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SEMICLASSICAL LIMIT OF SCHRÖDINGER-POISSON SYSTEM AND CLASSICAL LIMIT OF QUANTUM EULER-POISSON EQUATIONS IN 2D

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ABSTRACT. We consider the semiclassical limit of Schrödinger-Poisson system in two dimensions and show a WKB approximation of a solution via an analysis of a hydrodynamical system. The classical limit of quantum Euler-Poisson equations is discussed at the same time.

1. INTRODUCTION

In this article we consider the Cauchy problem of the following Schrödinger-Poisson system

$$\text{(SP)} \qquad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}}^\varepsilon u^\varepsilon, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |u^\varepsilon|^2, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \end{cases}$$

where u^ε is a $\mathbb{C}$-valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^2$, $\varepsilon > 0$ is a parameter, and $\lambda \in \mathbb{R}$ is a given constant. We also treat the following quantum Euler-Poisson equations

$$\text{(qEP)} \qquad \begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \lambda\rho^\varepsilon\nabla V_{\mathrm{P}}^\varepsilon = \varepsilon^2\rho^\varepsilon\nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right), \\ -\Delta V_{\mathrm{P}}^\varepsilon = \rho^\varepsilon, \\ \rho^\varepsilon(0,x) = \rho_0^\varepsilon(x), \quad v^\varepsilon(0,x) = v_0^\varepsilon(x), \end{cases}$$

where ρ^ε is a nonnegative function of $(t,x) \in \mathbb{R}_+\times\mathbb{R}^2$ and v^ε is an $\mathbb{R}^2$-valued function of (t,x). Our goal is to determine an asymptotic behavior of u^ε and of $(\rho^\varepsilon, v^\varepsilon)$ in the limit $\varepsilon \downarrow 0$.

In the above systems, the Poisson equation is posed with the following boundary condition:

$$\nabla V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \qquad \nabla V_{\mathrm{P}}^\varepsilon \in L^\infty(\mathbb{R}^2), \qquad V_{\mathrm{P}}^\varepsilon(0) = 0.$$

Under this condition a solution of $-\Delta V_{\mathrm{P}}^\varepsilon = f$ is given by a formula

$$V_{\mathrm{P}}^\varepsilon(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|x-y|}{|y|} f(y)\,dy.$$

For details, see Appendix A or [15, Section 2].

1.1. Madelung's transform and modified Madelung's transform. Madelung pointed out the relation between (nonlinear) Schrödinger equation and quantum Euler equation. Let u^ε be a solution to

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = N(|u^\varepsilon|)u^\varepsilon,$$

where N denotes a real-valued nonlinearity. Typical examples of N are $N(|u|) = |u|^\alpha$ ($\alpha > 0$) and $N(|u|) = (|x|^{-\gamma} * |u|^2)$. We now set

$$\rho^\varepsilon := |u^\varepsilon|^2, \qquad v^\varepsilon := \varepsilon\nabla \arg u^\varepsilon = \varepsilon \operatorname{Im}\left(\frac{\nabla u^\varepsilon}{u^\varepsilon}\right). \tag{1.1}$$

Then, the pair $(\rho^\varepsilon, v^\varepsilon)$ solves, at least formally,

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \rho^\varepsilon \nabla N(\sqrt{\rho^\varepsilon}) = \varepsilon^2 \rho^\varepsilon \nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right). \end{cases}$$

It can been seen, for example, by plugging $u^\varepsilon = \sqrt{\rho^\varepsilon} e^{i \arg u^\varepsilon}$ with the above Schrödinger equation. This transform is called Madelung's transform.

On the other hand, in [11] Grenier introduce a modified Madelung's transform

$$u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}, \tag{1.2}$$

where a^ε is complex-valued and ϕ^ε is real-valued. Plugging (1.2) with the Schrödinger equation, we obtain

$$\begin{aligned} -a^\varepsilon &\left(\partial_t \phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + N(|a^\varepsilon|)\right) \\ &+ i\varepsilon\left(\partial_t a^\varepsilon + \nabla\phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2}a^\varepsilon \Delta\phi^\varepsilon\right) + \frac{\varepsilon^2}{2}\Delta a^\varepsilon = 0. \end{aligned} \tag{1.3}$$

According to the order of ε, we assume that $(a^\varepsilon, \phi^\varepsilon)$ solves

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon \Delta\phi^\varepsilon = \dfrac{i}{2}\varepsilon\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + N(|a^\varepsilon|) = 0. \end{cases} \tag{1.4}$$

Of course, there are several ways to split (1.3) into a system for $(a^\varepsilon, \phi^\varepsilon)$. What is important here is that a^ε is supposed to be complex-valued. If a^ε is real-valued then this formulation is impossible and one has to separate (1.3) into its real part and imaginary part, in which case the resulting system is nothing but the one given by Madelung's transform (1.1) since $a^\varepsilon = \sqrt{\rho^\varepsilon}$ and $\nabla\phi^\varepsilon = \varepsilon\nabla \arg u^\varepsilon$.

The system (1.4) can be regarded as a symmetric hyperbolic system (more precisely, a system for $(\operatorname{Re} a^\varepsilon, \operatorname{Im} a^\varepsilon, \nabla\phi^\varepsilon)$ is a symmetric hyperbolic system). An analysis of (1.4) gives us a rigorous justification of *WKB approximation*

$$u^\varepsilon \sim e^{i\frac{\phi(t,x)}{\varepsilon}}(a_0(t,x) + \varepsilon a_1(t,x) + \cdots + \varepsilon^n a_n(t,x) + \cdots)$$

of solutions to nonlinear Schrödinger equations. This method is introduced by Grenier [11] with a class of nonlinearity which is essentially cubic type nonlinearity $N(|u|^2)u$ (however, the first mathematical justification of WKB approximation is given by Gerard [10] with another technique). This method is extended to several types of nonlinear Schrödinger equations in [2, 3, 13] (see also [5, 7, 9]). For the WKB approximation of the Schrödinger-Poisson system for other dimensions, we refer the reader to [14] (one dimension) and [1, 6, 16] (three dimensions and higher). In [12, 19, 20], the same limit is treated in one and two dimensions by Wigner measures.

