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On the Number of Iterations of Dantzig's Simplex Method*

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1 Introduction

We analyze the primal simplex method with the most negative coefficient pivoting rule (Dantzig's rule) under the condition that the primal problem has an optimal solution. We give an upper bound for the number of different basic feasible solutions (BFSs) generated by the simplex method. The bound is

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil,$$

where m is the number of constraints, n is the number of variables, δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs, respectively, and $\lceil a\rceil$ is the smallest integer bigger than $a\in\Re$. When the primal problem is nondegenerate, it becomes a bound for the number of iterations. Note that the bound depends only on the constraints of LP, but not the objective function.

Our work is motivated by a recent research by Ye [4]. He shows that the simplex method is strongly polynomial for the Markov Decision Problem. We apply the analysis in [4] to general LPs and obtain the upper bound. Our results include his strong polynomiality.

When we apply our result to an LP where a constraint matrix is totally unimodular and a constant vector b of constraints is integral, the number of different solutions generated by the simplex method is at most

$$n\lceil m\|b\|_1\log(m\|b\|_1)\rceil.$$

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2 The Simplex Method for LP

In this paper, we consider the linear programming problem of the standard form

$$\begin{array}{ll} \min & c^T x, \\ \text{subject to} & Ax = b, \ x \geq 0, \end{array} \tag{1}$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$ and $c \in \Re^n$ are given data, and $x \in \Re^n$ is a variable vector. The dual problem of (1) is

$$\begin{array}{ll} \max & b^T y, \\ \text{subject to} & A^T y + s = c, \ s \geq 0, \end{array} \tag{2}$$

where $y \in \Re^m$ and $s \in \Re^n$ are variable vectors.

We assume that $\text{rank}(A) = m$, the primal problem (1) has an optimal solution and an initial BFS x^0 is available. Let x^* be an optimal basic feasible solution of (1), (y^*, s^*) be an optimal solution of (2), and z^* be the optimal value of (1) and (2).

Given a set of indices $B \subset \{1, 2, \ldots, n\}$, we split the constraint matrix A, the objective vector c, and the variable vector x according to B and $N = \{1, 2, \ldots, n\} - B$ like

$$A = [A_B, A_N], \ c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \ x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}.$$

Define the set of bases

$$\mathcal{B} = \{B \subset \{1, 2, \ldots, n\} \mid |B| = m, \ \det(A_B) \neq 0\}.$$

Then a primal basic feasible solution for $B \in \mathcal{B}$ and $N = \{1, 2, \ldots, n\} - B$ is written as

$$x_B = A_B^{-1} b \geq 0, \ x_N = 0.$$

Let δ and γ be the minimum and the maximum values of all the positive elements of BFSs. Hence for any BFS $\hat{x}$ and any $j \in \{1, 2, \ldots, n\}$, if $\hat{x}_j \neq 0$, we have

$$\delta \leq \hat{x}_j \leq \gamma. \tag{3}$$

Note that these values depend only on A and b, but not on c.

Let $B^t \in \mathcal{B}$ be the basis of the t-th iteration of the simplex method and set $N^t = \{1, 2, \ldots, n\} - B^t$. Problem (1) can be written in the dictionary form:

$$\begin{array}{ll} \min & c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}, \\ \text{subject to} & x_{B^t} = A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t}, \\ & x_{B^t} \geq 0, \ x_{N^t} \geq 0. \end{array} \tag{4}$$

Table 1: Notations

x^*	:	an optimal basic feasible solution of (1)
(y^*, s^*)	:	an optimal solution of (2)
z^*	:	the optimal value of (1)
x^t	:	the t-th iterate of the simplex method
B^t	:	the basis of x^t
N^t	:	the nonbasis of x^t
$\bar{c}_{N^t}$	:	the reduced cost vector at t-th iteration
Δ^t	:	$-\min_{j \in N_t} \bar{c}_j$

The coefficient vector $\bar{c}_{N^t} = c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t}$ is called a reduced cost vector. When $\bar{c}_{N^t} \geq 0$, the current solution is optimal. Otherwise we conduct a pivot. That is, we choose one nonbasic variable (entering variable) and increase the variable until one basic variable (leaving variable) becomes zero. Then we exchange the two variables. Under the most negative rule, we choose an entering variable whose reduced cost is minimum. To put it precisely, we choose an index

$$j_{MN}^t = \arg \min_{j \in N_t} \bar{c}_j.$$

We set $\Delta^t = -\bar{c}_{j_{MN}^t}$.

We summarize the notations in Table 1.

