |\beta| \ge 2$ there exists a positive constant $C_{\alpha\beta}$ such that
$$|\partial_x^\alpha \partial_y^\beta \phi(x,y)| \le C_{\alpha\beta}.$$

A2 Let $(\partial_{y_j}\partial_{y_k}\phi(x,y))$ be the $n\times n$ square matrix with (j,k) element $\partial_{y_j}\partial_{y_k}\phi(x,y)$. Assume that there exists a positive constant C such that
$$|\det(\partial_{y_j}\partial_{y_k}\phi(x,y))| \ge C > 0$$
for any $(x,y) \in (\mathbf{R}^m \times \mathbf{R}^n)$. Here det means the determinant.

A3 The amplitude function $a(x,y)$,together with its all derivatives, is uniformly bounded on $\mathbf{R}^m \times \mathbf{R}^n$.

Theorem 3.2 (cf. [1]). *Under conditions A1, A2 and A3, the oscillatory integral $I(x)$ exists. Moreover there exist a positive constant C such that*
$$|I(x)| \le Cv^{-n/2} \max_{|\alpha|\le n+1} \sup_{y\in\mathbf{R}^n} |\partial_x^\alpha a(x,y)|.$$

Assumptions A1 and A2 assure that the value $\exp iv\phi(x,y)$ actually oscillates. This fact follows from the following Global implicit function theorem of Hadamard. cf. [18]

Theorem 3.3. *Let $\zeta_j(x,y) = \partial_{y_j}\phi(x,y)$, $j = 1,2,\ldots,n$. Consider for any fixed x the map $\Phi_x : \mathbf{R}^n \ni y = (y_1,y_1,\ldots,y_n) \to \zeta(y) = (\zeta_1(x,y),\zeta_2(x,y),\ldots,\zeta_n(x,y)) \in \mathbf{R}^n$. Then Φ_x is a global diffeomorphism. $y^*(x) = \Phi_x^{-1}(0)$ is the unique critical point of $\phi(x,y)$ with respect to y. Moreover there exists a positive constant C independent of x such that for any points $y,y' \in \mathbf{R}^n$ there holds inequality*
$$C^{-1}|y-y'| \le |\Phi_x(y) - \Phi_x(y')| \le C|y-y'|.$$
For any non zero multi-index α there exists constant C_α such that
$$|\partial_y^\alpha \zeta|, \quad |\partial_\zeta y| \le C_\alpha.$$

§ 4. Stationary Phase Method.

Assume A1,A2 and A3. Then stationary phase method is also valid. Let $H(x,y^*(x))$ be the Hessian matrix of $\phi(x,y)$ with respect to y at $y=y^*(x)$, i.e.,$H(x,y^*(x))$ is the $n\times n$ symmetric matrix of which the (j,k) element is $\partial_{y_j}\partial_{y_k}\phi(x,y^*(x))$.

Theorem 4.1 (Stationary phase method). *Assume the assumptions* A1, A2 *and* A3. *We have the following asymptotic formula as* $\nu\longrightarrow\infty$:

$$I(x)=\left(\frac{2\pi}{\nu}\right)^{n/2}|\det H(x,y^*(x))|^{-1/2}\exp\frac{\pi i}{4}[n-2\,\mathrm{Ind}(H(x,y^*(x)))]$$
$$\times e^{i\nu\phi(x,y^*(x))}\left(a(x,y^*(x))+\nu^{-1}r(\nu,x)\right).$$

Here $\mathrm{Ind}(H(x,y^*(x)))$ *is the number of negative eigenvalues of matrix* $H(x,y^*(x))$. *The remainder term* $r(\nu,x)$ *satisfies the following estimate: For any non-negative integer* k, *there exist positive number* $K(k)$ *and positive constant* C_k *such that for any multi-index* α *with* $|\alpha|\le k$ *there holds inequality*

$$|\partial_x^\alpha r(\nu,x)|\le C_k\max_{\substack{|\beta_1|\le K(k)\\|\beta_2|\le K(k)}}\sup_{y\in\mathbf{R}^n}|\partial_x^{\beta_1}\partial_y^{\beta_2}a(x,y)|. \tag{4.1}$$

cf. [13] and [1] for more information.

§ 5. Property of Classical Action.

Let $[a,b]$ be an interval of time. We now discuss Feynman path integral. Our assumption for potential $V(t,x)$ is the following (cf. W. Pauli [17]).

Assumption 5.1. 1. $V(t,x)$ is a real valued function of (t,x) which is continuous in (t,x) and infinite differentiable with respect to x.

2. For any non-negative integer m there exists a positive constant v_m such that

$$\max_{|\alpha|=m}\sup_{(t,x)\in[0,T]\times\mathbf{R}^d}|\partial_x^\alpha V(t,x)|\le v_m(1+|x|)^{\max\{2-m,0\}}.$$

First we discuss piecewise classical path γ_Δ. For the sake of simplicity we assume that $d=1$. We can discuss the case of $d\ge 2$ similarly, but notation will become cumbersome.

Classical path satisfies Euler equation.

$$\frac{d^2}{dt^2}\gamma(t)+\partial_x V(t,\gamma(t))=0,$$
$$\gamma(b)=x,\quad \gamma(a)=y.$$

One can prove the following

Theorem 5.2. *Let* μ_0 *be a positive number which satisfies*

$$\frac{\mu_0^2 d v_2}{8}<1. \tag{5.1}$$

If $|b-a|\le\mu_0$, *then for any* $x,y\in\mathbf{R}$ *there exists a unique classical path* γ *starting from* y *at time* a *and reaching* x *at time* b.

We always assume $|b-a| < \mu_0$ below. Let γ be classical path γ starting from y at time a and reaching x at time b. The action of classical path γ is a function of (b,a,x,y), and we denote it by $S(b,x,a,y)$. It is called the classical action.

$$S(b,a,x,y) = \int_a^b L(t,\dot\gamma(t),\gamma(t))dt = \int_a^b \frac{1}{2}\left(\frac{d}{dt}\gamma(t)\right)^2 - V(t,\gamma(t))dt.$$

One can prove the following Proposition, cf. [4].

Proposition 5.3. *If $|b-a| \le \mu_0$, the classical action $S(b,a,x,y)$ is of the following form:*

$$S(b,a,x,y) = \frac{|x-y|^2}{2(b-a)} + (b-a)\phi(b,a,x,y).$$

The function $\phi(b,a,x,y)$ is a function of (b,a,x,y) of class C^1 and estimated with some constant C

$$|\phi(b,a,x,y)| \le C(1+|x|^2+|y|^2).$$

Moreover, for any fixed a and b $\phi(b,a,x,y)$ is a C^∞ function of (x,y) and for any positive integer $m \ge 2$ we have

$$\max_{2\le|\alpha|+|\beta|\le m} \sup_{(x,y)\in\mathbf{R}^2} |\partial_x^\alpha \partial_y^\beta \phi(b,a,x,y)| = \kappa_m < \infty.$$

In particular, we know

$$\kappa_2 \le \frac{v_2}{2}\left(1 - \frac{v_2\mu_0^2}{8}\right)^{-1}.$$

Proof is banal.

§ 6. Time Slicing Approximation in the Case $J = 1$.