1.2. **Results.** In this article we introduce some results in [15] concerned with (SP) and some consequent results on (qEP). We first define the Zhidkov $X^s(\mathbb{R}^2)$ by

$$X^s(\mathbb{R}^2) := \{f \in L^\infty(\mathbb{R}^2); \nabla f \in H^{s-1}(\mathbb{R}^2)\},$$
$$\|f\|_{X^s(\mathbb{R}^2)} := \|f\|_{L^\infty(\mathbb{R}^2)} + \|\nabla f\|_{H^{s-1}(\mathbb{R}^2)}$$

for $s > 1$. This space is introduced in [21] (see also [8]). We sometimes write $L^p = L^p(\mathbb{R}^2)$, $H^s = H^s(\mathbb{R}^2)$, and $X^s = X^s(\mathbb{R}^2)$, for short. We say $f \in L^{p+}$ if there exists $\delta_0 > 0$ such that $f \in L^{p+\delta}$ holds for all $\delta \in (0, \delta_0]$.

Theorem 1.1 (Local unique existence, [15]). *Suppose $s > 2$. Let $u_0^\varepsilon(x) = A^\varepsilon(x)e^{i\Phi(x)/\varepsilon}$ with $A^\varepsilon \in H^s(\mathbb{R}^2)$, $\Phi \in C^4(\mathbb{R}^2)$, and $\nabla\Phi \in X^{s+1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $p \in (2, \infty)$. Suppose that A^ε is uniformly bounded in H^s with respect to ε. Then, there exist an existence time T independent of ε and a unique solution $u^\varepsilon \in C([0,T]; H^s(\mathbb{R}^2)) \cap C^1((0,T); H^{s-2}_{\mathrm{loc}}(\mathbb{R}^2))$ of* (SP). *The solution depends continuously on the data and conserves $\|u\|_{L^2}$.*

Remark 1.2. In [18, 17], global well-posedness of (SP) is proven in the space $\{u \in H^1(\mathbb{R}^d) | \sqrt{\log\langle x\rangle} u \in L^2(\mathbb{R}^d)\}$ for any $d \geqslant 1$ in which class an energy

$$E[u^\varepsilon(t)] = \frac{1}{2}\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} - \frac{\lambda}{4\pi}\iint_{\mathbb{R}^{2+2}} \log|x-y||u^\varepsilon(y)|^2|u^\varepsilon(x)|^2 dxdy$$

makes sense. The solution given in Theorem 1.1 does not necessarily have a finite energy.

Theorem 1.3 (WKB approximation, [15]). *In addition to the assumption of Theorem 1.1, we suppose $s > 3 + 4N$ for some $N \geqslant 1$ and $A^\varepsilon = A_0 + \varepsilon A_1 + \cdots + \varepsilon^N A_{2N} + o(\varepsilon^{2N})$ in H^s. We also suppose that there exists $\alpha > 0$ such that $|x|^\alpha A_j \in L^2$ for $j \leqslant N$, then there exist ϕ_0 and β_j such that the solution u^ε of* (SP) *given in Theorem 1.1 satisfies*

$$u^\varepsilon(t,x) = e^{i\frac{\phi_0(t,x)}{\varepsilon}}(\beta_0(t,x) + \varepsilon\beta_1(t,x) + \cdots + \varepsilon^{N-1}\beta_{N-1}(t,x) + o(\varepsilon^{N-1}))$$

in $L^\infty([0,T]; L^2)$.

Now we turn to results on (qEP).

Theorem 1.4 (Local existence). *Let $s > 3$. Let $\rho_0 \geqslant 0$ satisfy $\sqrt{\rho_0} \in H^s$. Let v_0 be an $\mathbb{R}^2$-valued function belonging to $L^p \cap X^{s+1}$ for some $p \in (2, \infty)$, and satisfying $\mathrm{rot} v_0 = 0$. Then, there exist an existence time T which is independent of ε, a nonnegative $\rho^\varepsilon \in C([0,T]; W^{s,1}) \cap C^1((0,T); W^{s-2,1})$, and an $\mathbb{R}^2$-valued $J^\varepsilon \in C([0,T]; W^{s,1}) \cap C^1((0,T); W^{s-3,1})$ with the following properties:*

- $(\rho^\varepsilon(0), J^\varepsilon(0)) = (\rho_0, \rho_0 v_0)$.
- *$(\rho^\varepsilon, J^\varepsilon)$ depends continuously on the data.*
- *$\partial_t\rho^\varepsilon + \mathrm{div} J^\varepsilon = 0$ holds for $t \in (0,T)$.*

- *Set* $\Omega^\varepsilon := \{(t,x) \in (0,T)\times\mathbb{R}^2 \mid \rho^\varepsilon(t,x) \neq 0\}$. *Then,* $v^\varepsilon = J^\varepsilon/\rho^\varepsilon$ *is an* $\mathbb{R}^2$ *valued function defined in* Ω^ε *which is continuous in* t *and continuously differentiable in* x. v^ε *satisfies* $\mathrm{rot} v^\varepsilon = 0$ *in* Ω^ε *and*
$$\partial_t(\rho^\varepsilon v^\varepsilon) + \mathrm{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \lambda\rho^\varepsilon\nabla V_{\mathrm{P}}^\varepsilon = \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)$$
in $(C_0^\infty(\Omega^\varepsilon))'$ *sense.*

Theorem 1.5 (Classical limit). *Let the assumption of Theorem 1.4 be fulfilled. Then, as* $\varepsilon \to 0$, *both*
$$\rho^\varepsilon \to \rho \in C([0,T];W^{s,1}), \qquad J^\varepsilon \to J \in C([0,T];W^{s,1})$$
hold in $L^\infty([0,T];W^{s-2,1})$ *sense. Set* $\Omega_t^0 := \{x \in \mathbb{R}^2 \mid \rho(t,x)\neq 0\}$ *for* $t\in(0,T)$ *and set* $\Omega^0 := (0,T)\times\Omega_t^0$. *Then,* $v := J/\rho \in C^1(\Omega^0)$ *belongs to* $(L^p\cap X^{s+1})(\Omega_t^0)$ *for each* $t\in(0,T)$. *One can extend* v *to* $\widetilde{v}\in C([0,T];(L^p\cap X^{s+1})(\mathbb{R}^2))$ *so that* $(\rho,\widetilde{v})$ *becomes a unique solution to the following Euler-Poisson equations:*
$$\text{(EP)}\qquad \begin{cases} \partial_t\rho + \mathrm{div}(\rho\widetilde{v}) = 0,\\ \partial_t\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v} + \lambda\nabla V_{\mathrm{P}} = 0\\ -\Delta V_{\mathrm{P}} = \rho, \qquad \mathrm{rot}\widetilde{v} = 0,\\ \rho(0) = \rho_0, \qquad \widetilde{v}(0) = v_0, \end{cases}$$

Remark 1.6. In Theorem 1.5, $v = J/\rho$ is smoother not only than v^ε (in $\Omega^\varepsilon\cap\Omega^0$) but also than J and ρ (in Ω^0).