3 Geometric Interpretation

If we write LP in the dictionary form (4), it can be seen as a problem in x_{N^t} space. The current point x^t corresponds to the origin (Figure 1). Assume that x^t is not optimal. Then at least one component of the reduced cost $\bar{c}_{N^t}$ is negative. From the definition of δ and γ, all the BFSs other than x^t are contained in the square with length γ minus the square with length δ. Under the most negative rule, a nonbasic variable whose coefficient of the reduced cost is minimum is chosen as the entering variable. In Figure 1, x_{N_2} is chosen as the entering variable. Then the iterate moves on the x_{N_2} axis. The objective function decreases $\Delta^t = \bar{c}_{N_2}$ per unit distance. In this case the iterate moves at least δ, thus the objective function decreases at least $\delta \Delta^t$. Then we get the following inequality.

$$c^T x^{t+1} \leq c^T x^t - \delta \Delta^t \tag{5}$$

We go on with Figure 1. All the BFSs are contained in the set $S = \{x_{N^t} | x_{N^t} \geq 0,\ e^T x_{N^t} \leq m\gamma\}$. Then the intercept of each axis is $m\gamma$. If we consider the contour at the intercept of the axis with the minimum coefficient (in this case x_{N_2}), the contour passes outside S. In ohter words,

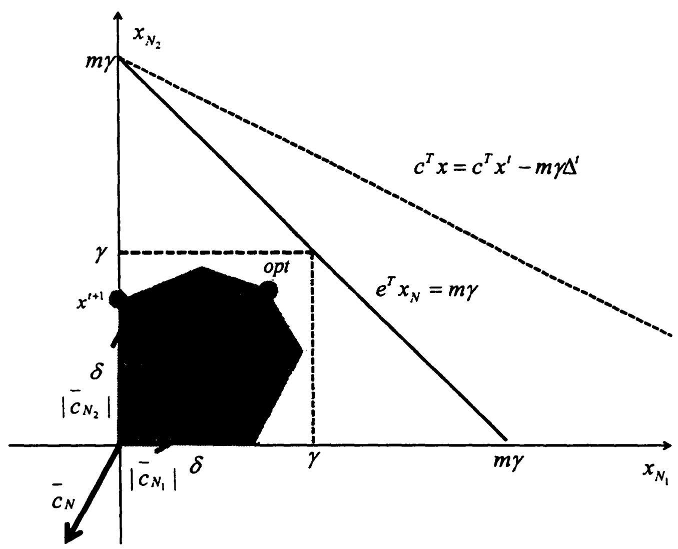

Figure 1: The feasible region seen from a nonoptimal BFS

$x \in S \Rightarrow c^T x \geq c^T x^t - m\gamma\Delta^t$. If we take an optimal BFS as x, we get the following lemma.

Lemma 3.1 *[2] Let z^* be the optimal value of Problem (1) and x^t be the t-th iterate generated by the simplex method with the most negative rule. Then we have*

$$z^* \geq c^T x^t - \Delta^t m\gamma. \tag{6}$$

By combining (5) and Lemma 3.1, we obtain the following theorem.

Theorem 3.1 *[2] Let x^t and x^{t+1} be the t-th and $(t+1)$-th iterates generated by the simplex method with the most negative rule. If $x^{t+1} \neq x^t$, then we have*

$$c^T x^{t+1} - z^* \leq (1 - \frac{\delta}{m\gamma})(c^T x^t - z^*). \tag{7}$$

Next we consider a dictionary at an optimal BFS (Figure 2).

$$\begin{array}{ll} \min & c_{B^*}^T A_{B^*}^{-1} b + \bar{c}_{N^*}^T x_{N^*}, \\ \text{s.t} & x_{B^*} = A_{B^*}^{-1} b - A_{B^*}^{-1} A_{N^*} x_{N^*} \geq 0, \\ & x_{N^*} \geq 0 \end{array}$$

In this case, all the components of the reduced cost $\bar{c}_{N^*}$ is nonnegative. The gap between the objective value of the current iterate x^t and z^* is $c^T x^t - z^* = \bar{c}_{N^*}^T x_{N^*}^t$ and we draw the contour at x^t. Then there exists an index $j \in N^*$ whose intercept is less than or equal to $m x_j^t$ (in Figure 2, N_1^* is such an index). The contour of $\bar{c}_{N^*}^T x_{N^*}$ is derived by reducing the contour at x^t by $\frac{c^T x - z^*}{c^T x^t - z^*}$ times, and the intercept reduces at the same rate. Thus we obtain the following lemma.