Let Δ_1 be the following simple division of $[a,b]$ with $J = 1$.

$$\Delta_1 : a = T_0 < T_1 < T_2 = b. \tag{6.1}$$

Then for this division Δ_1

$$\begin{aligned} &I[F_{\Delta_1}](\Delta_1;\nu,b,a,x,y) \\ &= \left(\frac{\nu}{2\pi i\tau_1}\right)^{1/2}\left(\frac{\nu}{2\pi i\tau_2}\right)^{1/2}\int_{\mathbf{R}} F_{\Delta_1}(x,x_1,y)e^{i\nu S_{\Delta_1}(x,x_1,y)}dx_1. \end{aligned}$$

The phase is

$$S_{\Delta_1}(x,x_1,y) = \frac{|x-x_1|^2}{2\tau_2} + \tau_2\phi(b,T_1,x,x_1) + \frac{|x_1-y|^2}{2\tau_1} + \tau_1\phi(T_1,a,x_1,y).$$

The critical point x_1^* is the solution of equation

$$\begin{aligned} 0 = x_1^* &- \frac{\tau_1 x + \tau_1 y}{\tau_1+\tau_2} \\ &+ \tau_1\tau_2\left(\frac{\tau_1}{\tau_1+\tau_2}\partial_{x_1}\phi(T_1,a,x_1^*,y) + \frac{\tau_2}{\tau_1+\tau_2}\partial_{x_1}\phi(b,T_1,x,x_1^*)\right). \end{aligned}$$

At the critical point x_1^* the Hessian $\mathrm{Hess}_{x_1^*} S_\Delta$ is

$$\begin{aligned}\mathrm{Hess}_{x_1^*} S_\Delta &= \frac{\tau_1+\tau_2}{\tau_1\tau_2} + \tau_1\partial_{x_1}^2\phi(T_1,a,x_1^*,y) + \tau_2\partial_{x_1}^2\phi(b,T_1,x,x_1^*)\\ &= \frac{\tau_1+\tau_2}{\tau_1\tau_2}\left(1+\tau_1\tau_2\{\frac{\tau_1}{\tau_1+\tau_2}\partial_{x_1}^2\phi(T_1,a,x_1^*,y) + \frac{\tau_2}{\tau_1+\tau_2}\partial_{x_1}^2\phi(b,T_1,x,x_1^*)\}\right).\end{aligned}$$

We define $D_{x_1^*}(\Delta_1;b,a,x,y)$ by

$$\begin{aligned}D_{x_1^*}(\Delta_1;b,a,x,y) =& \frac{\tau_1\tau_2}{\tau_1+\tau_2}\mathrm{Hess}_{x_1^*} S_\Delta\\ =& 1+\tau_1\tau_2\{\frac{\tau_1}{\tau_1+\tau_2}\partial_{x_1}^2\phi(T_1,a,x_1^*,y) + \frac{\tau_2}{\tau_1+\tau_2}\partial_{x_1}^2\phi(b,T_1,x,x_1^*)\}.\end{aligned}$$

We write

$$D_{x_1^*}(\Delta_1;b,a,x,y) = 1+\tau_1\tau_2 d(\Delta_1;b,a,x,y), \tag{6.2}$$

where

$$d(\Delta_1;b,a,x,y) = \frac{\tau_1}{\tau_1+\tau_2}\partial_{x_1}^2\phi(T_1,a,x_1^*,y) + \frac{\tau_2}{\tau_1+\tau_2}\partial_{x_1}^2\phi(b,T_1,x,x_1^*).$$

For any $K \geq 0$ there exists a positive constant C_K such that if $|\alpha|, |\beta| \leq K$, then we have the estimate

$$|\partial_x^\alpha \partial_y^\beta d(\Delta_1;b,a,x,y)| \leq C_K. \tag{6.3}$$

We apply the stationary phase method then we have the following important fact:

Lemma 6.1. *Let Δ_1 be the division* (6.1). *Using stationary phase method, we have*

$$\begin{aligned}&I[F_{\Delta_1}](\Delta_1;\nu,b,a,x,y)\\ &= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} D_{x_1^*}(\Delta_1,b,a,x,y)^{-1/2}\\ &\times \left[F_{\Delta_1}(x,x_1^*,y) + \frac{i\tau_1\tau_2\partial_{x_1}^2 F_{\Delta_1}(x,x_1^*,y)}{2\nu(b-a)D_{x_1^*}(\Delta_1,b,a,x,y)} + \nu^{-1}\tau_1\tau_2 b(\Delta_1;\nu,x,y)\right].\end{aligned} \tag{6.4}$$

Moreover, for any nonnegative integer m, there exist positive constant C_m and a natural number $M(m)$ such that as far as $|\alpha_2|$, $|\alpha_0| \leq m$ there holds the estimate:

$$\left|\partial_{x_2}^{\alpha_2}\partial_{x_0}^{\alpha_0} b(\Delta_1;\nu;x,y)\right| \leq C_m \max \sup_{x_1\in\mathbf{R}} |\partial_{x_2}^{\beta_2}\partial_{x_1}^{\beta_2}\partial_{x_0}^{\beta_0} F_{\Delta_1}(x,x_1,y)|. \tag{6.5}$$

Here max *is taken for all β_1 with $|\beta_1| \leq M(m)$ and $\beta_2 \leq \alpha_2$, $\beta_0 \leq \alpha_0$.*

Corollary 6.2. *If $F(\gamma) \equiv 1$,*

$$\begin{aligned}&I[1](\Delta_1;\nu,b,a,x,y)\\ &= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} D_{x_1^*}(\Delta_1,b,a,x,y)^{-1/2}\left[1+\nu^{-1}\tau_1\tau_2 b(\Delta_1;\nu,x,y)\right].\end{aligned} \tag{6.6}$$

Here $b(\Delta_1,\nu,x,y)$ satisfies the following estimate: for any α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_y^\beta b(\Delta_1;\nu,x,y)| \leq C_{\alpha\beta}.$$

§7. Time Slicing Approximation is Oscillatory Integral.

We will discuss time slicing approximation corresponding to general division of $[a,b]$.

We assume that $V(t,x)$ satisfies the Assumption 5.1. We assume that

$$|b-a| \leq \mu_0. \tag{7.1}$$

Let Δ be the division of time interval $[a,b]$

$$\Delta : a = T_0 < T_1 < \cdots < T_J < T_{J+1} = b.$$

Assumption 7.1. We assume that for any $\{\alpha_j\}$ there exists positive constant C such that

$$|\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)| \leq C.$$

Here C may depend on $\{\alpha_j\}$ and on Δ.

We discuss the time slicing approximation of path integral.

$$\begin{aligned} &I[F_\Delta](\Delta; \nu, b, a, x, y) \\ &= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j}\right)^{1/2} \int_{\mathbf{R}^J} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) \exp\left(i\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)\right) \prod_{j=1}^{J} dx_j. \end{aligned} \tag{7.2}$$

We claim this satisfies conditions A1, A2 and A3 of §3.
Condition A3 is clearly satisfied. We check condition A1.

$$\begin{aligned} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) &= S(\gamma_\Delta)(x_{J+1}, x_J, \ldots, x_1, x_0) \\ &= \sum_{j=1}^{J+1} S(T_j, T_{j-1}, x_j, x_{j-1}) = \sum_{j=1}^{J+1} \left(\frac{|x_j - x_{j-1}|^2}{2\tau_j} + \tau_j \phi(T_j, T_{j-1}, x_j, x_{j-1})\right). \end{aligned}$$

Note that

$$\begin{aligned} \partial_{x_j} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) &= \frac{x_j - x_{j-1}}{\tau_j} + \frac{x_j - x_{j+1}}{\tau_{j+1}} \\ &+ \tau_j \partial_{x_j} \phi_j(x_j, x_{j-1}) + \tau_{j+1} \partial_{x_j} \phi_{j+1}(x_{j+1}, x_j). \end{aligned} \tag{7.3}$$

Here we used abbreviation:

$$\phi_j(x_j, x_{j-1}) = \phi(T_j, T_{j-1}, x_j, x_{j-1}).$$

It follows from (7.3) and Proposition 5.3 that condition A1 is satisfied.