Remark 1.7. If $\rho_0 > 0$ then $\Omega^0 = (0,T)\times\mathbb{R}^2$. Namely, $\rho(t,x) > 0$ for all $(t,x)\in[0,T)\times\mathbb{R}^2$.

2. Analysis of a system obtained by (1.2)

In this section, we establish an existence result on the system
$$\text{(2.1)}\qquad \begin{cases} \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon,\\ \partial_t\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0,\\ V_{\mathrm{P}}^\varepsilon(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{|y|}\right)|a^\varepsilon(t,y)|^2dy,\\ a^\varepsilon(0,x) = A^\varepsilon(x), \quad \phi^\varepsilon(0,x) = \Phi(x) \end{cases}$$
which appears when we plug (1.2) with (SP).

Assumption 2.1. *We assume the following for some* $s > 2$:

- $A^\varepsilon \in H^s(\mathbb{R}^2)$ *and* $\|A^\varepsilon\|_{H^s(\mathbb{R}^2)}$ *is uniformly bounded.*
- $\Phi\in C^1([0,T)\times\mathbb{R}^2)$ *with* $\nabla\Phi\in X^{s+1}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ *for some* $p\in(2,\infty)$.

Theorem 2.2 ([15]). *Let Assumption 2.1 be satisfied. Then, there exist* $T>0$ *independent of* ε *and a unique solution*
$$a^\varepsilon \in C([0,T);H^s(\mathbb{R}^2))\cap C^1((0,T);H^{s-2}(\mathbb{R}^2)),$$
$$\phi^\varepsilon\in C^1([0,T)\times\mathbb{R}^2) \text{ with } \nabla\phi^\varepsilon\in C([0,T);X^{s+1}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2))$$

of (2.1). *Moreover,* $\|a^\varepsilon\|_{H^s}$ *and* $\|\nabla\phi^\varepsilon\|_{X^{s+1}\cap L^p}$ *are uniformly bounded, and the data-to-solution mapping* $(A^\varepsilon, \nabla\Phi) \mapsto (a^\varepsilon, \nabla\phi^\varepsilon)$ *is continuous from* $H^s \times (X^{s+1} \cap L^p)$ *to* $C([0,T); H^{s-1} \times (X^s \cap L^p))$. *The mass* $\|a^\varepsilon(t)\|_{L^2(\mathbb{R}^2)}$ *is conserved and it holds that*

$$\limsup_{|x|\to\infty} \frac{|\phi^\varepsilon(t,x) - \Phi(x)|}{\log|x|} \leqslant \frac{t|\lambda|}{2\pi} \|A^\varepsilon\|_{L^2}^2 .$$

Furthermore, if $s > 3$ *and* $A_0 := \lim_{\varepsilon\to 0} A^\varepsilon$ *exists in* $H^s(\mathbb{R}^2)$ *then the following properties hold:*

- $(a_0, \phi_0) := (a^\varepsilon, \phi^\varepsilon)_{\varepsilon=0}$ *exists in the same class and solves*

$$\text{(2.2)} \qquad \begin{cases} \partial_t a_0 + \nabla\phi_0 \cdot \nabla a_0 + \frac{1}{2} a_0 \Delta\phi_0 = 0, \\ \partial_t \phi_0 + \frac{1}{2}|\nabla\phi_0|^2 + \lambda V_{\mathrm{P}} = 0, \\ V_{\mathrm{P}}(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\log\frac{|x-y|}{|y|}\right)|a_0(t,y)|^2 dy, \\ a_0(0,x) = A_0(x), \quad \phi_0(0,x) = \Phi(x) \end{cases}$$

  and $(a^\varepsilon, \nabla\phi^\varepsilon)$ *converges to* $(a_0, \nabla\phi_0)$ *as* $\varepsilon \to 0$ *in the* $C([0,T]; H^{s-2} \times (X^{s-2} \cap L^{2+}))$ *topology.*
- *For any bounded domain* $\Omega \subset \mathbb{R}^2$, *it holds that* $\|\phi^\varepsilon - \phi_0\|_{L^\infty([0,T]\times\Omega)} \to 0$ *as* $\varepsilon \to 0$.

Proof of the Theorem 2.2. We only show the existence of the solution, which is omitted in [15]. For uniqueness part and convergence, consult [15]. Put $w^\varepsilon := \nabla\phi^\varepsilon$ and consider

$$\text{(2.3)} \qquad \begin{cases} \partial_t a^\varepsilon + w^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot w^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \quad a^\varepsilon(0,x) = A_0^\varepsilon(x); \\ \partial_t w^\varepsilon + (w^\varepsilon \cdot \nabla) w^\varepsilon + \lambda \nabla V_{\mathrm{P}}^\varepsilon = 0, \quad w^\varepsilon(0,x) = \nabla\Phi_0(x); \\ V_{\mathrm{P}}^\varepsilon = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|x-y|}{|y|}|a^\varepsilon(y)|^2 dy, \end{cases}$$

Step 1. In this step, we assume $2 > s > 1$. Set the "partial energy"

$$E_{\mathrm{p}}(t) := \|a^\varepsilon\|_{H^s}^2 + \|\nabla w^\varepsilon\|_{H^s}^2 .$$

