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Weighted energy estimates for viscous conservation law with non-convex flux and its applications

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1 Introduction

In this note, we consider the initial-boundary value problem on the half line for a damped wave equation with a nonlinear convection term:

$$\begin{cases} u_{tt}-u_{xx}+u_t+f(u)_x=0, & x>0,\ t>0, \\ u(0,t)=u_-, & t>0, \\ \lim_{x\to\infty} u(x,t)=u_+, & t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x>0, \end{cases} \tag{1.1}$$

where the function $f(u)$ is a given C^2 function satisfying $f(0)=0$ and $u_\pm$ are given constants $u_-<u_+=0$ and we assume that

$$|f'(0)|<1, \quad f(u)>0 \quad for \quad u\in[u_-,0). \tag{1.2}$$

In Ueda-Kawashima [8], it is pointed out that by applying the Chapman-Enskog expansion to (1.1), the viscous conservation law

$$w_t+f(w)_x=(\mu(w)w_x)_x, \tag{1.3}$$

is derived as the second order approximation of the expansion, where $\mu(w)=1-(f'(w))^2$. They also pointed out that the sub-characteristic condition $|f'(w)|<1$ implies the parabolicity of (1.3) and this means that the dissipative structure for $t\to\infty$ of (1.1) is similar to the one for viscous conservation laws. Actually, they showed in [8] that in the level of diffusion waves, the asymptotic behavior of the solution of damped wave equation with convection term is well approximated by the one of the viscous conservation law of the form (1.3).

The purpose of this note is to show that in the level of stationary waves, behavior of the solution for damped wave equation is well approximated by the viscous conservation law.

The asymptotic stability of stationary waves for viscous conservation law in the half space is investigated by Liu-Matsumura-Nishihara [3], Liu-Nishihara [4], Ueda-Nakamura-Kawashima [9] and Hashimoto-Matsumura [1]. Liu-Matsumura-Nishihara [3] deal the case that the flux $f(u)$ is convex ($f''(u)>0$) and showed not only the stability of stationary waves but also the one of superposition of stationary waves and rarefaction wave. On the other hand, Liu-Nishihara [4] treated the case that the flux is not necessarily convex and showed the asymptotic stability of non-degenerate stationary waves. The asymptotic stability of higher order degenerate stationary waves are investigated by Ueda-Nakamura-Kawashima in [9] and they dealt the case where the flux is convex except for infinite direction. Hashimoto-Matsumura [1] showed the asymptotic stability of degenerate stationary waves under the condition where the flux is not necessarily convex.

According to the investigation of Ueda-Kawashima [8], we can expect that the solution of (1.1) and (1.2) with the following flux of three types (I), (II), (III):

(I) $f''(0) > 0, \ |f'(0)| < 1,$
(II) $f(u) = \frac{C_{q+1}}{q+1}(-u)^{q+1} + O(|u|^{q+2}), \quad$ for $u \to 0$, $(q \in \mathbb{N})$,
(III) $0 < |f'(0)| < 1.$

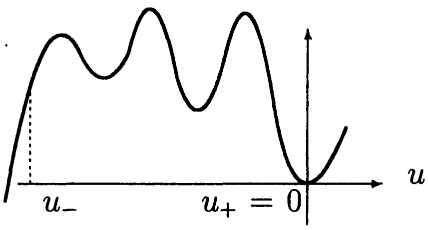

figure of (I) and (II)

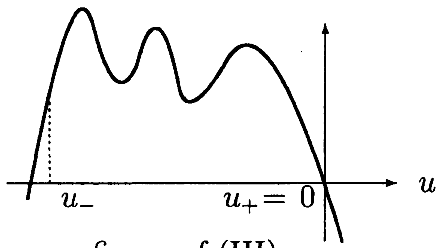

figure of (III)

is also tends to the stationary wave as time tends to infinity. Indeed, Ueda [7] showed that if the flux $f(u)$ satisfies

$$f''(u) > 0, \quad |f'(u)| < 1 \quad \text{for} \quad u \in [u_-, 0], \tag{1.4}$$

then the solution of (1.1) tends toward the stationary solution ϕ, provided that the initial perturbation is suitably small. Here, the stationary solution $\phi = \phi(x)$ is defined by the solution of the stationary problem corresponding to (1.1):

$$\begin{cases} f(\phi) = \phi_x, & x > 0, \\ \phi(0) = u_-, & \lim_{x \to \infty} \phi(x) = 0. \end{cases} \tag{1.5}$$

This note is direct consequence of Ueda [7] and Hashimoto-Matsumura [1]. That is, we apply the result of Ueda [7] to the general convection term (I), (II), (III). We also clear that for the asymptotic stability of stationary waves, it is enough to assume the sub-characteristic condition only at the far field, that is $|f'(0)| < 1$.

The second purpose of this paper is to derive the time decay estimate of the difference $u(x,t) - \phi(x)$, by employing the space-time weighted energy estimate used in Kawashima-Matsumura [2] and Nishikawa [6].

Before closing this section, we give some notations used in this note. For $1 \leq p < \infty$, We denote by $L^2 = L^2(\mathbb{R}_+)$ the usual Lebesgue space over $\mathbb{R}_+$ with the norm $\|\cdot\|_{L^2}$, and $H^1 = H^1(\mathbb{R}_+)$ the corresponding first order Sobolev space with the norm $\|\cdot\|_{H^1}$. Moreover, $H_0^1 = H_0^1(\mathbb{R}_+)$ denotes the space of functoins $f \in H^1$ with $f(0) = 0$, as a subspace of H^1. For $\alpha > 0$, $L_\alpha^2 = L_\alpha^2(\mathbb{R}_+)$ denotes the polynomially weighted L^2 space with the norm $\|f\|_{L_\alpha^2} := \|(1+x)^\alpha f(x)\|_{L^2}$, while $L_{\alpha,exp}^2 = L_{\alpha,exp}^2(\mathbb{R}_+)$ denotes the exponentially weighted L^2 space with the norm $\|f\|_{L_{\alpha,exp}^2} := \|e^{\alpha x} f(x)\|_{L^2}$. Let $H_\alpha^s =$

$H^s_\alpha(\mathbb{R}_+)$, which denotes the weighted Sobolev space corresponding to L^2_α, that is

$$H^s_\alpha := \{u \in L^2_\alpha; \partial^k_x u \in L^2_\alpha \ for \ 0 \le k \le s\}.$$

2 Main results and Reformulation of the problem

In this section, we give the statement of our main theorems. We state results separately in terms of the convection condition (I), (II) and (III). To complete this procedure, we first review the fundamental properties of the stationary solution $\phi(x)$ which satisfies (1.5). For its proof, we refer the reader to [3, 4, 7].