Now we check condition A2. Consider $J \times J$ matrix Ψ whose (j,k) element is

$$\Psi_{jk} = \partial_{x_j} \partial_{x_k} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0).$$

Then

$$\Psi_{jk} = \begin{cases} \dfrac{1}{\tau_j} + \dfrac{1}{\tau_{j+1}} + \tau_j \partial_{x_j}^2 \phi_j(x_j, x_{j-1}) + \tau_{j+1} \partial_{x_j}^2 \phi_{j+1}(x_{j+1}, x_j) & \text{if} \quad j = k \\ \dfrac{-1}{\tau_j} + \tau_j \partial_{x_k} \partial_{x_j} \phi_j(x_j, x_{j-1}) & \text{if} \quad k = j-1 \\ \dfrac{-1}{\tau_k} + \tau_k \partial_{x_j} \partial_{x_k} \phi_k(x_k, x_{k-1}) & \text{if} \quad k = j+1 \\ 0 & \text{if} \quad |j-k| \geq 2. \end{cases}$$

We can divide the matrix Ψ into two parts.

$$\Psi = H_\Delta + W_\Delta,$$

where

$$H_\Delta = \begin{pmatrix} \frac{1}{\tau_1}+\frac{1}{\tau_2} & -\frac{1}{\tau_2} & 0 & 0 \cdots & 0 & 0 \\ -\frac{1}{\tau_2} & \frac{1}{\tau_2}+\frac{1}{\tau_3} & -\frac{1}{\tau_3} & 0 \cdots & 0 & 0 \\ 0 & -\frac{1}{\tau_3} & \cdots & \cdots\cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots\cdots & \vdots & -\frac{1}{\tau_J} \\ 0 & 0 & 0 & \cdots 0 & -\frac{1}{\tau_J} & \frac{1}{\tau_J}+\frac{1}{\tau_{J+1}} \end{pmatrix}$$

and W_Δ is a matrix whose $(j.k)$ element is

$$w_{jk} = \begin{cases} \partial_{x_j}^2(\tau_j\phi_j + \tau_{j+1}\phi_{j+1}) & \text{if} \quad j=k \\ \partial_{x_k}\partial_{x_j}\tau_j\phi_j & \text{if} \quad k=j-1 \\ \partial_{x_j}\partial_{x_k}\tau_k\phi_k & \text{if} \quad k=j+1 \\ 0 & \text{if} \quad |j-k|\ge 2. \end{cases} \tag{7.4}$$

The matrix H_Δ is a constant matrix with determinant

$$\det H_\Delta = \frac{\tau_1+\tau_2+\cdots+\tau_{J+1}}{\tau_1\tau_2\ldots\tau_{J+1}} = \frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}}.$$

It has it inverse H_Δ^{-1}. Regarding W_Δ as an perturbation, we write

$$\Psi = H_\Delta(I + H_\Delta^{-1}W_\Delta).$$

We will prove that $H_\Delta^{-1}W_\Delta$ is very small. Since $H_\Delta^{-1}W_\Delta$ is a $J\times J$ square matrix, it defines a linear map from $\mathbf{R}^J$ into itself. For any $\xi = (\xi_1,\xi_2,\ldots,\xi_J)$ let

$$\|\xi\|_\infty = \max_{1\le j\le J}\{|\xi_j|\}.$$

Then $\|\xi\|_\infty$ is a norm in $\mathbf{R}^J$. $H_\Delta^{-1}W_\Delta$ is very small in the following sense. For any ξ we have

$$\|H_\Delta^{-1}W_\Delta\xi\|_\infty \le \kappa_2(\tau_1+\cdots+\tau_J)^2\|\xi\|_\infty.$$

The following proposition states that condition A2 is satisfied for $I[F_\Delta](\Delta;\nu,b,a,x,y)$. cf. [6].

Proposition 7.2. *Let $0<\mu_1$ be so small that $\mu_1\le\mu_0$ and that $\kappa_2\mu_1^2<1$. Let $|b-a|\le\mu_1$. Then for any $(x_{J+1},x_J,\ldots,x_1,x_0)\in\mathbf{R}^{J+2}$ we have estimates*

$$(1-\kappa_2\mu_1^2)^J \le \det(I+H_\Delta^{-1}W_\Delta) \le (1+\kappa_2\mu_1^2)^J,$$

and

$$(1-\kappa_2\mu_1^2)^J\frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}} \le \det\Psi = \det(H_\Delta+W_\Delta) \le (1+\kappa_2\mu_1^2)^J\frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}}.$$

As a conclusion, conditions A1, A2 and A3 of §3 are satisfied, $I[F_\Delta](\Delta;\nu,b,a,x,y)$ has a definite value if Δ is fixed in the case $|b-a|\le\mu_1$. We answered Q1 of §2.

§ 8. stationary point of the phase function

Let μ_1 be as in Proposition 7.2. We assume that $|b-a| \leq \mu_1$ in the following. The stationary point $(x_J^*, \cdots, x_1^*)$ of the phase function $S_\Delta(x_{J+1}, x_J, \ldots, x_0)$ exists uniquely. It is the solution of system of equations:

$$\partial_{x_j} S_\Delta(x_{J+1}, x_J^*, \cdots, x_1^*, x_0) = 0, \text{for any} j = 1, 2, \ldots, J.$$

This equations mean that

$$\partial_{x_j} S(T_j, T_{i-1}, x_j^*, x_{j-1}^*) + \partial_{x_j} S(T_{j+1}, T_j, x_{j+1}^*, x_j^*) = 0 \quad \text{for any} \quad j = 1, 2, \ldots, J.$$

Here we set $x_{J+1}^* = x_L, x_0^* = x_0$. These x_j^*、 $j = 1, 2, \ldots, J$ are functions of $(x_{J+1}, x_0) = (x, y)$.

Let γ_Δ^* be the piecewise classical path which connects (T_j, x_j^*). Then the following proposition is well known.

Proposition 8.1. *The piecewise classical path γ_Δ^* coincides with the classical path γ^* which starts $x_0 = y$ at time a and reaching $x_{J+1} = x$ at time b. The piecewise classical path γ_Δ^* is a smooth path.*

Corollary 8.2. *The value of the phase function at the stationary point equals*

$$S_\Delta(x_{J+1}, x_J^*, \cdots, x_1^*, x_0) = S(b, a, x, y).$$

We can apply stationary phase method to the oscillatory integral $I[F_\Delta](\Delta; \nu, b, a, x, y)$, if $|b-a| < \mu_1$. Since $IndH_\Delta = 0$, stationary phase method gives

Theorem 8.3. *If $|b-a| \leq \mu_1$, we obtain*

$$I[F_\Delta](\Delta; \nu, b, a, x, y)$$
$$= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} \left(\det(I + H_\Delta^{-1} W_\Delta^*)\right)^{-1/2} p(\Delta, \nu, b, a, x, y)$$

with some function $p(\Delta, \nu, b, a, x, y)$. Here W_Δ^* is W_Δ evaluated at $y = y^*(x)$.

How does $p(\Delta, \nu, b, a, x, y)$ behave as $|\Delta| \to 0$? This is the core of the problem.

The next theorem was known earlier. cf. [14].

Theorem 8.4 (Kumano-go,H. & Taniguchi,K.). *Assume $|b-a| \leq \mu_0$. Assume that F_Δ satisfies the following property: For any non negative integer K there exits a positive constant A_K such that as long as $|\alpha_0| \leq K, |\alpha_1| \leq K, \ldots, |\alpha_{J+1}| \leq K$ one has*

$$|\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_J}^{\alpha_J} \ldots \partial_0^{\alpha_0} F_\Delta(x_{J+1}, \ldots, x_0)| \leq A_K.$$

Then we have

$$I[F_\Delta](\Delta; \nu, b, a, x, y) = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} p(\Delta; \nu, b, a, x, y).$$

Moreover for any nonnegative integer k there exist positive integer $K(k)$ and positive constant C_k such that as long as $|\alpha_0| \leq k, |\alpha_{J+1}| \leq k$, there holds estimate

$$(8.1) \qquad |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} p(\Delta; \nu, b, a, x, y)| \leq C_k^J A_{K(k)}.$$

Here $K(k)$、 C_k are independent of Δ and of J.

If we let $J \to \infty$ then the bound $C_k^J A_{K(k)}$ obtained by (8.1) may go to ∞. In order to answer Q2 of §2 we have to improve the above Kumano-go, H.& Taniguchi Theorem.

§ 9. Stationary phase method for integrals over a space of large dimension

We can improve stationary phase method so that we can let $|\Delta| \to 0$.

First we improve estimate in Proposition7.2 by using result of §6.

Assume $|b-a| \le \mu_1$. Let γ^* be the unique classical path starting from y at time a and reaching x at time b. Let $x_j^* = \gamma^*(T_j)$ for $j = 0, 1, 2, \dots, J+1$. We set

$$D(\Delta; b, a, x, y) = \det(I + H_\Delta^{-1} W_\Delta^*)$$
$$= \left(\frac{\tau_1 \tau_2 \dots \tau_{J+1}}{(b-a)}\right) \det \mathrm{Hess}_{x_J^*, x_{J-1}^*, \dots, x_1^*} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0).$$

Here $\mathrm{Hess}_{x_J^*, x_{J-1}^*, \dots, x_1^*} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ is the Hessian matrix of $S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ at $(x_J^*, x_{J-1}^*, \dots x_1^*)$

Now we have

Theorem 9.1. *The function $D(\Delta; b, a, x, y)$ is of the following form:*

$$D(\Delta; b, a, x, y) = 1 + (b-a)^2 d(\Delta; b, a, x, y). \tag{9.1}$$

Here for any $K \ge 0$ there exists a positive constant C_K independent of Δ such that if $|\alpha|, |\beta| \le K$, then

$$|\partial_x^\alpha \partial_y^\beta d(\Delta; b, a, x, y)| \le C_K. \tag{9.2}$$

Proof. First we use the following property of Hessian, of which proof is omitted here. cf. for ex. [6].