We estimate $\frac{d}{dt}E_{\mathrm{p}}(t)$. Denote $\Lambda = (1-\Delta)^{1/2}$. Then,

$$\begin{aligned} \frac{d}{dt}\|a^\varepsilon\|_{H^s}^2 &= 2\,\mathrm{Re}\,\langle \Lambda^s \partial_t a^\varepsilon, \Lambda^s a^\varepsilon\rangle \\ &= -2\,\mathrm{Re}\,\langle \Lambda^s (w^\varepsilon\cdot\nabla) a^\varepsilon, \Lambda^s a^\varepsilon\rangle - \mathrm{Re}\,\langle \Lambda^s (a^\varepsilon \nabla\cdot w^\varepsilon), \Lambda^s a^\varepsilon\rangle + 0 \\ &= \int_{\mathbb{R}^2} (\nabla\cdot w^\varepsilon)|\Lambda^s a^\varepsilon|^2 dx - 2\,\mathrm{Re}\,\langle [\Lambda^s, w^\varepsilon\cdot\nabla] a^\varepsilon, \Lambda^s a^\varepsilon\rangle \\ &\quad - \mathrm{Re}\,\langle \Lambda^s (a^\varepsilon \nabla\cdot w^\varepsilon), \Lambda^s a^\varepsilon\rangle \\ &\leqslant C E_{\mathrm{p}}(t)^{\frac{3}{2}} \end{aligned}$$

by the Sobolev embedding $H^s \hookrightarrow L^\infty$ and the commutator estimate

$$\|[\Lambda^s, w^\varepsilon\cdot\nabla]a^\varepsilon\|_{L^2} \leqslant C(\|\nabla w^\varepsilon\|_{L^\infty}\|\nabla a^\varepsilon\|_{H^{s-1}} + \|\nabla w^\varepsilon\|_{W^{s-1,\frac{2}{s-1}}}\|\nabla a^\varepsilon\|_{L^{\frac{2}{2-s}}}).$$

Similarly,

$$\frac{d}{dt}\|\nabla w^\varepsilon\|_{H^s}^2 = -2\langle \Lambda^s\nabla((w^\varepsilon\cdot\nabla)w^\varepsilon),\Lambda^s w^\varepsilon\rangle - \lambda\left\langle \Lambda^s\nabla^2 P,\Lambda^s\nabla w^\varepsilon\right\rangle \leqslant CE_{\mathrm{p}}(t)^{\frac{3}{2}},$$

where we have employed the following estimate on the nonlinearity:

$$\left\|\nabla^2 P\right\|_{H^s} \leqslant C\|\Delta P\|_{H^s} \leqslant C\left\||a^\varepsilon|^2\right\|_{H^s} \leqslant C\|a^\varepsilon\|_{H^s}^2$$

by the L^2-boundedness of the Riesz transform. Therefore, we end up with the following estimate; there exist positive constants T and C such that

$$\sup_{t\in[0,T]} E_{\mathrm{p}}(t) \leqslant CE_{\mathrm{p}}(0).$$

Step 2. We next estimate the "total energy"

$$E(t) := E_{\mathrm{p}}(t) + \|w^\varepsilon(t)\|_{L^p}^2 + \|w^\varepsilon(t)\|_{L^\infty}^2.$$

By the integral version of the second line of (2.3), we deduce that

$$\begin{aligned}\|w^\varepsilon(t)\|_{L^p} &\leqslant \|\nabla\Phi_0\|_{L^p} + T\|w^\varepsilon\|_{L_T^\infty L^p}\|w^\varepsilon\|_{L_T^\infty L^p} + T|\lambda|\,\|\nabla P\|_{L_T^\infty L^p}\\ &\leqslant \|\nabla\Phi_0\|_{L^p} + (TCE_{\mathrm{p}}(0))\,\|\nabla w^\varepsilon\|_{L_T^\infty L^\infty} + T|\lambda|CE_{\mathrm{p}}(0).\end{aligned}$$

Letting T so small that $TCE_{\mathrm{p}}(0) \leqslant 1/2$ and taking the supremum over $[0,T]$ in time, we obtain

$$\|w^\varepsilon\|_{L_T^\infty L^p} \leqslant 2\|\nabla\Phi_0\|_{L^p} + CE_{\mathrm{p}}(0).$$

The same argument yields the bound of $\|w^\varepsilon\|_{L_T^\infty L^\infty}$, which leads us to the desired estimate: There exist positive constants T and C such that

$$\sup_{t\in[0,T]} E(t) \leqslant CE(0).$$

Now, by a standard argument, we obtain the solution $(a^\varepsilon, w^\varepsilon) \in C([0,T]; H^s\times X^{s+1}\cap L^p)$ of (2.3). Moreover, the solution is bounded in this class uniformly in ε because so is $E(0)$ by assumption.

Step 3. We construct ϕ^ε as

$$\phi^\varepsilon(t) := \Phi - \int_0^t \left(\frac{1}{2}|v^\varepsilon(s)|^2 + \lambda P(s)\right) ds.$$

One verifies that $(a^\varepsilon, \nabla\phi^\varepsilon)$ solves (2.3) and so that $\nabla\phi^\varepsilon = v^\varepsilon$ by uniqueness. It is clear from above representation to see that, for any bounded domain $\Omega\subset\mathbb{R}^2$, $\phi^\varepsilon - \Phi$ is bounded in $L^\infty([0,T]\times\Omega)$ uniformly in ε. Moreover,

$$\begin{aligned}\|\phi^\varepsilon - \phi_0\|_{L^\infty([0,T\times\Omega])} &\leqslant \frac{T}{2}\left\||v^\varepsilon|^2 - |v_0|^2\right\|_{L^\infty([0,T]\times\Omega)}\\ &\quad + |\lambda|T\,\|P_\varepsilon - P_0\|_{L^\infty([0,T]\times\Omega)}\\ &\leqslant C_\Omega(\|v^\varepsilon - v_0\|_{L^\infty([0,T]\times\Omega)} + \|a^\varepsilon - a_0\|_{L^\infty([0,T]\times\Omega)})\\ &\to 0\end{aligned}$$

as $\varepsilon\to 0$. □

Remark 2.3. Even in the case where the initial data $w^\varepsilon(0,x)$ is not rotation-free, that is, even if $\mathrm{rot}w^\varepsilon(0,x)\not\equiv 0$, the existence, uniqueness, continuous dependence, and convergence of the solution $(a^\varepsilon, w^\varepsilon)$ to (2.3) is still valid. Of course, in that case, there does not exist ϕ^ε such that $\nabla\phi^\varepsilon = w^\varepsilon$.