Lemma 2.1. *Suppose that* (1.2). *Then the stationary problem* (1.5) *has a unique smooth solution $\phi(x)$ satisfying $u_- < \phi(x) < 0$ and $\phi_x(x) > 0$ for $x > 0$. Moreover, for the non-degenerate case $f'(0) < 0$, we have*

$$|\partial^k_x \phi(x)| \le Ce^{-cx}, \quad x \ge 0$$

for each nonnegative integer k. On the other hand, for the degenerate case $f'(0) = 0$, we obtain

$$|\partial^k_x \phi(x)| \le C(1+x)^{-k-1}, \quad x \ge 0$$

for each nonnegative integer k.

Now, we state our main theorems. The first theorem is asymptotic stability of the solution to (1.1) with the flux (I).

Theorem 2.2 (The case I).
(i)(Asymptotic stability) Suppose that (1.2) *hold true. Assume that $u_0 - \phi \in H^1$ and $u_1 \in L^2$. Let $\phi(x)$ be the stationary solution satisfying the problem* (1.5). *Then there exists a positive constant ε_1 such that, if $\|u_0 - \phi\|_{H^1} + \|u_1\|_{L^2} \le \varepsilon_1$, then the initial-boundary value problem* (1.1) *has a unique global solution in time u satisfying*

$$u - \phi \in C^0([0,\infty); H^1_0), \qquad (u-\phi)_x, u_t \in L^2(0,\infty; L^2),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0. \tag{2.1}$$

(ii)(Polynomial decay rate) Suppose that $f'(0) < 0$ and (1.2) *hold true. Let $\phi(x)$ be the stationary wave of the problem* (1.5), *and $u(x,t)$ be the global solution to the problem* (1.1) *which is constructed in (i). If $u_0 - \phi \in H^1_\alpha$ and $u_1 \in L^2_\alpha$ for $\alpha \ge 0$, then we have*

$$\|u(t) - \phi\|_{H^1} \le CE_\alpha(1+t)^{-\alpha/2} \tag{2.2}$$

for $t \ge 0$, where C is a positive constant and $E_\alpha := \|u_0 - \phi\|_{H^1_\alpha} + \|u_1\|_{L^2_\alpha}$.
(iii)(Exponential decay rate) Suppose that the same conditions as in (ii) hold true. Then, if $u_0 - \phi \in H^1_{\alpha,exp}$ and $u_1 \in L^2_{\alpha,exp}$ for $\alpha > 0$, then we obtain

$$\|u(t) - \phi\|_{H^1} \le CE_{\alpha,exp}e^{-\beta t}$$

for $t \ge 0$, where β is a positive constant depending on α, C is a positive constant and $E_{\alpha,exp} := \|u_0 - \phi\|_{H^1_{\alpha,exp}} + \|u_1\|_{L^2_{\alpha,exp}}$.

The second theorem is concern about asymptotic stability of q−th order degenerate stationary waves for (1.1) with the flux condition (II).

Theorem 2.3 (The case II). *Suppose that (1.2) and (II) hold true. Assume that* $u_0 - \phi \in H^1_\alpha$ *and* $u_1 \in L^2_\alpha$ *for* α *with* $1 \le \alpha < \alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q$. *Let* $\phi(x)$ *be the stationary solution satisfying the problem* (1.5). *Then there exists a positive constant* ε_2 *such that, if* $\|u_0 - \phi\|_{H^1_1} + \|u_1\|_{L^2_1} \le \varepsilon_2$, *then the initial-boundary value problem* (1.1) *has a unique global solution in time* u *satisfying*

$$u - \phi \in C^0([0,\infty); H^1_\alpha) \cap C^1([0,\infty); L^2_\alpha),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0.$$

Moreover, the solution verifies the following decay estimate

$$\|u(t) - \phi\|_{H^1} \le C M_\alpha (1+t)^{-\alpha/4} \tag{2.3}$$

for $t \ge 0$, *where* C *is a positive constant and* $M_\alpha := \|u_0 - \phi\|_{H^1_\alpha} + \|u_1\|_{L^2_\alpha}$.

The third theorem is concern about asymptotic stability of non-degenerate stationary waves for (1.1) with the flux condition (III).

Theorem 2.4 (The case III).
(i)(Asymptotic stability)

Suppose that (1.2) and (III) hold true. Assume that $u_0 - \phi \in H^1 \cap L^1$, $u_1 \in L^2 \cap L^1$, $z_0 := -\int_x^\infty u_0(y) - \phi(y)dy \in L^2$ *and* $z_1 := -\int_x^\infty u_1(y)dy \in L^2$. *Let* $\phi(x)$ *be the stationary solution satisfying the problem* (1.5). *Then there exists a positive constant* ε_3 *such that, if* $\|z_0\|_{H^2} + \|z_1\|_{H^1} \le \varepsilon_3$, *then the initial-boundary value problem* (1.1) *has a unique global solution in time* u *satisfying*

$$u - \phi \in C^0([0,\infty); H^1_0) \cap C^1([0,\infty); L^2) \cap L^2(0,\infty; L^2),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0.$$

(ii)(Polynomial decay rate) Suppose that (1.2) and (III) hold true. Let $\phi(x)$ *be the stationary wave of the problem* (1.5), *and* $u(x,t)$ *be the global solution to the problem* (1.1) *which is constructed in (i). If* $u_0 - \phi \in H^1_\alpha$ *and* $u_1 \in L^2_\alpha$ *for* $\alpha \ge 0$, *then we have*

$$\|u(t) - \phi\|_{H^1} \le C E_\alpha (1+t)^{-\alpha/2} \tag{2.4}$$

for $t \ge 0$, *where* C *is a positive constant and* $E_\alpha := \|u_0 - \phi\|_{H^1_\alpha} + \|u_1\|_{L^2_\alpha}$.
(iii) (Exponential decay rate) Suppose that the same condition as in (ii) hold true. Then, if $u_0 - \phi \in H^1_{\alpha,exp}$ *and* $u_1 \in L^2_{\alpha,exp}$ *for* $\alpha > 0$, *then we obtain*

$$\|u(t) - \phi\|_{H^1} \le C E_{\alpha,exp} e^{-\beta t}$$

for $t \ge 0$, *where* β *is a positive constant depending on* α, C *is a positive constant and* $E_{\alpha,exp} := \|u_0 - \phi\|_{H^1_{\alpha,exp}} + \|u_1\|_{L^2_{\alpha,exp}}$.