Proposition 9.2. *Assume that $\phi(x,y)$ is a real valued function of $(x,y) \in \mathbf{R}^m \times \mathbf{R}^n$ of class C^∞. Assume further that there exists a C^∞ map $y^\# : \mathbf{R}^m \ni x \longrightarrow y^\#(x) \in \mathbf{R}^n$ such that*

$$\partial_y \phi(x, y^\#(x)) \equiv 0, \quad \det \mathrm{Hess}_y \phi(x,y)|_{y=y^\#(x)} \ne 0.$$

Furthermore assume that

$$\phi^\# : \mathbf{R}^m \ni x \longrightarrow \phi(x, y^\#(x)) \in \mathbf{R}$$

is critical at $x = x^*$, i.e.,

$$\partial_x \phi^\#(x)|_{x=x^*} = 0.$$

Then $(x^*, y^*) = (x^*, y^\#(x^*)) \in \mathbf{R}^m \times \mathbf{R}^n$ is a critical point of function $\phi(x,y)$ and the following equality holds:

$$\det \mathrm{Hess}_{(x^*, y^*)} \phi = \det \mathrm{Hess}_{x^*} \phi^\# \times \det \mathrm{Hess}_y \phi(x,y)|_{(x,y)=(x^*,y^*)}.$$

In order to use the proposition, we introduce notations. For $k > j$ let $S_{k,j}(x_k, x_j)$ be abbreviation of classical action $S(T_k, T_j, x_k, x_j)$. For $0 < k < m$ let $(x_{m-1}^*, \dots, x_{k+1}^*)$ be the critical point of the function

$$(x_{m-1}, \dots, x_{k+1}) \longrightarrow S_{m,m-1}(x_m, x_{m-1}) + \dots + S_{k+1,k}(x_{k+1}, x_k).$$

$(x^*_{m-1},\dots,x^*_{k+1})$ is a function of (x_m,x_k) and equality

$$S_{m,k}(x_m,x_k)=S_{m,m-1}(x_m,x^*_{m-1})+\dots+S_{k+1,k}(x^*_{k+1},x_k)$$

holds.

We define $D_{x^*_{m-1},\dots,x^*_{k+1}}(S_{m,m-1}+\cdots+S_{k+1,k};x_m,x_k)$ by

$$\begin{aligned}&\det[\mathrm{Hess}_{(x^*_{m-1},\dots,x^*_{k+1})}(S_{m,m-1}+\cdots+S_{k+1,k})]\\&=\frac{\tau_{k+1}+\cdots+\tau_m}{\tau_m\tau_{m-1}\cdots\tau_{k+1}}D_{x^*_{m-1}\cdots x^*_{k+1}}(S_{m,m-1}+\cdots+S_{k+1,k};x_m,x_k).\end{aligned}$$

In this notation

$$D(\Delta;b,a,x,y)=D_{x^*_{L-1}\cdots x^*_1}(S_{J+1,J}+\cdots+S_{1,0};x_{J+1},x_0).$$

Applying proposition 9.2 repeatedly, we can prove the following fact:

Theorem 9.3. *The following equality holds:*

$$D(\Delta;b,a,x_{J+1},x_0)=\prod_{k=2}^{J+1}D_{x^*_{k-1}}(S_{k,k-1}+S_{k-1,0};x_k,x_0)|_{x_k=x^*_k}. \tag{9.3}$$

As a result of (6.2) and 6.3 in §6, we obtain the following

$$D_{x^*_{k-1}}(S_{k,k-1}+S_{k-1,0};x_k,x_0)=1+\tau_k(\tau_1+\cdots+\tau_{k-1})d_{k,0}(x_k,x_0), \tag{9.4}$$

where for any α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$|\partial^\alpha_{x_0}\partial^\beta_{x_k}d_{k,0}(x_k,x_0)|\le C_{\alpha\beta}. \tag{9.5}$$

(9.1) follows from (9.3) and (9.4). (9.2) follows from (9.3) and (9.5). Theorem 9.1 is now proved. □

Assuming a new assumption about the amplitude $F(\gamma)$, now we improve stationary phase method so that we can let $|\Delta|\to 0$.

Assumption 9.4. The functional $F(\gamma)$ satisfies the following condition: For any nonnegative integer K there exist positive constants A_K and X_K such that for any division Δ and α_j satisfying $|\alpha_j|\le K\quad(0\le j\le J+1)$ we have

$$|\partial^{\alpha_0}_{x_0}\partial^{\alpha_1}_{x_1}\dots\partial^{\alpha_{J+1}}_{x_{J+1}}F_\Delta(x_{J+1},x_J,\dots,x_1,x_0)|\le A_KX_K^{J+1}. \tag{9.6}$$

Here A_K,X_K may depend on K but are independent of Δ and of J.

Remark. $F(\gamma)\equiv 1$ satisfies the above assumption 9.4.

The next theorem states that the stationary phase method is valid even in the case $|\Delta|\to 0$. cf. [6] and

Theorem 9.5. [4] *Assume that $F(\gamma)$ satisfies the above Assumption 9.4. Further we assume $|b-a|\le\mu_1$. Then*

$$\begin{aligned}&I[F_\Delta](\Delta;\nu,b,a,x,y)\\&=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(\gamma^*)}\\&\times D(\Delta;b,a,x,y)^{-1/2}\big(F(\gamma^*)+\nu^{-1}(b-a)r(\Delta;\nu,b,a,x,y)\big).\end{aligned}$$

[4] A sharper result is given in [10].

The following estimate for $r(\Delta;\nu,b,a,x,y)$ holds: For any integer $K \geq 0$ there exist $M(K) \geq 0$ and a constant $C_K > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta r(\Delta;\nu,b,a,x,y)| \leq C_K A_{M(K)} \tag{9.7}$$

if $|\alpha|, |\beta| \leq K$. Both $M(K)$ and C_K may depend on K but are independent of Δ and of J.

Theorem 9.6. *As a particular case, we have*

$$\begin{aligned} &I[1](\Delta;\nu,b,a,x,y) \\ &= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} \\ &\times D(\Delta;b,a,x,y)^{-1/2}\left(1+\nu^{-1}(b-a)^2 r(\Delta;\nu,b,a,x,y)\right). \end{aligned}$$

Here $r(\Delta;\nu,b,a,x,y)$ satisfies the same estimate as (9.7) with $A_K = 1$.

Corollary 9.7. *Under the same assumption as in Theorem9.5, we can have*

$$\begin{aligned} &I[F_\Delta](\Delta;\nu,b,a,x,y) \\ &= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} D(\Delta;b,a,x,y)^{-1/2} g(\Delta;\nu,b,a,x,y). \end{aligned}$$

Here $g(\Delta;\nu,b,a,x,y)$ is a function with the following property: For any integer $m \geq 0$ there exists $M(m)$ and C_m independent of Δ,J such that if $\alpha,\beta \leq m$ then

$$|\partial_x^\alpha \partial_y^\beta g(\Delta;\nu,b,a,x,y)| \leq C_K A_{M(K)}. \tag{9.8}$$

The right hand side of (9.8) remains bounded if $|\Delta| \to 0$. Hence this corollary improves Kumano-go& Taniguchi theorem.

Now we prove Theorem 9.5. In order to get $I[F_\Delta](\Delta;\nu,b,a,x,y)$ we successively perform integration by x_1, x_2, x_3 $\dots$,x_J on the right hand side of (7.2). At each step we apply stationary phase method. In doing so, we use a small trick in treating remainder terms, which is explained below.