3. WKB APPROXIMATION OF SOLUTIONS TO (SP)

Now we briefly outline the proof of Theorems 1.1 and 1.3.

3.1. **Unique existence.** We have obtained a unique solution $(a^\varepsilon, \phi^\varepsilon)$ to the system (2.1). One then easily verifies that $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solves (SP) in the L^2 sense because the first line and the second line of (2.1) are satisfied in the L^2 sense and in the classical sense, respectively. $u^\varepsilon \in L^\infty([0,T]; H^s)$ follows from the following lemma:

Lemma 3.1. *For any $s > 1$,*

$$\left\| a e^{i\phi} \right\|_{H^s} \leqslant C \left\| a \right\|_{H^s} (1 + \left\| \nabla^2 \phi \right\|_{H^{\max(s-2,0)}})(1 + \left\| \nabla \phi \right\|_{L^\infty}^{\lceil s \rceil}), \tag{3.1}$$

where $\lceil s \rceil$ denotes the minimum integer larger than or equal to s.

Moreover, thanks to the following lemma, one can show $u^\varepsilon \in C([0,T]; H^s)$.

Lemma 3.2. *Let $s > 0$. Assume $\|a\|_{H^s}$ is bounded. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\nabla \phi\|_{L^\infty} + \left\|\nabla^2 \phi\right\|_{H^{\max(s-2,0)}} + |\phi(0)| < \delta$ then $\left\| a(e^{i\phi} - 1) \right\|_{H^s} \leqslant \varepsilon$.*

The uniqueness of (SP) is driven from that of (2.1):

Lemma 3.3. *Let (A^ε, Φ) satisfy Assumption 2.1. Set*

$$A := C([0,T); H^s(\mathbb{R}^2)) \cap C^1((0,T); H^{s-2}_{\mathrm{loc}}(\mathbb{R}^2))$$
$$B := \left\{ \phi \in C^1([0,T) \times \mathbb{R}^2); \nabla\phi \in X^{s+1}(\mathbb{R}^2) \cap L^{2+}(\mathbb{R}^2) \right\}.$$

Then, the following two statements are equivalent:

1. *The system (2.1) has a unique solution $(a^\varepsilon, \phi^\varepsilon) \in A \times B$.*
2. *The system*

$$\begin{cases} i\partial_t u^\varepsilon + \dfrac{1}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}}^\varepsilon u^\varepsilon, \\ V_{\mathrm{P}}^\varepsilon = -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \left(\log \dfrac{|x-y|}{|y|} \right) |u^\varepsilon(y)|^2 dy, \\ \partial_t \psi^\varepsilon + \dfrac{1}{2} |\nabla \psi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, \\ u(0,x) = u_0^\varepsilon(x), \quad \psi(0,x) = \Phi \end{cases} \tag{3.2}$$

has a unique solution $(u, \psi) \in A \times B$.

It is worth mentioning that this lemma suggests that all solution u^ε of (SP) is written as $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ with a solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.1).

3.2. **Approximation.** We see in Theorem 2.2 that the system (2.1) has a unique solution

$$a^\varepsilon \in C([0,T); H^s(\mathbb{R}^2)) \cap C^1((0,T); H^{s-2}(\mathbb{R}^2)),$$
$$\phi^\varepsilon \in C^1([0,T) \times \mathbb{R}^2) \text{ with } \nabla\phi^\varepsilon \in C([0,T); X^{s+1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$$

and that it converges (with two derivative loss) to (a_0, ϕ_0) solving (2.2) if $s > 3$ and $H^s \ni A_0 = \lim_{\varepsilon\to 0} A^\varepsilon$ exists. Then, one verifies that $(b^\varepsilon, \psi^\varepsilon) =$

$((a^\varepsilon - a_0)/\varepsilon, (\phi^\varepsilon - \phi_0)/\varepsilon)$ solves a system similar to (2.1). Thus, mimicking the proof of Theorem 2.2, we can prove that

$$b^\varepsilon \in C([0,T); H^{s-2}(\mathbb{R}^2)) \cap C^1((0,T); H^{s-4}(\mathbb{R}^2)),$$
$$\psi^\varepsilon \in C^1([0,T) \times \mathbb{R}^2) \text{ with } \nabla\psi^\varepsilon \in C([0,T); X^{s-1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$$

exist and are uniformly bounded if $b^\varepsilon(0) = (A^\varepsilon - A_0)/\varepsilon$ is uniformly bounded in H^{s-2}. Moreover, $(a_1, \phi_1) := \lim_{\varepsilon\to 0}(b^\varepsilon, \psi^\varepsilon)$ is defined in the same space as $(b^\varepsilon, \psi^\varepsilon)$ when $A_1 = \lim_\varepsilon b^\varepsilon(0) \in H^{s-2}$ exists. Repeating this argument N_0 times, we obtain the following. For details of the proof, consult [6, 11].

Assumption 3.4. *Let $N_0 \geqslant 1$ and assume the following for some $s > 3 + 2N_0$:*

- *$A^\varepsilon \in H^s(\mathbb{R}^2)$ and $\|A^\varepsilon\|_{H^s(\mathbb{R}^2)}$ is uniformly bounded. Moreover, there exists $A_k \in H^{s-2k}$ such that A^ε is expanded as*
$$A^\varepsilon = A_0 + \varepsilon A_1 + \cdots + \varepsilon^k A_k + o(\varepsilon^k) \quad \text{in } H^{s-2k}(\mathbb{R}^2)$$
for all $k \in [0, N_0]$.
- *$\Phi \in C^1([0,T) \times \mathbb{R}^2)$ with $\nabla\Phi \in X^{s+1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $p \in (2, \infty)$.*