In what follows, we state the scheme of the proof. Let $\phi(x)$ be the stationary solution satisfying (1.5). Then we reformulate our problem (1.1) by introducing the perturbation $v(x,t)$ by

$$u(x,t) = \phi(x,t) + v(x,t). \tag{2.5}$$

This is the standard strategy for solving our stability problem. Then, we rewrite our original problem (1.1) as

$$\begin{cases} v_{tt} - v_{xx} + v_t + \{f(\phi+v) - f(\phi)\}_x + h = 0, & x > 0, \quad t > 0, \\ v(0,t) = 0, & t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x > 0. \end{cases} \tag{2.6}$$

where we put $v_0(x) := u_0(x) - \phi_0(x)$ and $v_1(x) := u_1(x)$. We will discuss this reformulated problem in Section 3, 4 and 5 to prove our main theorems.

3 Proof of Case I

The aim of this section is to prove Theorem 2.2. In order to derive the existence of the global solution in time described in Theorem 2.2, we need the local existence theorem. For this purpose, we define the solution space for any interval $I \subseteq \mathbb{R}_+$ and $M > 0$ by

$$X_M(I) := \{v \in C^0(I; H_0^1(\mathbb{R}_+)) \ ; \ v_t \in C^0(I; L^2(\mathbb{R}_+)), \ \sup_{t \in I}(\|v(t)\|_{H^1} + \|v_t(t)\|_{L^2}) \le M\}.$$

For the solution space $X_M(I)$, the local existence theorem of the solution v for (2.6) is stated as follows.

Proposition 3.1 (local existence). *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ such that if $\|v_0\|_{H^1} + \|v_1\|_{L^2} \le M$, then the initial boundary value problem* (2.6) *has a unique solution $v \in X_{2M}([0, t_0])$.*

3.1 A priori estimate

To construct a global solution, it is important to derive the following *a priori* estimate of solutions v for (2.6) in the Sobolev space H^1.

Proposition 3.2 (*a priori* estimate). *Suppose that the same assumptions as in Theorem 2.2 hold true. Then, there exists a positive constant ε_1 such that if $v \in X_{\varepsilon_1}([0,T])$ is the solution of the problem* (2.6) *for some $T > 0$, then it holds*

$$\begin{aligned} &\|v(t)\|_{H^1}^2 + \|v_t(t)\|_{L^2}^2 + \int_0^t (\|v_t(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{L^2}^2 + \|\sqrt{\phi_x} v(\tau)\|_{L^2}^2)\, d\tau \\ &\qquad \le C(\|v_0\|_{H^1}^2 + \|v_1\|_{L^2}^2) \end{aligned} \tag{3.1}$$

for $t \in [0,T]$, where C is a positive constant independent of T.

Before proceeding to the proof of Proposition 5.2, we give some preparations concerning a weight function. Since $f''(0) > 0$ and $|f'(0)| < 1$ by the condition (I), there exist positive constants r and ν such that

$$f''(u) \ge \nu \quad \text{and} \quad |f'(u)| < 1 \qquad \text{for} \qquad |u| \le r.$$

In this situation, we choose the weight function as

$$w(u) = f(u) + \delta g(u) \quad \text{for} \quad u \in [u_-, r], \tag{3.2}$$

where $g(u)$ is defined by $g(u) = -u^{2m} + r^{2m}$, and δ and m are positive constants determined later. Then, we obtain the following lemma.

Lemma 3.3 (Hashimoto-Matsumura [1]). *Suppose that $f(u)$ satisfies (1.2) and* (I). *Let $w(u)$ be the weight function defined in* (3.2). *Then, for suitably small $\delta > 0$ and suitably large integer m, there exist positive constants c and C such that*

$$c \le w(u) \le C, \qquad (f''w - fw'')(u) \ge c \tag{3.3}$$

for $u \in [u_-, r]$.

For the proof, readers are referred to [1]. Furthermore, we prepare the key lemma for the weight function (3.2) as follows.

Lemma 3.4. *Suppose that $f(u)$ satisfies (1.2) and* (I). *Let $w(u)$ be the weight function defined in* (3.2) *which satisfies the condition* (3.3). *Then, for suitably small $\delta > 0$, we obtain the inequality*

$$(f'w - fw')(u)^2 < w(u)^2 \tag{3.4}$$

for $u \in [u_-, r]$.

Proof. By the definition of w, we rewrite (3.4) as

$$\delta^2\{(f'g - fg')(u)\}^2 < \{(f + \delta g)(u)\}^2. \tag{3.5}$$

Thus, the inequality (3.5) is enough to derive the inequality (3.4). In order to get the inequality (3.5), we divide the interval $[u_-, r]$ into $[u_-, -r]$ and $[-r, r]$. We first consider the interval $[-r, r]$. By the condition $|f'(u)| < 1$ and $(fg)(u) \ge 0$ for $u \in [-r, r]$, we choose δ suitably small, obtaining

$$\begin{aligned}
&\{(f + \delta g)(u)\}^2 - \delta^2\{(f'g - fg')(u)\}^2 \\
&\quad = \delta^2 g(u)^2\big(1 - f'(u)^2\big) + f(u)^2\big(1 - \delta^2 g'(u)^2\big) + 2\delta(fg)(u)\big(1 + \delta(f'g')(u)\big) \\
&\quad \ge \delta^2 g(u)^2\big(1 - f'(u)^2\big) + f(u)^2\big\{1 - \delta^2 \max_{u\in[-r,r]} |g'(u)|^2\big\} \\
&\qquad\qquad + 2\delta(fg)(u)\big\{1 - \delta \max_{u\in[-r,r]} |f'g'(u)|\big\} \\
&\quad > 0 \quad \text{for} \quad u \in [-r, r].
\end{aligned}$$

Next, we consider the interval $[u_-, -r]$. Taking δ sufficiently small, we have

$$(f + \delta g)(u) \ge \min_{u\in[u_-,-r]} f(u) - \delta \max_{u\in[u_-,-r]} |g(u)| \ge \frac{1}{2} \min_{u\in[u_-,-r]} f(u)$$

for $u \in [u_-, -r]$. Therefore, using the inequality

$$\delta^2\{(f'g - fg')(u)\}^2 \le \delta^2 \max_{u\in[u_-,-r]} |(f'g - fg')(u)|^2$$

and choosing δ suitably small such that

$$\delta \max_{u\in[u_-,-r]} |(f'g - fg')(u)| \le \frac{1}{2} \min_{u\in[u_-,-r]} f(u),$$

we obtain the desired inequality (3.5) for $u \in [u_-, -r]$ and complete the proof. □

Using Lemmas 3.3 and 3.4, we give proof of Proposition 3.2.