First we treat integration by x_1. The part of the right hand side of (7.2) which is related to x_1 is

$$I_1 = \left(\frac{\nu}{2\pi i\tau_2}\right)^{1/2}\left(\frac{\nu}{2\pi i\tau_1}\right)^{1/2}\int_{\mathbf{R}} F_\Delta(x_{J+1},x_J,\dots,x_2,x_1,x_0)e^{i\nu(S_{2,1}(x_2,x_1)+S_{1,0}(x_1,x_0))}\,dx_1. \tag{9.9}$$

As we did in §6, we regard $\nu(\tau_1^{-1}+\tau_2^{-1})$ as a large parameter and apply stationary phase method to this integral. Then

$$\begin{aligned} I_1 = &\left(\frac{\nu}{2\pi i(\tau_1+\tau_2)}\right)^{1/2} e^{i\nu S_{2,0}(x_2,x_0)} \\ &\left(P_1[F](x_{J+1},x_J,\dots,x_2,x_0)+R_1[F](x_{J+1},x_J,\dots,x_2,x_0)\right). \end{aligned} \tag{9.10}$$

Here $P_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is the main term and $R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is the remainder. Let Δ_2 be the division of $[a,b]$ such that

$$\Delta_2 : a = T_0 < T_2 < T_3 < \cdots < T_J < T_{J+1} = b. \tag{9.11}$$

Then the main term can be expressed as

$$P_1[F](x_{J+1},x_J,\dots,x_2,x_0) = F_{\Delta_2}(x_{J+1},x_J,\dots,x_2,x_0)D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1/2}.$$

As a result of (6.4) and (6.5) in Lemma 6.1, the remainder $R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ can be written

$$\begin{aligned}(9.12)\qquad R_1[F](x_{J+1},x_J,\dots,x_2,x_0) &= D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1/2}\\ &\times\left(\frac{\tau_1\tau_2}{2\nu(\tau_1+\tau_2)}(D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1}\partial_{x_1}^2 F_\Delta(x_{J+1},x_J,\dots,x_2,x_1^*,x_0)))\right.\\ &\left.+\frac{(\tau_1\tau_2)}{\nu}b(\nu,x_{J+1},x_J,\dots,x_2,x_0)\right).\end{aligned}$$

$R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is a complicated function with respect to x_2 but is relatively simple with respect to variables $(x_{J+1},x_J,\dots,x_3,x_0)$. In fact, we have the following fact. For any $m\ge 0$ there exist positive constant C_m and positive integer $M(m)$ such that as long as $|\alpha_0|,|\alpha_2|\le m$, we have for any $\beta_{J+1},\beta_J,\dots\beta_3$

$$\begin{aligned}(9.13)\qquad &|\partial_{x_{J+1}}^{\beta_{J+1}}\partial_{x_J}^{\beta_J}\dots\partial_{x_3}^{\beta_3}\partial_{x_2}^{\alpha_2}\partial_{x_0}^{\alpha_0}b(\nu,x_{J+1},x_J,\dots,x_2,x_0)|\\ &\le C_m \max_{|\gamma|\le M(m),\alpha_0'\le\alpha_0,\alpha_2'\le\alpha_2}\ \sup_{x_1\in\mathbf{R}}|\partial_{x_{J+1}}^{\beta_{J+1}}\partial_{x_J}^{\beta_J}\dots\partial_{x_3}^{\beta_3}\partial_{x_2}^{\alpha_2'}\partial_{x_0}^{\alpha_0'}\partial_{x_1}^{\gamma}F(\gamma_\Delta)|.\end{aligned}$$

Here we must note that the differential operator with respect to x_j for $j\ge 3$ is the same on both sides of the above (9.13).

Remark. The magnitude of the remainder term (9.12) is small, roughly speaking, of order $O(\nu^{-1}\min\{\tau_1,\tau_2\})$. In particular if $F(\gamma)\equiv 1$ the remainder term is $O(\nu^{-1}\tau_1\tau_2)$.

Next we treat integration with respect to variable x_2. In doing so, we use the following trick. The main term $P_1[F]$ is a relatively simple function of x_2. We integrate it by x_2 and apply stationary phase method.

On the other hand the remainder term $R_1[F]$ is a complicated function with respect to x_2 but is a relatively simple function with respect to x_3. Thus we postpone integration of $R_1[F]$ with respect to x_2 until later and we do integrate it with respect to x_3 beforehand.

We integrate $P_1[F]$ by x_2 and apply stationary phase method. Then we obtain the main term and the remainder:

$$(9.14)\qquad \left(\frac{\nu}{2\pi i\tau_3}\right)^{1/2}\left(\frac{\nu}{2\pi i(\tau_1+\tau_2)}\right)^{1/2}\int_{\mathbf{R}} e^{i\nu(S_{3,2}(x_3,x_2)+S_{2,0}(x_2,x_0))}$$

$$(9.15)\qquad P_1[F](x_{J+1},x_J,\dots,x_3,x_2,x_0)\,dx_2$$

$$\begin{aligned}(9.16)\qquad &=\left(\frac{\nu}{2\pi i(\tau_1+\tau_2+\tau_3)}\right)^{1/2}e^{i\nu S_{3,1}(x_3,x_0)}\\ &\big(P_2P_1[F](x_{J+1},\dots,x_3,x_0)+R_2P_1[F](x_{J+1},\dots,x_3,x_0)\big).\end{aligned}$$

Let Δ_3 be the division

$$\Delta_3 : a = T_0 < T_3 < T_4 < \dots < T_{J+1}.$$

Then the main term is

$$
\begin{aligned}
&P_2P_1[F](x_{J+1},\dots,x_3,x_0)\\
&\qquad = D_{x_2^*}(S_{3,2}+S_{2,0};x_3,x_0)^{-1/2}P_1[F](x_{J+1},\dots,x_3,x_2^*,x_0).\\
&\qquad = D_{x_2^*}(S_{3,2}+S_{2,0};x_3,x_1)^{-1/2}D_{x_1^*}(S_{2,1}+S_{1,0};x_2^*,x_0)^{-1/2}F(\gamma_{\Delta_3}(x_{J+1},\dots,x_3,x_0).
\end{aligned}
$$

Here $x_2^*=\gamma_{\Delta_3}(T_2)$ is the critical point with respect to x_2 for fixed (x_3,x_0). Using (9.3), we have

$$D_{x_2^*}(S_{3,2}+S_{2,0};x_3,x_0)D_{x_1^*}(S_{2,1}+S_{1,0};x_2^*,x_0)=D_{x_2^*x_1^*}(S_{3,2}+S_{2,1}+S_{1,0};x_3,x_0).$$

Therefore,

$$P_2P_1[F](x_{J+1},\dots,x_3,x_0)=D_{x_2^*x_1^*}(S_{3,2}+S_{2,1}+S_{1,0};x_3,x_0)^{-1/2}F_{\Delta_3}(x_{J+1},\dots,x_3,x_0).$$

The remainder term

$$R_2P_1[F](x_{J+1},\dots,x_3,x_0)$$

is a function which is very complicated with respect to x_3 but relatively simple with respect to x_4.

When we treat integration by x_3, we perform integration of the terms $P_2P_1[F]$ and $R_1[F]$. But we postpone integration of $R_2P_1[F]$ by x_3 until later and integrate it with respect x_4 beforehand.

In this manner, we successively perform integration by x_j $(j=1,2,\dots,J)$ in equality (7.2) which define $I[F](\Delta;\nu,b,a,x,y)$. In integrating by x_j we apply stationary phase and get main term and the remainder. We perform integration of the main term by x_{j+1}. But as to the remainder, we skip integration of it by x_{j+1} and perform integration of it by x_{j+2} beforehand.

Repeating this operation, $I[F_\Delta](\Delta;\nu,b,a,x,y)$ is expressed as a sum of many terms.

$$I[F](\Delta;\nu,b,a,x,y)=A_0(\Delta;\nu,b,a,x,y)+\sum{}' A_{j_{s_k},j_{s_{k-1}},\dots,j_{s_1}}. \tag{9.17}$$

Here $A_0(\Delta;\nu,b,a,x,y)$ is the main term through all steps, i.e.

$$A_0(\Delta;\nu,b,a,x,y)=P_JP_{J-1}\dots P_1[F].$$

The sum $\sum'$ is the sum over sequences $\{j_{s_k},j_{s_{k-1}},\dots,j_{s_1}\}$ which is a subsequence of the sequence $\{J,J-1,J-2,\dots,1\}$ and $A_{j_{s_k},j_{s_{k-1}},\dots,j_{s_1}}$ is the term which came from skipping integration with respect to variables $x_{j_{s_k}},x_{j_{s_{k-1}}},\dots,x_{j_{s_1}}$.