Proposition 3.5. *Let Assumption 3.4 be satisfied. Then, the unique solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.1) given by Theorem 2.2 has the following expansion:*

$$\begin{aligned} a^\varepsilon &= a_0 + \varepsilon a_1 + \cdots + \varepsilon^{N_0} a_{N_0} + o(\varepsilon^{N_0}), \quad &&\text{in } L^\infty([0,T); H^{s-2N_0}) \\ \phi^\varepsilon &= \phi_0 + \varepsilon\phi_1 + \cdots + \varepsilon^{N_0}\phi_{N_0} + o(\varepsilon^{N_0}), \quad &&\text{in } L^\infty([0,T); L^\infty_{\mathrm{loc}}), \\ \nabla\phi^\varepsilon &= \nabla\phi_0 + \cdots + \varepsilon^{N_0}\nabla\phi_{N_0} + o(\varepsilon^{N_0}), \quad &&\text{in } L^\infty([0,T); X^{s+1-2N_0} \cap L^{2+}), \end{aligned}$$

where, for all $j \in [0, N_0]$, $a_j \in C([0,T); H^{s-2j})$ and $\phi_j \in C^1([0,T) \times \mathbb{R}^2)$ with $\nabla\phi_j \in X^{s+1-2j} \cap L^{2+}$.

An immediate consequence of this proposition is the WKB approximation

$$u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}} \sim e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots \varepsilon^{N_0-1}\beta_{N_0-1} + o(\varepsilon^{N_0-1})),$$

where $\beta_0 = e^{i\phi_1}a_0$, $\beta_1 = e^{i\phi_1}(a_1 + ia_0\phi_2)$, and so on. This asymptotic holds in $L^\infty([0,T); H^{s-2N_0}_{\mathrm{loc}})$. The reason why this asymptotic does not holds in $L^\infty([0,T); H^{s-2N_0})$ but in $L^\infty([0,T); H^{s-2N_0}_{\mathrm{loc}})$ is that ϕ_j grows at the spatial infinity because of Poisson terms. Hence, $\beta_1 \notin L^2$, in general.

Nevertheless, an additional assumption gives us the WKB approximation in $L^\infty([0,T); L^2(\mathbb{R}^2))$ as in Theorem 1.3. One can show that the growths of ϕ_j is at most the logarithmic order. More precisely, we cancel this logarithmic growth by polynomial decay of amplitude term which is obtained from the following lemma.

Lemma 3.6. *Let $N \geqslant 1$ be an integer and let Assumption 3.4 be satisfied for some $N_0 = 2N$. Let a_j ($j \in [0, 2N]$) be given in Proposition 3.5. Let $\alpha \in (0, 1]$. If*

$$(1 + |x|)^{\alpha/2^j}|a_j(t)|^2 \in L^1(\mathbb{R}^2), \qquad j = 0, 1, \ldots, N \tag{3.3}$$

holds at the initial time $t = 0$, then (3.3) holds for all $t \in [0, T)$.

4. Classical limit of solutions to (qEP)

4.1. **Existence.** We now prove Theorem 1.4 via the analysis of (2.1).

Proof of Theorem 1.4. First set $A^\varepsilon = \sqrt{\rho_0} \in H^s$ and

$$\Phi(x) = \int_{\Gamma_x} v_0(y)dy,$$

where Γ_x is a smooth contour from the origin to x. $\Phi(x)$ does not depends on Γ_x since v_0 is irrotational. Let $(a^\varepsilon, \phi^\varepsilon)$ be a solution to (2.1) with $(a^\varepsilon(0), \phi^\varepsilon(0)) = (A^\varepsilon(x), \Phi(x))$ given by Theorem 2.2. Define

$$\begin{aligned} &\rho^\varepsilon(t,x) = |a^\varepsilon(t,x)|^2 \in C([0,T]; W^{s,1}), \\ &J^\varepsilon(t,x) = |a^\varepsilon(t,x)|^2 \nabla\phi^\varepsilon + \varepsilon \operatorname{Im} \overline{a^\varepsilon} \nabla a^\varepsilon \in C([0,T]; W^{s-1,1}). \end{aligned} \tag{4.1}$$

One easily verifies that $\partial_t \rho^\varepsilon + \operatorname{div} J^\varepsilon = 0$ is valid in L^2-sense. Obviously, $\rho^\varepsilon \geqslant 0$ holds. Note that, however, ρ^ε is not positive, that is, ρ^ε may be equal to zero at some points. At the initial time $t = 0$, we have

$$\rho^\varepsilon(0,x) = |A^\varepsilon|^2 = \rho_0, \qquad J^\varepsilon(0,x) = |A^\varepsilon|^2 v_0 = \rho_0 v_0.$$

We now set $Z^\varepsilon = \{(t,x) \in [0,T] \times \mathbb{R}^2 \mid a^\varepsilon(t,x) = 0\}$ which is uniquely determined by the initial data. Since a^ε is continuous both in time and space, Z is a closed set. Consequently, $\Omega^\varepsilon := ((0,T) \times \mathbb{R}^2) \setminus Z^\varepsilon$ is an open set uniquely determined by the initial data. In Ω^ε, we define $\arg a^\varepsilon$ so that it is continuous both in time and space. It follows that $\arg a^\varepsilon$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t in Ω^ε. Set $S^\varepsilon = \phi^\varepsilon + \varepsilon \arg a^\varepsilon$ and $u^\varepsilon = \sqrt{\rho^\varepsilon} e^{iS^\varepsilon/\varepsilon} (= a^\varepsilon e^{i\phi^\varepsilon/\varepsilon})$. Introduce an $\mathbb{R}^2$-valued function v^ε on Ω^ε by

$$v^\varepsilon = \nabla S^\varepsilon = \frac{J^\varepsilon}{\rho^\varepsilon}.$$

Since u^ε solves (SP), we deduce that

$$\partial_t S^\varepsilon + \frac{1}{2}|\nabla S^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = \frac{\varepsilon^2}{2} \frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}$$

holds in Ω^ε. Then, taking ∇, one sees that v^ε solves

$$\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \lambda \nabla V_{\mathrm{P}}^\varepsilon = \frac{\varepsilon^2}{2} \nabla \left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right)$$

in $(C_0^\infty(\Omega^\varepsilon))'$ sense. Multiplying by ρ^ε and adding $v^\varepsilon(\partial_t\rho^\varepsilon + \operatorname{div} J^\varepsilon) = 0$, we see that

$$\partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \lambda \rho^\varepsilon \nabla V_{\mathrm{P}}^\varepsilon = \frac{\varepsilon^2}{2} \rho^\varepsilon \nabla \left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right)$$

holds in $(C_0^\infty(\Omega^\varepsilon))'$ sense. Since $v^\varepsilon = \nabla S^\varepsilon$, we have $\operatorname{rot} v^\varepsilon = 0$ in Ω^ε. □

4.2. **Classical limit.** Let us proceed to the proof of the classical limit.