Proof of Proposition 3.2. First, put

$$N(T) = \sup_{0<t<T} \big(\|v(t)\|_{H^1} + \|v_t(t)\|_{L^2}\big),$$

and then we suppose $N(T) < 1$ throughout this section. We introduce a new unknown function $\tilde{v}$ as

$$v(x,t) = w\big(\phi(x,t)\big)\tilde{v}(x,t), \tag{3.6}$$

where w is the weight function defined by (3.2). Substituting (3.6) into the equation of (2.6), we obtain

$$\big(w(\phi)\tilde{v}\big)_{tt} - \big(w(\phi)\tilde{v}\big)_{xx} + \big(w(\phi)\tilde{v}\big)_t + \big\{f(\phi + w(\phi)\tilde{v}) - f(\phi)\big\}_x = 0. \tag{3.7}$$

Multiplying (3.7) by $\tilde{v}$, we get

$$\begin{aligned}&\Big\{\frac{1}{2}(w + w_t)(\Phi)\tilde{v}^2 + w(\phi)\tilde{v}_t\tilde{v}\Big\}_t - w(\phi)\tilde{v}_t^2 + w(\phi)\tilde{v}_x^2 + \frac{1}{2}(w_{tt} - w_{xx} + w_t)(\phi)\tilde{v}^2\\ &+ \phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta) - f'(\phi)d\eta + \phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta)w'(\phi)\eta d\eta + \mathcal{F}_x = 0,\end{aligned} \tag{3.8}$$

where we define $\mathcal{F}$ as

$$\mathcal{F} = -\frac{1}{2}w(\phi)_x\tilde{v}^2 - w(\phi)\tilde{v}\tilde{v}_x + \big(f(\phi + w(\phi)\tilde{v}) - f(\phi)\big)\tilde{v} - \int_0^{\tilde{v}} f(\phi + w(\phi)\eta) - f(\phi)d\eta.$$

By using the condition $\phi_x = f(\phi)$, we find that

$$\begin{aligned} w_{xx}(\phi) &= w'(\phi)(-\phi_{xx}) + w''(\phi)\phi_x^2\\ &= -w'(\phi)f(\phi)_x - w''(\phi)\phi_x^2\\ &= -\{w''(\phi)\phi_x + (f'w')(\phi)\}\phi_x\\ &= -(fw'' + f'w')(\phi)\phi_x.\end{aligned} \tag{3.9}$$

Moreover, by the straightforward calculation, we have

$$\begin{aligned}&\phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta) - f'(\phi)d\eta + \phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta)w'(\phi)\eta d\eta\\ =&\frac{1}{2}(f''w + f'w')(\phi)\,\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2.\end{aligned} \tag{3.10}$$

Therefore substituting (3.9) and (3.10) into the equality (3.8), we obtain

$$\begin{aligned}&\Big\{\frac{1}{2}w(\phi)\tilde{v}^2 + w(\phi)\tilde{v}_t\tilde{v}\Big\}_t + w(\phi)\tilde{v}_x^2 + \frac{1}{2}(f''w - fw'')(\phi)\phi_x\tilde{v}^2 - w(\phi)\tilde{v}_t^2 + \mathcal{F}_x\\ &= O(|\tilde{v}|)\phi_x\tilde{v}^2.\end{aligned} \tag{3.11}$$

Next, we multiply (3.7) by $2\tilde{v}_t$, obtaining

$$\mathcal{G}_t + 2w(\phi)\tilde{v}_t^2 + \mathcal{H} - \big(2w(\phi)\tilde{v}_t\tilde{v}_x\big)_x = 0, \tag{3.12}$$

where $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\begin{aligned}\mathcal{G} = {}& w(\phi)\tilde{v}_t^2 + w(\phi)\tilde{v}_x^2 + (w_x(\phi)\tilde{v}^2)_x - w_{xx}(\phi)\tilde{v}^2\\ &+ 2\phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta) - f'(\phi)d\eta + 2\phi_x\int_0^{\tilde{v}} f'(\phi + w(\phi)\eta)w'(\phi)\eta d\eta,\end{aligned}$$

$$\mathcal{H} = \big\{f'(\phi + w(\phi)\tilde{v})w(\phi) - w_x(\phi)\big\}\tilde{v}_t\tilde{v}_x.$$

Applying the relations (3.9) and (3.10), we rewrite $\mathcal{G}$ as

$$\mathcal{G} = w(\phi)\tilde{v}_t^2 + w(\phi)\tilde{v}_x^2 + (w_x(\phi)\tilde{v}^2)_x + (f''w - fw'')(\phi)\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2. \tag{3.13}$$

On the other hand, making use of the equality

$$f'(\phi + w(\phi)\tilde{v})w(\phi) - w_x(\phi) = (f'w - fw')(\phi) + O(|\tilde{v}|),$$

we have

$$\mathcal{H} = 2(f'w - fw')(\phi)\tilde{v}_t\tilde{v}_x + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x. \tag{3.14}$$

Summing up (3.11) and (3.12), and substituting (3.13) and (3.14) into the resultant equation, we obtain

$$\big(E + R_1\big)_t + D + F_x = R_1 + R_2, \tag{3.15}$$

where E, D, F, R_1 and R_2 are defined by

$$\begin{aligned}
E &= w(\phi)\Big(\frac{1}{2}\tilde{v}^2 + \tilde{v}_t^2 + \tilde{v}_x^2 + \tilde{v}\tilde{v}_t\Big) + (f''w - fw'')(\phi)\phi_x\tilde{v}^2,\\
D &= w(\phi)(\tilde{v}_x^2 + \tilde{v}_t^2) + 2(f'w - fw')(\phi)\tilde{v}_t\tilde{v}_x + \frac{1}{2}(f''w - fw'')(\phi)\phi_x\tilde{v}^2,\\
F &= -\frac{1}{2}w(\phi)_x\tilde{v}^2 - w(\phi)(\tilde{v}\tilde{v}_x + 2\tilde{v}_t\tilde{v}_x)\\
&\qquad + \big(f(\phi + w(\phi)\tilde{v}) - f(\phi)\big)\tilde{v} - \int_0^{\tilde{v}} f(\phi + w(\phi)\eta) - f(\phi)d\eta,\\
R_1 &= O(|\tilde{v}|)\phi_x\tilde{v}^2, \quad R_2 = O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x.
\end{aligned}$$