By Proposition 9.2, $P_JP_{J-1}\dots P_1[F]$ coincides with the main term of stationary phase method of $I[F_\Delta](\Delta;\nu,b,a,x,y)$ with respect to variables $(x_J,x_{J-1},\dots,x_1)$.That is

$$P_JP_{J-1}\dots P_1[F](\Delta;b,a,x,y)=\prod_{j=1}^{J}D(S_{j+1,j}+S_{j,0})|_{x_j=x_j^*}^{-1/2}F(\gamma^*)=D(\Delta;b,a,x,y)F(\gamma^*).$$

The term $A_{j_{s_k},j_{s_{k-1}},\dots,j_{s_1}}$ is of the following form:

$$
\begin{aligned}
A_{j_{s_\ell},j_{s_{\ell-1}},\dots,j_{s_1}} &= \nu^{-\ell}\prod_{k=1}^{\ell}\Big(\frac{\nu}{2\pi i(T_{j_{s_k}+1}-T_{j_{s_k}})}\Big)^{1/2}\\
&\times\int_{\mathbf{R}^\ell}e^{\nu i S_{j_{s_\ell},j_{s_{\ell-1}},\dots,j_{s_1}}(x_{j_{s_\ell}},x_{j_{s_{\ell-1}}},\dots,x_{j_{s_1}})}a_{j_{s_\ell},j_{s_{\ell-1}},\dots,j_{s_1}}(x_{J+1},x_{j_{s_\ell}},\dots,x_{j_{s_1}},x_0)\prod_{k=1}^{\ell}dx_{j_{s_k}}.
\end{aligned}
$$

Here

$$S_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)=\sum_{k=1}^{\ell}\left(S_{js_{k+1},js_k}(x_{js_{k+1}},x_{js_k})+S_{js_k,js_{k-1}}(x_{js_k},x_{js_{k-1}})\right).$$

And $a_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)$ is a function satisfying the following estimate. For any integer $m\ge 0$ there exists a positive integer $K(m)$ and a positive constant $C(m)$ such that as long as $|\alpha_{js_k}|\le m,(k=1,2,\dots,\ell)$ and $|\alpha_0|\le m,|\alpha_{J+1}|\le m$ we have

$$(9.18)\quad |\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}\prod_{k=1}^{\ell}\partial_{x_{js_k}}^{\alpha_{js_k}}a_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)|$$

$$\le C(m)\left(\prod_{k=1}^{\ell}\tau_{js_k}\right)A_{K(m)}X_{K(m)}^{\ell}.$$

Now we apply Kumano-go & Taniguchi theorem to the right hand side of (9.18). We can prove that

$$A_{js_\ell,js_{\ell-1},\dots,js_1}=\nu^{-\ell}\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}b_{js_\ell,js_{\ell-1},\dots,js_1}(\Delta;\nu,b,a,x,y),$$

and we have the estimate

$$|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}b_{js_\ell,js_{\ell-1},\dots,js_1}(\Delta;\nu,b,a,x,y)|\le C_1(m)^{\ell}C(m)A_{K(m)}X_{K(m)}^{\ell}\prod_{k=1}^{\ell}\tau_{js_k}.$$

From here we have

$$\sum' A_{js_k,js_{k-1},\dots,js_1}=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}c(\Delta;\nu,b,a,x,y),$$

where

$$c(\Delta;\nu,b,a,x,y)=\sum'\nu^{-\ell}b_{js_\ell,js_{\ell-1},\dots,js_1}(\Delta;\nu,b,a,x,y),$$

and we have that

$$\begin{aligned}|\partial_x^{\alpha_{J+1}}\partial_y^{\alpha_0}c(\Delta;\nu,b,a,x,y)|&\le\sum'\nu^{-\ell}C_1(m)^{\ell}C(m)A_{K(m)}X_{K(m)}^{\ell}\prod_{k=1}^{\ell}\tau_{js_k}\\&\le C(m)A_{K(m)}\left[\prod_{j=1}^{J}\left(1+\nu^{-1}C_1(m)X_{K(m)}\tau_j\right)-1\right]\\&\le\nu^{-1}C'(m)A_{K(m)}X_{K(m)}(b-a).\end{aligned}$$

with some constant C'(m) independent of Δ and of J.

Theorem 9.5 is now proved. Similarly we can prove Theorem 9.6.

§ 10. Convergence of Feynman Path Integral.

We shall prove that the limit

$$\lim_{|\Delta|\to 0}I[1](\Delta;\nu,b,a,x,y)$$

exists. cf. [8] and also [5]. Existence of $\lim_{|\Delta|\to 0} I[F](\Delta;\nu,b,a,x,y)$ for more general $F(\gamma)$ is proved in [15]. See also [9].

We begin with

Theorem 10.1. *The limit*

$$D(b,a,x,y) = \lim_{|\Delta|\to 0} D(\Delta,b,a,x,y) \tag{10.1}$$

exists and

$$D(b,a,x,y) = 1 + (b-a)^2 d(b,a,x,y). \tag{10.2}$$

For any $K \geq 0$ there exits constants $C_K > 0$ such that for any α,β with $|\alpha|,|\beta| \leq K$ there holds estimate:

$$|\partial_x^\alpha \partial_y^\beta d(b,a,x,y)| \leq C_K.$$

Remark. As one can see from next Theorem, $D(\Delta,b,a,x,y)$ converges uniformly together with its all derivatives with respect to (x,y).

To prove Theorem 10.1, we have only to prove the following Theorem. cf. [7], [12] or [8].

Theorem 10.2. *Assume $|b-a| \leq \mu_1$. Let Δ be an arbitrary division of $[a,b]$. Let Δ' be an arbitrary refinement of Δ. We define $d(\Delta,\Delta';x,y)$ by the following equality.*

$$\frac{D(\Delta';b,a,x,y)}{D(\Delta;b,a,x,y)} = 1 + |\Delta|(b-a)d(\Delta,\Delta';x,y).$$

Then for any α and β, there exists a positive constant $C_{\alpha,\beta}$ which is independent of Δ,Δ' and of (a,b,x,y) such that

$$|\partial_x^\alpha \partial_y^\beta d(\Delta,\Delta';x,y) \leq C_{\alpha,\beta}. \tag{10.3}$$

Proof. We prove Theorem 10.2 through several steps. Let Δ be

$$\Delta : a = T_0 < T_1 < T_2 < \cdots < T_J < T_{J+1} = b$$

and its refinement Δ' be

$$\begin{aligned}\Delta' : a = T_0 = T_{1,0} < T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} = T_1 = T_{2,0} < T_{2,1} < \dots \\ \cdots < T_{2,p_2} < T_{2,p_2+1} = T_2 = T_{3,0} < \cdots < T_J < T_{J+1,1} < T_{J+1,2} < \dots \\ \cdots < T_{J+1,p_{J+1}} < T_{J+1,p_{J+1}+1} = T_{J+1} = b.\end{aligned}$$

Set $\tau_j = T_j - T_{j-1}$ $\tau_{j,k} = T_{j,k} - T_{j,k-1}$.

The piecewise classical path corresponding to division Δ' is denoted by

$$\gamma_{\Delta'}(x_{J+1}, x_{J+1,p_{J+1}}, \dots, x_J, \dots, x_1, x_{1,p_1}, \dots, x_{1,1}, x_0)(t),$$

which will be abbreviated to $\gamma_{\Delta'}(t)$. The action of $\gamma_{\Delta'}(t)$ is

$$S_{\Delta'}(x_{J+1}, x_{J+1,p_{J+1}}, \dots, x_J, \dots, x_1, x_{1,p_1}, \dots, x_{1,1}, x_0).$$

In the following, we use a special sequence of refinements $\{\Delta^{(k)}\}_{k=0,1,2,\dots,J+1}$ of Δ such that $\Delta^{(0)} = \Delta, \Delta^{(J+1)} = \Delta'$ and $\Delta^{(k)}$ is a refinement of $\Delta^{(k-1)}$.