Proof of Theorem 1.5. As seen in the proof of Theorem 1.4, ρ^ε and J^ε are given by

$$\rho^\varepsilon = |a^\varepsilon|^2, \qquad J^\varepsilon = |a^\varepsilon|^2\nabla\phi^\varepsilon + \varepsilon \operatorname{Im} \overline{a^\varepsilon}\nabla a^\varepsilon,$$

where $(a^\varepsilon, \rho^\varepsilon)$ is a solution to (2.1). By the latter part of Theorem 2.2, as ε tends to zero,

$$a^\varepsilon \to a \in C([0,T]; H^s),$$
$$\nabla\phi^\varepsilon \to \nabla\phi \in C([0,T]; L^p \cap X^{s+1})$$

hold in $L^\infty([0,T]; H^{s-2})$ and $L^\infty([0,T]; L^{2+} \cap X^{s-1})$ sense, respectively. Then, it is easy to see that

$$\rho^\varepsilon = |a^\varepsilon|^2 \to |a|^2 =: \rho$$

holds as $\varepsilon \to 0$ in $L^\infty([0,T]; W^{s-2,1})$. Similarly, as $\varepsilon \to 0$,

$$|a^\varepsilon|^2\nabla\phi^\varepsilon \to |a|^2\nabla\phi \qquad \text{in } L^\infty([0,T]; W^{s-2,1}).$$

Moreover, the uniform boundedness of a^ε in $L^\infty([0,T]; H^s)$ give us

$$\varepsilon \operatorname{Im} \overline{a^\varepsilon}\nabla a^\varepsilon \to 0 \qquad \text{in } L^\infty([0,T]; W^{s-1,1})$$

as $\varepsilon \to 0$. Therefore, setting $J = |a|^2\nabla\phi$, we see that the convergence $J^\varepsilon \to J$ holds in $L^\infty([0,T]; W^{s-2,1})$ as $\varepsilon \to 0$. Let Ω_t^0 and Ω^0 be as in Theorem 1.5. Hence, we can define

$$v = \frac{J}{\rho} = \nabla\phi$$

in Ω^0. Hence, $v \in C([0,T]; L^p \cap X^{s+1}(\Omega_t^0))$ for $t \in [0,T]$. Obviously, we can extend v into a function defined $\widetilde{v}$ in whole $[0,T] \times \mathbb{R}^2$ by

$$\widetilde{v} = \nabla\phi \in C([0,T]; L^p \cap X^{s+1}(\mathbb{R}^2)).$$

Since (a, ϕ) solves (2.2), one sees that $(\rho, \widetilde{v})$ solves (EP). $\square$

4.3. **Further expansion.** Since ρ^ε and J^ε are given by (4.1) and since asymptotic expansions of a^ε and $\nabla\phi^\varepsilon$ is shown in Proposition 3.5, an expansions of ρ^ε and J^ε are easily deduced. We now suppose that the Assumption of Theorem 1.4 is satisfied for $s > 2N_0 + 3$, where $N_0 \geqslant 1$ is an integer. Then, we have the following asymptotic expansion:

$$\rho^\varepsilon = \rho + \varepsilon\rho^{(1)} + \cdots + \varepsilon^{N_0-1}\rho^{(N_0-1)} + o(\varepsilon^{N_0-1}) \quad \text{in } L^\infty([0,T]; W^{s-2N_0,1}),$$
$$J^\varepsilon = J + \varepsilon J^{(1)} + \cdots + \varepsilon^{N_0-1}J^{(N_0-1)} + o(\varepsilon^{N_0-1}) \quad \text{in } L^\infty([0,T]lW^{s-2N_0,1}),$$

where, for $l \in [1, N_0]$,

$$\rho^{(l)}(t,x) = \sum_{i+j=l} \overline{a_i(t,x)}a_j(t,x) \in C([0,T]; W^{s-2k,1})$$

and

$$J^{(l)}(t,x) = \sum_{i+j+k=l} \overline{a_i(t,x)}a_j(t,x)\nabla\phi_k(t,x) + \sum_{i+j=k-1} \overline{a_i(t,x)}\nabla a_j(t,x) \in C([0,T]; W^{s-2k,1}).$$

Appendix A. On the Poisson equation in $\mathbb{R}^2$

In this article, we have considered the Poisson equation

$$-\Delta P = f \quad \text{in } \mathbb{R}^2 \tag{A.1}$$

with the conditions

$$|\nabla P| \to 0 \text{ as } |x| \to \infty, \quad P(0) = 0, \tag{A.2}$$

$$\nabla P \in L^\infty(\mathbb{R}^2). \tag{A.3}$$

Now let us give some remark on the Poisson equation in $\mathbb{R}^2$ (see also [15]). It is well known that, when we consider in $\mathbb{R}^d$ for $d \geqslant 3$, the solution P is defined by the Fourier transform or by the Newtonian potential as

$$P(x) = \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2}\mathcal{F}f(\xi)\right](x) \tag{A.4}$$

$$= \frac{1}{n(n-2)\omega_n}(|x|^{2-n} * f)(x), \tag{A.5}$$

where ω_n denotes the volume of the unit sphere in $\mathbb{R}^d$. In this case, it can be said that (A.1) in $\mathbb{R}^d$ is posed with not the condition (A.2)–(A.3) but

$$P \to 0 \text{ as } |x| \to \infty, \quad P \in L^\infty(\mathbb{R}^d). \tag{A.6}$$

For a good f, say $f \in \mathcal{S}(\mathbb{R}^d)$, the solution P defined by (A.4) or (A.5) satisfies (A.6), and is unique by Liouville's theorem.