Therefore, integrating the equation (3.15) over $\mathbb{R}_+$, we get the energy equality

$$\frac{d}{dt}\int_0^\infty E + R_1 dx + \int_0^\infty D\,dx = \int_0^\infty R_1 + R_2\,dx. \tag{3.16}$$

Here, calculating the discriminants and using Lemma 3.4, we have the condition

$$\int_0^\infty E\,dx \sim \|(\tilde{v}, \tilde{v}_x, \tilde{v}_t, \sqrt{\phi_x}\tilde{v})\|_{L^2}^2, \quad \int_0^\infty D\,dx \sim \|(\tilde{v}_x, \tilde{v}_t, \sqrt{\phi_x}\tilde{v})\|_{L^2}^2. \tag{3.17}$$

Therefore, integrating (3.16) over $(0, t)$, and taking $N(T)$ sufficiently small, we obtain

$$\|\tilde{v}\|_{H^1}^2 + \|\tilde{v}_t\|_{L^2}^2 + \int_0^t \|\tilde{v}_x\|_{L^2}^2 + \|\tilde{v}_t\|_{L^2}^2 + \|\sqrt{\phi_x}\tilde{v}\|_{L^2}^2 d\tau \le C\big(\|\tilde{v}_0\|_{H^1}^2 + \|\tilde{v}_1\|_{L^2}^2\big).$$

Finally, by the positivity of w and the simple relations $v_x = w_x\tilde{v} + w\tilde{v}_x$ and $v_t = w\tilde{v}_t$, we find that $\|v\|_{L^2} \sim \|\tilde{v}\|_{L^2}$. Thus, we have the desired estimate (3.1) and complete the proof of Proposition 3.2. □

3.2 Convergence rates of stationary solutions

In this section, we prove Theorems 2.2-(i). The main idea of the proofs are due to Ueda [7]. We use the space-time weighted energy method introduced in Kawashima-Matsumura [2]. Before stating the proofs, we give a preparation. The following lemma is concerning the inequality of the nonlinear term f and the weight function w.

Lemma 3.5. *Suppose the same condition as in Theorems 2.2-(i). Let $w(u)$ be the weight function defined by* (3.2)*. Then, for suitably large integer m, there exists a positive constant c such that*

$$(fw' - f'w)(u) \ge c \tag{3.18}$$

for $u \in [u_-, 0]$.

Proof. By the definition of weight function w, we have

$$(fw' - f'w)(u) = \delta(fg' - f'g)(u).$$

In order to derive the desired inequality, we decompose the interval $[u_-, 0]$ into $[u_-, -r]$, $[-r, -r/2]$ and $[-r/2, 0]$. We first consider the case $[u_-, -r]$. For $u \in [u_-, -r]$, we have

$$\begin{aligned}(fg' - f'g)(u) &= -\{2mu^{2m-1}f(u) + f'(u)(-u^{2m} + r^{2m})\} \\ &= -2mu^{2m-1}\Big\{f'(u)\Big(-1 + \Big|\frac{r}{u}\Big|^{2m}\Big)\frac{u}{2m} + f(u)\Big\}.\end{aligned} \tag{3.19}$$

Here, we note that $|r/u| \le 1$ and $f(u) \ge c_0$, $|f'(u)| < C$ for $u \in [u_-, -r]$, where c_0 and C are positive constants. Thus we can choose m sufficiently large such that

$$f'(u)\Big(-1 + \Big|\frac{r}{u}\Big|^{2m}\Big)\frac{u}{2m} + f(u) \ge \frac{c_0}{2}. \tag{3.20}$$

Therefore, (3.19) and (3.20) imply the following inequality

$$(fg' - f'g)(u) \ge c_0 m r^{2m-1} > 0. \tag{3.21}$$

For the case $u \in [-r, -r/2]$, since $f > 0, g' > 0$ and $f' < 0 \le g$, it immediately holds

$$(fg' - f'g)(u) \ge (fg')(u) \ge (fg')(-r/2) > 0. \tag{3.22}$$

Finally, for the case $u \in [-r/2, 0]$, since $f > 0, g' \ge 0$ and $f' < 0 < g$, we get

$$(fg' - f'g)(u) \ge (f'g)(u) \ge \min_{u\in[-r/2,0]} |(f'g)(u)| > 0. \tag{3.23}$$

Thus combining (3.21), (3.22) and (3.23), we obtain the desired estimate (3.18). □

Now, we prove Theorem 2.2-(ii) and (iii). Applying Lemma 3.5 to F, we calculate F as

$$\begin{aligned}-F &= \frac{1}{2}(fw' - f'w)(\phi)\tilde{v}^2 + w(\phi)(\tilde{v}\tilde{v}_x + 2\tilde{v}_t\tilde{v}_x) + O(|\tilde{v}|^3) \\ &\ge c\tilde{v}^2 - C(\tilde{v}_x^2 + \tilde{v}_t^2) + O(|\tilde{v}|^3),\end{aligned} \tag{3.24}$$

where c and C are positive constants.

Let γ and β be any positive constants satisfying $0 \le \gamma, \beta \le \alpha$. We multiply the equality (3.15) by $(1+t)^\gamma(1+x)^\beta$, obtaining

$$\begin{aligned}&\{(1+t)^\gamma(1+x)^\beta(E + R_1)\}_t - \gamma(1+t)^{\gamma-1}(1+x)^\beta(E + R_1) + (1+t)^\gamma(1+x)^\beta D \\ &\quad + \{(1+t)^\gamma(1+x)^\beta F\}_x - \beta(1+t)^\gamma(1+x)^{\beta-1}F = (1+t)^\gamma(1+x)^\beta(R_1 + R_2).\end{aligned} \tag{3.25}$$

Substituting (5.29) into (5.30), integrating the resultant inequality over $\mathbb{R}_+ \times (0, t)$ and taking $\sup_{0\le t\le T}\|v(t)\|_{L^\infty}$ sufficiently small, we have

$$\begin{aligned}&(1+t)^\gamma\|(\tilde{v}, \tilde{v}_t, \tilde{v}_x)(t)\|^2_{L^2_\beta} + \int_0^t (1+\tau)^\gamma\big(\|(\tilde{v}_t, \tilde{v}_x, \sqrt{\phi_x}\tilde{v})(\tau)\|^2_{L^2_\beta} + \beta\|\tilde{v}(\tau)\|^2_{L^2_{\beta-1}}\big)d\tau \\ &\le CE_\beta^2 + \gamma C\int_0^t (1+\tau)^{\gamma-1}\|(\tilde{v}, \tilde{v}_t, \tilde{v}_x)(\tau)\|^2_{L^2_\beta}d\tau + \beta C\int_0^t (1+\tau)^\gamma\|(\tilde{v}_t, \tilde{v}_x)(\tau)\|^2_{L^2_{\beta-1}}d\tau\end{aligned}$$

for an arbitrary γ and β with $0 \le \gamma$, $\beta \le \alpha$, where C is a constant independent of γ and β. For the above estimate, applying the induction argument, we can obtain the desired estimate (2.4) in Theorem 2.2-(ii). For the details, we refer the readers to [6, 7].