We define $\Delta^{(1)}$ by

$$\Delta^{(1)} : a = T_0 = T_{1,0} < T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} = T_1 < T_2 < \cdots < T_J < T_{J+1} = b.$$

$\Delta^{(1)}$ is different from Δ only in $[T_0, T_1]$ where $\Delta^{(1)}$ and Δ' has the same division points. We write by $\gamma_{\Delta^{(1)}}(x_{J+1}, x_J, \ldots, x_1, x_{1,p_1}, \ldots, x_{1,1}, x_0)$ the piecewise classical path corresponding to division $\Delta^{(1)}$.

We define $\Delta^{(2)}$ so that $\Delta^{(2)}$ is different from $\Delta^{(1)}$ only in $[T_1, T_2]$ and it has the same division points as Δ' in $[T_1, T_2]$. $\Delta^{(2)}$ is

$$\begin{aligned}\Delta^{(2)} : a = T_0 = T_{1,0} < T_{1,1} < \cdots < T_{1,p_1} < T_{1,p_1+1} = T_1 = T_{2,0} < T_{2,1} < \ldots \\ < \cdots < T_{2,p_2} < T_{2,p_2+1} = T_2 < \cdots < T_J < T_{J+1} = b.\end{aligned} \tag{10.4}$$

Similarly , $\Delta^{(j)}$ is defined for $j = 3, 4, \ldots, J$.

We compare $D(\Delta; b, a, x, y)$ and $D(\Delta^{(1)}; b, a, x, y)$.

Let δ_1 be the division of $[T_0, T_1]$ defined by

$$\delta_1 : a = T_0 = T_{1,0} < T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} = T_1. \tag{10.5}$$

Let $\gamma_{\delta_1}(x_{1,p_1+1}, x_{1,p_s}, \ldots, x_{1,1}, x_{1,0})$ be the piecewise classical path which pass $x_{1,j}$ at time $T_{1,j}$ for $j = 0, 1, \ldots, p_s + 1$. We write its action by

$$S_{\delta_1}(x_{1,p_1+1}, x_{1,p_s}, \ldots, x_{1,1}, x_{1,0}) = \sum_{k=1}^{p_1+1} S(T_{1,k}, T_{1,k-1}, x_{1,k}, x_{1,k-1}). \tag{10.6}$$

The action of $S(\gamma_{\Delta^{(1)}})$ is written

$$\begin{aligned} S(\gamma_{\Delta^{(1)}}) &= S_{\Delta^{(1)}}(x_{J+1}, x_J, \ldots, x_1, x_{1,p_1}, \ldots, x_{1,1}, x_0) \\ &= \left(\sum_{j=2}^{J+1} S(T_j, T_{j-1}, x_j, x_{j-1})\right) + \sum_{k=1}^{p_1+1} S(T_{1,k}, T_{1,k-1}, x_{1,k}, x_{1,k-1}) \\ &= \left(\sum_{j=2}^{J+1} S(T_j, T_{j-1}, x_j, x_{j-1})\right) + S_{\delta_1}(x_{1,p_1+1}, x_{1,p_s}, \ldots, x_{1,1}, x_{1,0}). \end{aligned} \tag{10.7}$$

In calculating $\det(\operatorname{Hess} S_{\Delta^{(1)}})$, we first fix $(x_{J+1}, x_J, \ldots, x_1, x_0)$ and consider critical point $(x^*_{1,p_s}, \ldots, x^*_{1,1})$ with respect to $(x_{1,p_s}, \ldots, x_{1,1})$. Then

$$\begin{aligned} &\det\left(\operatorname{Hess}_{(x^*_{1,p_s}, \ldots, x^*_{1,1})} S_{\Delta^{(1)}}(x_{J+1}, x_J, \ldots, x_1, x^*_{1,p_1}, \ldots, x^*_{1,1}, x_0)\right) \\ &= \det\left(\operatorname{Hess}_{(x^*_{1,p_s}, \ldots, x^*_{1,1})} S_{\delta_1}(x_{1,p_1+1}, x^*_{1,p_s}, \ldots, x^*_{1,1}, x_{1,0}\right) \\ &= \frac{\tau_1}{\prod_{k=1}^{p_1+1} \tau_{1,j}} D(\delta_1; T_1, T_0, x_1, x_0). \end{aligned} \tag{10.8}$$

Since

$$S_{\delta_1}(x_{1,p_1+1}, x^*_{1,p_s}, \ldots, x^*_{1,1}, x_{1,0}) = S(T_1, T_0, x_1, x_0),$$

we know that for fixed $(x_{J+1},\ldots,x_1,x_0)$

$$\begin{aligned}&S_{\Delta^{(1)}}(x_{J+1},x_J,\ldots,x_1,x^*_{1,p_1},\ldots,x^*_{1,1},x_0)\\&=\left(\sum_{j=2}^{J+1}S(T_j,T_{j-1},x_j,x_{j-1})\right)+S(T_1,T_0,x_1,x_0)\\&=S_\Delta(x_{J+1},x_J,\ldots,x_1,x_0).\end{aligned}$$

Therefore using Proposition 9.2, we obtain

$$\begin{aligned}&\det\left(\mathrm{Hess}_{x^*_J,\ldots,x^*_1,x^*_{1,p_s},\ldots,x^*_{1,1}}S_{\Delta^{(1)}}\right)\\&=\det\left(\mathrm{Hess}_{x^*_J,\ldots,x^*_1}S_\Delta\right)\times\det\left(\mathrm{Hess}_{(x^*_{1,p_s},\ldots,x^*_{1,1})}S_{\delta_1}|_{x_1=x^*_1}\right).\end{aligned}$$

It follows from this and Theorem 9.1 applied to δ_1 that

$$\begin{aligned}D(\Delta^{(1)};b,a,x,y)&=D(\Delta;b,a,x,y)D(\delta_1;T_1,T_0,x^*_1,y)\\&=D(\Delta;b,a,x,y)\left(1+\tau_1^2d(\delta_1;T_1,T_0,x,y)\right).\end{aligned}\tag{10.9}$$

For any α,β there exists a positive constant such that

$$|\partial_x^\alpha\partial_y^\beta d(\delta_1;T_1,T_0,x,y)|\le C_{\alpha\beta}.\tag{10.10}$$

Similarly we can prove that

$$\begin{aligned}D(\Delta^{(j)};b,a,x,y)&=D(\Delta^{(j-1)};b,a,x,y)D(\delta_j;T_j,T_{j-1},x^*_j,x^*_{j-1})\\&=D(\Delta^{(j-1)};b,a,x,y)\left(1+\tau_j^2d(\delta_j;T_j,T_{j-1},x,y)\right).\end{aligned}\tag{10.11}$$

For any any α,β there exists a positive constant such that

$$|\partial_x^\alpha\partial_y^\beta d(\delta_j;T_j,T_{j-1},x,y)|\le C_{\alpha\beta}.\tag{10.12}$$

Here δ_j denotes the division of $[T_{j-1},T_j]$

$$\delta_j: T_{j-1}=T_{j,0}<T_{j,1}<\cdots<T_{j,p_j}<T_{j,p_j+1}=T_{j+1}.\tag{10.13}$$

Finally it follows from (10.11) that

$$D(\Delta';b,a,x,y)=D(\Delta;b,a,x,y)\prod_{j=1}^{J+1}\left(1+\tau_j^2d(\delta_j;T_j,T_{j-1},x,y)\right).$$

We define $d(\Delta,\Delta';b,a,x,y)$ by

$$\prod_{j=1}^{J+1}\left(1+\tau_j^2d(\delta_j;T_j,T_{j-1},x,y)\right)=1+|\Delta|(b-a)d(\Delta,\Delta';b,a,x,y).\tag{10.14}$$

Then estimate 10.3 holds. Theorem 10.2 is proved.

□

Next we prove existence of $\lim_{|\Delta|\to0}I[1](\Delta;\nu,b,a,x,y)$. Existence of $\lim_{|\Delta|\to0}I[F](\Delta;\nu,b,a,x,y)$ for more general $F(\gamma)$ is proved in [15]. See also [9].