In the two dimensional case, it is not possible to define the solution by (A.4) in general (even in the distribution sense) because of the singularity of $|\xi|^{-2}$. In [4, 19, 12], the definition (A.4) is employed under several assumption on f which provides $\mathcal{F}f(\xi) = O(|\xi|)$ as $\xi \to 0$. To realize it, it is almost necessary to suppose the following neutrality condition:

$$2\pi\mathcal{F}f(0) = \int_{\mathbb{R}^2} f(x)dx = 0. \tag{A.7}$$

This condition is, however, very restrictive in some cases. For example, in our systems (SP) or (qEP), this condition excludes all nontrivial solutions because we know $f \geqslant 0$. To avoid such a situation, we observe the fact that

$$\mathcal{F}^{-1}\left[\frac{-i\xi}{|\xi|^2}\mathcal{F}f(\xi)\right](x) \tag{A.8}$$

(which may be equal to ∇P) is well-defined even in the two-dimensional case, and we modify the condition (A.6) into (A.2)–(A.3), so that the Poisson equation (A.1) has a solution. We denote $p^* = 2p/(2-p)$ for $p < 2$. p^* is increasing in p, and $1^* = 2$.

Theorem A.1 ([15]). • *If $f \in L^{p_0}(\mathbb{R}^2)$ for some $p_0 \in (1,2)$, then*

$$P(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{|y|}\right)f(y)dy \tag{A.9}$$

is well-defined for all $x \in \mathbb{R}^2$ and is a weak solution of (A.1) *in such a sense that its weak derivative*

$$\nabla P(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}f(y)dy \in L^{p_0^*}(\mathbb{R}^2) \tag{A.10}$$

satisfies $\langle \nabla P, \nabla \varphi\rangle = -\langle f, \varphi\rangle$ *for all* $\varphi \in \mathcal{S}(\mathbb{R}^2)$*. Moreover, this solution satisfies* (A.2) *and if* $f \in L^1(\mathbb{R}^2)$ *then*

$$\|P\|_{\mathrm{BMO}} \leqslant C \|f\|_{L^1} \tag{A.11}$$

and

$$\limsup_{|x|\to\infty} \frac{|P(x)|}{\log\langle x\rangle} \leqslant \frac{\|f\|_{L^1}}{2\pi}. \tag{A.12}$$

- *If, in addition,* f *is continuous and* $\nabla f \in L^{q_0}(\mathbb{R}^2)$ *for some* $q_0 > 2$*, then* P *is in* $C^2(\mathbb{R}^2)$ *and is the unique classical solution of* (A.1) *with* (A.2)-(A.3)*. Moreover,* P *satisfies* $\nabla P \in L^r(\mathbb{R}^2)$ *for* $r \in [p_0^*, \infty]$*,* $\nabla^2 P \in L^p(\mathbb{R}^2)$ *for* $p \in [p_0, \infty]$*, and* $\nabla^3 P \in L^{q_0}(\mathbb{R}^2)$*.*

Remark A.2. The operator $\nabla(-\Delta)^{-1} := -\mathcal{F}^{-1} i\xi/|\xi|^2 \mathcal{F}$ is defined as a bounded operator from $L^{p_0}(\mathbb{R}^2)$ to $L^{p_0^*}(\mathbb{R}^2)$ for $p_0 \in (1,2)$. Remark that both (A.8) and (A.9) make sense for $f \in L^{p_0}(\mathbb{R}^2)$, $p_0 \in (1,2)$. Therefore, it can be said that (A.9) is one of the "proper" integral of (A.8). Remark that, from this point of view, the Newtonian potential $-(2\pi)^{-1}(\log|x| * f)$ is not proper (see Proposition A.4 and the consequent remarks, below).

Remark A.3. Note that $\nabla P \in L^2(\mathbb{R}^2)$ only if f satisfies the neutrality condition $-2\pi\mathcal{F}f(0) = \int_{\mathbb{R}^2} f dx = 0$. This is because $\|\nabla P\|_{L^2} = \left\||\xi|^{-1}\mathcal{F}f\right\|_{L^2}$.

A.1. **The Newtonian potential.** We can also give a rigorous meaning of the Newtonian potential

$$\widetilde{P}(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} (\log|x-y|) f(y) dy \tag{A.13}$$

as a solution of the Poisson equation. Notice that $-\frac{1}{2\pi}\log|x|$ is the Newtonian kernel in two dimensions and so that $\widetilde{P}$ is a two-dimensional version of (A.5).

Proposition A.4. *Let* $f \in L^{p_0}(\mathbb{R}^2)$ *for some* $p_0 \in (1,2)$ *and let* $\widetilde{P}$ *be as in* (A.13)*. If* $\widetilde{P}(x)$ *is finite at some* $x \in \mathbb{R}^2$*, then it is finite for all* $x \in \mathbb{R}^2$ *and, moreover,* $\widetilde{P}(x) = P(x) + \widetilde{P}(0)$*, where* P *is the solution of* (A.1) *with* (A.2)-(A.3) *given by Theorem A.1.*

Notice that the proposition implies the following:

- If $\widetilde{P}(x)$ diverges at some $x \in \mathbb{R}^2$ then it necessarily diverges for all $x \in \mathbb{R}^2$ under the same assumption on f.
- The difference between P and $\widetilde{P}$ is merely a constant $\widetilde{P}(0)$. However, when we consider $\widetilde{P}$, we need an additional assumption on f only for saying that this constant is finite.
- If $f \in L^{p_0}(\mathbb{R}^2)$ is so that $\widetilde{P}(0)$ is finite, then $\widetilde{P}$ is a weak solution of (A.1) with the condition $\widetilde{P}(0) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}(\log|y|)f(y)dy$ and $|\nabla\widetilde{P}| \to 0$ as $|x| \to \infty$.

The proof of this proposition is obvious: It suffices to mention that, for any $f \in L^{p_0}(\mathbb{R}^2)$ ($p_0 \in (1,2)$) and x_1, $x_2 \in \mathbb{R}^2$,

$$-\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\frac{|x_1-y|}{|x_2-y|}\right) f(y)dy$$

is finite.

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