□

Finally, we prove Theorem 2.2-(iii) by using the space-time weighted energy method.

Proof of Theorem 2.2-(iii). Let α, $\beta > 0$. Multiplying (3.15) by $e^{\beta t}e^{\alpha x}$, we obtain

$$\begin{aligned}&\{e^{\beta t}e^{\alpha x}(\bar{E}+\bar{R}_1)\}_t - \beta e^{\beta t}e^{\alpha x}(\bar{E}+\bar{R}_1) + e^{\beta t}e^{\alpha x}\bar{D}\\ &+\{e^{\beta t}e^{\alpha x}\bar{F}\}_x - \alpha e^{\beta t}e^{\alpha x}\bar{F} = e^{\beta t}e^{\alpha x}(\bar{R}_1+\bar{R}_2).\end{aligned} \tag{3.26}$$

Substituting (5.29) into (5.36), integrating the resultant inequality over $\mathbb{R}_+ \times (0,t)$ and taking $\sup_{0\le t\le T}\|v(t)\|_{L^\infty}$ sufficiently small, we get

$$\begin{aligned}&e^{\beta t}\|(\tilde{v}_t,\tilde{v}_x,\tilde{v})(t)\|^2_{L^2_{\alpha,exp}} + \int_0^t e^{\beta\tau}\|(\tilde{v}_t,\tilde{v}_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau + \alpha\int_0^t e^{\beta\tau}\|\tilde{v}(\tau)\|^2_{L^2_{\alpha,exp}}d\tau\\ &\le CE^2_{\alpha,exp} + \beta C_0\int_0^t e^{\beta\tau}\|\tilde{v}(\tau)\|^2_{L^2_{\alpha,exp}}d\tau + (\alpha+\beta)C_1\int_0^t e^{\beta\tau}\|(\tilde{v}_t,\tilde{v}_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau,\end{aligned}$$

where C_0, C_1 and C are positive constants independent of α and β. Taking $\alpha > 0$ and $\beta > 0$ suitably small such that $\beta C_0 \le \alpha$ and $(\alpha+\beta)C_1 \le 1$, we obtain the desired estimate in Theorem 2.2-(iii) and complete the proof. □

4 Proof of Case II

The aim of this section is to prove Theorem 2.3, which is a direct application of the work of Ueda-Nakamura-Kawashima [9]. In order to derive the existence of the global solution in time described in Theorem 2.3, we need the local existence theorem. For this purpose, we define the solution space for any interval $I \subseteq \mathbb{R}_+$, $M > 0$ and positive constant α with $1 \le \alpha \le \alpha_*(q)$ by

$$X_M(I) := \{v \in C^0(I; H^1_\alpha(\mathbb{R}_+))\ ;\ v_t \in C^0(I; L^2_\alpha(\mathbb{R}_+)),\ \sup_{t\in I}(\|v(t)\|_{H^1_\alpha} + \|v_t(t)\|_{L^2_\alpha}) \le M\}.$$

For the solution space $X_M(I)$, the local existence theorem of the solution v for (2.6) is stated as follows.

Proposition 4.1 (local existence). *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ such that if $\|v_0\|_{H^1_\alpha} + \|v_1\|_{L^2_\alpha} \le M$, then the initial boundary value problem* (2.6) *has a unique solution $v \in X_{2M}([0,t_0])$.*

4.1 Energy estimate

To construct a global solution, it is important to derive the following estimate of solutions v of (2.6) in the Sobolev space H^1_β, where β is a positive constant satisfying $0 \le \beta \le \alpha$.

Proposition 4.2 (space-time energy estimate). *Suppose that the same assumptions as in Theorem 2.3 hold true. Then, there exists a positive constant ε_2 such that if*

$v \in X_{\varepsilon_2}([0,T])$ is the solution of the problem (2.6) for some $T>0$, then it holds

$$\begin{aligned}
&(1+t)^\gamma \|(v,v_t,v_x)(t)\|^2_{L^2_\beta} + \int_0^t (1+\tau)(\|(v_t+v_x)(\tau)\|^2_{L^2_\beta} + \|v(\tau)\|^2_{L^2_{\beta-2}})d\tau \\
&\le CM_\beta^2 + \gamma C\int_0^t (1+\tau)^{\gamma-1}\|(v,v_t,v_x)(\tau)\|^2_{L^2_\beta}d\tau \\
&\quad + \beta C\int_0^t (1+\tau)^\gamma(\|(v_t,v_x)(\tau)\|^2_{L^2_\beta} + \|v(\tau)\|^2_{L^2_{-2}})d\tau
\end{aligned} \tag{4.1}$$

for $\gamma \ge 0$ and β with $0 \le \beta \le \alpha$, where $M_\beta := \|v_0\|_{H^1_\beta} + \|v_1\|_{L^2_\beta}$ and C is a positive constant independent of γ, β and T.