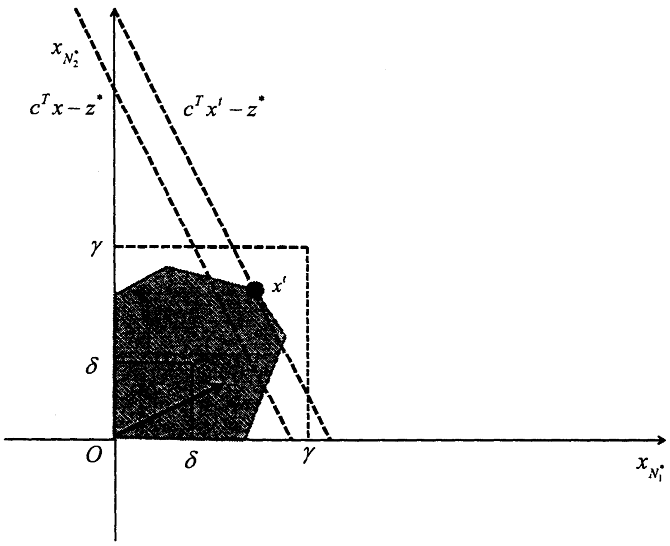

Figure 2: Feasible region seen from an optimal BFS

Lemma 3.2 *[2] Let x^t be the t-th iterate generated by the simplex method. If x^t is not optimal, there exists $\bar{j} \in B^t$ such that $x^t_{\bar{j}} > 0$ and for any k, the k-th iterate x^k satisfies*

$$x^k_{\bar{j}} \leq \frac{m(c^T x^k - z^*)}{c^T x^t - z^*} x^t_{\bar{j}}.$$

Figure 3 is useful to show how we obtain the bound for the number of different basic solutions. If the primal problem (1) is nondegenerate, we

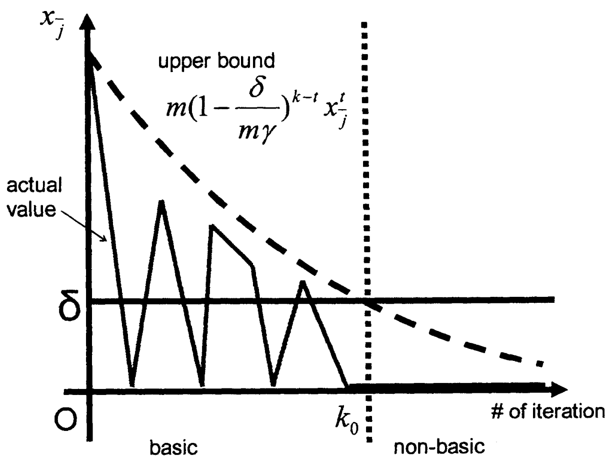

Figure 3: Changes in $x_{\bar{j}}$

have $x^{t+1} \neq x^t$ for all t. Then by theorem 3.1, the upper bound of $x_{\bar{j}}$

implied in Lemma 3.2 becomes $m(1-\frac{\delta}{m\gamma})^{k-t}x^t_{\bar{j}}$. If k is bigger than k_0, the upper bound is less than δ, meaning $x^k_{\bar{j}}$ is zero from the definition of δ. More generally, we have the following result.

Lemma 3.3 *[2] Let x^t be the t-th iterate generated by the simplex method with the most negative rule. Assume that x^t is not an optimal solution. Then there exists $\bar{j} \in B^t$ satisfying the following two conditions.*

1. $x^t_{\bar{j}} > 0$.
2. *If the simplex method generates $\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$ different basic feasible solutions after t-th iterate, then $x_{\bar{j}}$ becomes zero and stays zero.*

The event described in Lemma 3.3 can occur at most once for each variable. Thus we get the following result.

Theorem 3.2 *[2] When we apply the simplex method with the most negative rule for LP (1) having optimal solutions, we encounter at most $n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$ different basic feasible solutions.*

Note that the result is valid even if the simplex method fails to find an optimal solution because of a cycling.

If the primal problem is nondegenerate, we have $x^{t+1} \neq x^t$ for all t. This observation leads to a bound for the number of iterations of the simplex method.

Corollary 3.1 *[2] If the primal problem is nondegenerate, the simplex method finds an optimal solution in at most $n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$ iterations.*

4 Applications to Special LPs

We have the following result for an LP whose constraint matrix A is totally unimodular and all the elements of b are integers. Recall that the matrix A is totally unimodular if the determinant of every nonsingular square submatrix of A is 1 or -1.

Corollary 4.1 *[2] Assume that the constraint matrix A of (1) is totally unimodular and the constraint vector b is integral. When we apply the simplex method with the most negative rule for (1), we encounter at most $n\lceil m\|b\|_1\log(m\|b\|_1)\rceil$ different basic feasible solutions.*

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