Theorem 10.3. [5] *The limit*

$$K(\nu,b,a,x,y) = \lim_{|\Delta|\to 0} I[1](\Delta,\nu,b,a,x,y)$$

exists. Moreover $K(\nu,b,a,x,y)$ is of the form.

$$K(\nu,b,a,x,y) = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} D(b,a,x,y)^{-1/2}(1+\nu^{-1}r(\nu,b,a,x,y)).$$

For any α,β there exist a positive constant $C_{\alpha\beta}$ such that we have

$$|\partial_x^\alpha \partial_y^\beta r(\nu,b,a,x,y)| \leq C_{\alpha\beta}.$$

Remark. Moreover $I[1](\Delta,\nu,b,a,x,y)$ converges uniformly together with it all derivatives with respect to (x,y). See the next theorem.

We have only to prove that $I[1](\Delta;\nu,b,a,x,y)$ is a Cauchy sequence with respect to $|\Delta|$.

Theorem 10.4. [6] *Assume that $|b-a| \leq \mu_1$. Let Δ be an arbitrary division of the interval $[a,b]$ and Δ' be its arbitrary refinement. Let S' be an abbreviation of the phase function corresponding to Δ'. Then*

$$\begin{aligned} &I[1](\Delta';\nu,b,a,x,y) - I[1](\Delta;\nu,b,a,x,y) \\ &= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} D(\Delta;b,a,x,y)^{-1/2} q(\Delta,\Delta',\nu,b,a,x,y) e^{i\nu S(b,a,x,y)}. \end{aligned}$$

Moreover, for arbitrary α,β there exists positive constant $C_{\alpha,\beta}$ such that there holds the estimate

$$|\partial_x^\alpha \partial_y^\beta q(\Delta,\Delta';\nu,b,a,x,y)| \leq C_{\alpha,\beta}|\Delta|(b-a). \tag{10.15}$$

Proof. The proof is along the same line as the proof of Theorem 10.2. We use the notations $\Delta,\Delta',\Delta^{(1)},\delta_1,\gamma_{\Delta^{(1)}}$ etc. given in the proof of Theorem 10.2.

First we compare $I[1](\Delta^{(1)};\nu,b,a,x,y)$ with $I[1](\Delta;\nu,b,a,x,y)$. Using (10.7), we have

$$\begin{aligned} &I[1](\Delta^{(1)};\nu,b,a,x,y) \\ &= \prod_{j=2}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2} \prod_{k=1}^{p_1+1}\left(\frac{\nu}{2\pi i\tau_{1,k}}\right)^{1/2} \int_{\mathbf{R}^J} \exp(i\nu \sum_{j=2}^{J+1} S(T_j,T_{j-1},x_j,x_{j-1})) \\ &\times \left[\int_{\mathbf{R}^{p_1}} \exp(i\nu \sum_{k=1}^{p_1+1} S(T_{1,k},T_{1,k-1},x_{1,k},x_{1,k-1})) \prod_{k=1}^{p_1} dx_{1,k}\right] \prod_{j=1}^{J} dx_j. \end{aligned} \tag{10.16}$$

Let $x_{1,k}^* = \gamma_\Delta(T_{1,k})$ for $1 \leq k \leq p_1$. Then it is the critical point with respect to $(x_{1,p_1},\ldots,x_{1,1})$ of action $S(\gamma_{\Delta^{(1)}}) = S_{\Delta^{(1)}}(x_{J+1},x_J,\ldots,x_1,x_{1,p_1},\ldots,x_{1,1},x_0)$.

We fix $(x_J,\ldots,x_1)$ and integrate with respect to $(x_{1,p_1},\ldots,x_{1,1})$ in (10.16) and we apply The-

[5] For more information see [8] and [11].
[6] cf.[7] and [12]

orem 9.5 in which $[a,b]$ and Δ are replaced by $[T_0,T_1]$ and δ_1, respectively. Then we have

$$\begin{aligned}
&I[1](\Delta^{(1)};\nu,b,a,x,y)\\
&=\prod_{j=1}^{J+1}(\frac{\nu}{2\pi i\tau_j})^{1/2}\int_{\mathbf{R}^J}F_{\Delta^{(1)}/\Delta}(x_{J+1},\ldots,x_0)\exp(i\nu S_\Delta(x_{J+1},x_j,\ldots,x_1,x_0))\prod_{j=1}^{J}dx_j\\
&=I[F_{\Delta^{(1)}/\Delta}](\Delta;\nu,b,a,x,y),
\end{aligned}$$

with

$$F_{\Delta^{(1)}/\Delta}(\nu,x_{J+1},\ldots,x_0)=D(\delta_1;T_1,T_0,x_1,y)^{-1/2}\left(1+\frac{\tau_1^2}{\nu}r_{\Delta^{(1)}/\Delta}(\nu,T_1,T_0,x_1,y))\right).$$

Here $D(\delta_1;T_1,T_0,x_1,y)$ is given by (9.1) and used in (10.9). So we know that it is of the following form :

$$D(\delta_1;T_1,T_0,x_1,y)=1+\tau_1^2 d(\delta_1;T_1,T_0,x_1,x_0). \tag{10.17}$$

This means that we have

$$F_{\Delta^{(1)}/\Delta}(\nu,x_{J+1},x_J,\ldots,x_1,x_0)=1+\tau_1^2 f_{\Delta^{(1)}/\Delta}(\nu,T_1,T_0,x_1,x_0).$$

And we have the estimate for $f_{\Delta^{(1)}/\Delta}(\nu,T_1,T_0,x_1,x_0)$: For any α,β there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}f_{\Delta^{(1)}/\Delta}(\nu,T_1,T_0,x_1,x_0)|\le C_{\alpha,\beta}.$$

Now we can write

$$\begin{aligned}
&I[1](\Delta^{(1)};\nu,b,a,x,y)-I[1](\Delta;\nu,b,a,x,y)\\
&=I[F_{\Delta^{(1)}/\Delta}-1](\Delta;\nu,b,a,x,y)=\tau_1^2 I[f_{\Delta^{(1)}/\Delta}](\Delta;\nu,b,a,x,y).
\end{aligned}\tag{10.18}$$

Now we can apply the stationary phase method Theorem 9.5 to the right hand side of above equation and obtain

$$\tau_1^2 I[f_{\Delta^{(1)}/\Delta}](\Delta,\nu,b,a,x,y)=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta^{(1)},\Delta;\nu,b,a,x,y).$$

Here $q(\Delta^{(1)},\Delta;\nu,b,a,x,y)$ has the following property: For any α,β there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}q(\Delta^{(1)},\Delta;\nu,b,a,x,y)|\le C_{\alpha,\beta}\tau_1^2. \tag{10.19}$$

This means that

$$\begin{aligned}
&I[1](\Delta^{(1)};\nu,b,a,x,y)-I[1](\Delta;\nu,b,a,x,y)\\
&=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta^{(1)},\Delta;\nu,b,a,x,y).
\end{aligned}\tag{10.20}$$

Similar discussion as above gives for $k=2,3,\ldots,J+1$

$$\begin{aligned}
&I[1](\Delta^{(k)};\nu,b,a,x,y)-I[1](\Delta^{(k-1)};\nu,b,a,x,y)\\
&=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y).
\end{aligned}\tag{10.21}$$

For any α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$|\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y)|\le C_{\alpha\beta}\tau_k^2.$$

Consequently, we have

$$\begin{aligned}(10.22)\qquad & I[1](\Delta';\nu,b,a,x,y)-I[1](\Delta;\nu,b,a,x,y)\\ &=\sum_{k=1}^{J+1}\left(I[1](\Delta^{(k)};\nu,b,a,x,y)-I[1](\Delta^{(k-1)};\nu,b,a,x,y)\right)\\ &=\sum_{k=1}^{J+1}\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y)\\ &=\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2}e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta,\Delta';\nu,b,a,x,y).\end{aligned}$$

Where

$$q(\Delta,\Delta';\nu,b,a,x,y)=\sum_{k=1}^{J+1}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y).$$

For any α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$(10.23)\qquad |\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}q(\Delta,\Delta';\nu,b,a,x,y)|\le C_{\alpha\beta}\sum_{k=1}^{J+1}\tau_k^2.$$

This proves Theorem 10.4. □

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