Proof. The proof is given by space-time weighted energy method. First, put

$$N(T) = \sup_{0<t<T}\big(\|v(t)\|_{H^1_\alpha} + \|v_t(t)\|_{L^2_\alpha}\big),$$

and then we suppose $N(T) < 1$ throughout this section. We start with the equality (2.6). Let $0 \le \beta \le \alpha$ and define a weight function as

$$\bar{w}(\phi) = (-\phi)^{\beta q}. \tag{4.2}$$

We multiply (2.6) by $\bar{w}(\phi)$ and obtain

$$(\bar{w}(\phi)(E+R_1))_t + (\bar{w}(\phi)(D+R_2) - \bar{w}'(\phi)\phi_x F) + (\bar{w}(\phi)F)_x = 0.$$

where E, D, F, R_1, and R_2 are the same polynomial defined in Section 3. By using Lemma 2.1, we see that $\bar{w}(\phi) \sim (1+x)^\beta$. We deconstruct as

$$\begin{aligned}
\tilde{D} &:= \bar{w}(\phi)(D+R_2) - \bar{w}'(\phi)\phi_x F = \bar{D} + \bar{R}, \\
\bar{D} &= \bar{w}\{w(\tilde{v}_x^2 + \tilde{v}_t^2) + 2(f'w - fw')\tilde{v}_t\tilde{v}_x + \frac{1}{2}(f''w - fw'')\phi_x\tilde{v}^2\}, \\
&\quad - \bar{w}'\phi_x\{\frac{1}{2}(f'w - fw')\tilde{v}^2 - w\tilde{v}\tilde{v}_x - 2w\tilde{v}_t\tilde{v}_x\}, \\
\bar{R} &= -\bar{w}'\phi_x O(|\tilde{v}|)\tilde{v}^2 + \bar{w}(\phi)\big(O(|\tilde{v}|)\phi_x\tilde{v}^2 + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x\big).
\end{aligned}$$

Then we have the following estimate for $x \in \mathbb{R}_+$.

$$\begin{aligned}
&c(\tilde{v}^2 + \tilde{v}_x^2 + \tilde{v}_t^2) \le E \le C(\tilde{v}^2 + \tilde{v}_x^2 + \tilde{v}_t^2), \\
&|\bar{R}| \le C(1+x)^{\beta-1}|\tilde{v}|^3 + C(1+x)^\beta|\tilde{v}|(\tilde{v}_t^2 + \tilde{v}_x^2) \\
&|R_1| \le C(1+x,)^\beta|\tilde{v}|^3, \\
&\bar{D} \ge c(1+x)^\beta(\tilde{v}_x^2 + \tilde{v}_t^2) + c(1+x)^{\beta-2}\tilde{v}^2 - \beta c\{\tilde{v}_t^2 + \tilde{v}_x^2 + (1+x)^{-2}\tilde{v}^2\},
\end{aligned} \tag{4.3}$$

where c and C are positive constants of β with $0 \le \beta \le \alpha$. Here, we assume that $N(T)$ is suitably small, then we can easily derive the estimate for E, $|\bar{R}|$ and $|R_1|$ in (4.3). In what follows, we show the estimate for $\bar{D}$. We further decompose $\bar{D}$ as

$$\begin{aligned}
\bar{D} &= \bar{D}_1 + \bar{D}_2, \\
\bar{D}_1 &= \bar{w}\big(w\tilde{v}_x^2 + w\tilde{v}_t^2 + \frac{1}{2}(f''w - fw'')\phi_x\tilde{v}^2\big) + \bar{w}'\phi_x\big(w\tilde{v}\tilde{v}_x - \frac{1}{2}(f'w - fw')\tilde{v}^2\big), \\
\bar{D}_2 &= 2\{\bar{w}(f'w - fw') + \bar{w}'w\phi_x\}\tilde{v}_t\tilde{v}_x.
\end{aligned} \tag{4.4}$$

By using the definition of $\bar{w}$, we observe that

$$
\begin{aligned}
&\bar{w}'(\phi) = \beta q(-\phi)^{-\beta q-1}, \quad \phi_x = \frac{C_{q+1}}{q+1}(-\phi)^{q+1}(1+O(|\phi|)),\\
&f'(\phi) = -C_{q+1}(-\phi)^q\big(1+O(|\phi|)\big), \quad f''(\phi) = qC_{q+1}(-\phi)^{q-1}\big(1+O(|\phi|)\big),\\
&\bar{w}'(\phi)\phi_x = \frac{\beta q}{q+1}C_{q+1}\bar{w}(\phi)(-\phi)^q\big(1+O(|\phi|)\big).
\end{aligned}
\tag{4.5}
$$

Substituting (4.5) into the definition of $\bar{D}_1$, then we have the following equality.

$$
\begin{aligned}
\bar{D}_1 =&\delta r^{2m}(-\phi)^{-\beta q}\{\tilde{v}_x^2+\tilde{v}_t^2+\beta\tilde{v}_x\cdot a_q(-\phi)^q\tilde{v}+\frac{q+1}{2q}(\beta+1)\cdot\big(a_q(-\phi)^q\tilde{v}\big)^2\}\\
&+(-\phi)^{-\beta q}\{O(|\phi|)\tilde{v}_x\cdot(-\phi)^q\tilde{v}+O(|\phi|)\big((-\phi)^q\tilde{v}\big)^2+O(|\phi|)\tilde{v}_t^2\},
\end{aligned}
$$

where $a_q = \frac{C_{q+1}\cdot q}{q+1}$. Therefore, if $0 \le \beta \le \alpha$ with $\alpha < \alpha_*(q)$, we estimate $\bar{D}$ from bellow as

$$
\bar{D} \ge c(1+x)^\beta(\tilde{v}_x^2+\tilde{v}_t^2)+c(1+x)^{\beta-2}\tilde{v}^2, \qquad for \quad x\to+\infty, \tag{4.6}
$$

where c is a positive constant independent of β. On the other hand, for any fixed large $K>0$, we see easily from the definition of $\bar{D}$ that

$$
\bar{D} \ge c(\tilde{v}^2+\tilde{v}_x^2+\tilde{v}_t^2)-C\beta(\tilde{v}^2+\tilde{v}_x^2+\tilde{v}_t^2), \tag{4.7}
$$

for $0 \le x \le K$, where c and C are positive constants depending on k but not on β. Combining (4.6) and (4.7), we derive the desired estimate (4.3). Once we have the estimate (4.3), by using the same strategy as in [9], we derive the desired inequality (4.1). For the details, we refer the readers to [9]. □

Once we derive the inequality (4.1), by applying the induction argument, we can derive the following two decay estimate in Proposition 4.3 and 4.4.

Proposition 4.3. *Let $M_\alpha := \|v_0\|_{H^1_\alpha}+\|v_1\|_{L^2_\alpha}$. Under the same assumption in Proposition 4.2, we have the following estimate for integer j with $0\le j\le[\alpha/2]$,*

$$
\begin{aligned}
&(1+t)^j\|(\tilde{v},\tilde{v}_x,\tilde{v}_t)(t)\|^2_{L^2_{\alpha-2}}\\
&+\int_0^t(1+\tau)^j(\|(\tilde{v}_x,\tilde{v}_t)\|^2_{L^2_{\alpha-2j}}+\|v(\tau)\|^2_{L^2_{\alpha-2j-2}})d\tau \le CM_\alpha^2, \quad for \quad 0\le t\le T,
\end{aligned}
\tag{4.8}
$$

where C is a positive constant.

Proposition 4.4. *Let γ be a positive constant with $\gamma>\alpha$, provided that $\alpha/2$ is not an integer. Then, under the same assumption as in 4.2, we have the following estimate,*

$$
\begin{aligned}
&(1+t)^\gamma\|(\tilde{v},\tilde{v}_x,\tilde{v}_t)(t)\|^2_{L^2}\\
&+\int_0^t(1+\tau)^\gamma(\|(\tilde{v}_x,\tilde{v}_t)\|^2_{L^2}+\|v(\tau)\|^2_{L^2_{-2}})d\tau \le CM_\alpha^2(1+t)^{\gamma-\alpha/2},
\end{aligned}
\tag{4.9}
$$

for $0\le t\le T$, where C is a positive constant.

By using these inequality (4.8) and (4.9), and noting that $v(x,t)=w(\phi)\tilde{v}(x,t)$, we derive decay estimate

$$
\|v(t)\|_{H^1}+\|v_t(t)\|_{L^2} \le CM_\alpha(1+t)^{-\alpha/4}. \tag{4.10}
$$

On the other hand, the inequality (4.8) with $\alpha=1$ and $j=0$ leads the following a priori estimate

$$\|(v, v_x, v_t)\|_{L_1^2}^2 + \int_0^t (\|(v_x, v_t)(\tau)\|_{L_1^2}^2 + \|v(\tau)\|_{L_{-1}^2}^2) d\tau \le CM_1^2. \qquad (4.11)$$

Combining the inequality (4.11) and local existence theorem, we have the time-global solution and show the Theorem 2.3-(i). Therefore, (4.10) verify the asymptotic rate (2.3) in Theorem 2.3-(ii). For the details, we refer the readers to [9]. We complete the proof of Theorem 2.3.

5 Proof of Case III

The aim of this section is to prove Theorem 2.4. All of the first, we reformulate the problem (2.6). We introduce a new unknown function $z(x,t)$ as

$$z(x,t) := -\int_x^\infty v(y,t)dy.$$

Here, we assume integrability of $z(x,t)$ over $\mathbb{R}_+$. We reformulate (1.1) in terms of $z(x,t)$ as

$$\begin{cases} z_{tt} - z_{xx} + z_t + \{f(\phi + z_x) - f(\phi)\} = 0, & x > 0, \quad t > 0, \\ z_x(0,t) = 0, & t > 0, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & x > 0. \end{cases} \qquad (5.1)$$

where we put $z_0(x) := -\int_x^\infty (u_0(y) - \phi(y))dy$ and $z_1(x) := -\int_x^\infty u_1(y)dy$. In this section, we will discuss this reformulated problem to prove our main theorem. In order to derive the existence of the global solution in time described in Theorem 2.4, we need the local existence theorem. For this purpose, we define the solution space for any interval $I \subseteq \mathbb{R}_+$ and $M > 0$ by

$$X_M(I) := \{z \in C^0(I; H^2 \cap H^1_{w(u)}),\ z_t \in C^0(I; H^1 \cap L^2_{w(u)});\ z_x \in L^2(I; H^2 \cap L^2_{w(u)}),$$
$$\sup_{t\in I}(\|z(t)\|_{H^2} + \|z_t(t)\|_{H^1}) \le M\}.$$

For the solution space $X_M(I)$, the local existence theorem of the solution v for (2.6) is stated as follows.

Proposition 5.1 (local existence). *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ such that if $z_0 \in C^0(I; H^2 \cap H^1_{w(u)})$, $z_1 \in C^0(I; H^1 \cap L^2_{w(u)})$ and $\|z_0\|_{H^2} + \|z_1\|_{H^1} \le M$, then the initial boundary value problem* (2.6) *has a unique solution $z \in X_{2M}([0, t_0])$.*

5.1 A priori estimate

To construct a global solution, it is important to derive the following *a priori* estimate of solutions z for (5.1) in the Sobolev space H^2.

Proposition 5.2 (*a priori* estimate). *Suppose that the same assumptions as in Theorem 2.4-(i) hold true. Then, there exists a positive constant ε_3 such that if $z \in$*

$X_{\varepsilon_3}([0,T])$ is the solution of the problem (5.1) *for some $T>0$, then it holds*

$$\begin{aligned}&\|z(t)\|_{H^2}^2+\|z_t(t)\|_{H^1}^2+\int_0^t(\|\sqrt{\phi_x}z(\tau)\|_{L^2}^2+\|z_t(\tau)\|_{H^1}^2+\|z_x(\tau)\|_{H^1}^2)\,d\tau\\&\le C(\|z_0\|_{H^2}^2+\|z_1\|_{H^1}^2),\end{aligned}\tag{5.2}$$

for $t\in[0,T]$, where C is a positive constant independent of T.

Before proceeding to the proof of Proposition 5.2, we give some preparations for a weight function. We introduce a weight function as

$$w(u):=\begin{cases}(-e^{Au}+1)/f(u), & u\in[u_-,0),\\ -A/f'(0), & u=0,\end{cases}\tag{5.3}$$

where A is a positive constant which is chosen later. For this weight function, we obtain the following lemma.

Lemma 5.3. *Suppose that $f(u)$ satisfies* (1.2) *and* (III). *Let $w(u)$ be the weight function defined in* (5.3). *Then if we take A sufficiently large, $w(u)$ satisfies the following conditions in $u\in[u_-,0]$ for some positive constant c and C.*

$$\text{(i) } c<w(u)<C,\qquad \text{(ii) } \Big((fw)'(u)\Big)^2<w^2,$$
$$\text{(iii) } (fw)''(u)<0,\qquad \text{(iv) } (fw)'(u)<-c.$$

Proof. Since (iii) and (iv) are clear, so we only prove (i) and (ii).
Proof of (i). We divide the interval $[u_-,0]$ into $[u_-,-r]$ and $[-r,0]$, for some positive constant r. We first consider the interval [-r,0]. By the definition of $w(u)$, we obtain

$$w(u)=\frac{-e^{Au}+1}{|f(u)|}=\frac{|1-(1+Ae^{A\theta u}u)|}{|f'(0)u+\frac12 f''(\theta u)u^2|}=\frac{|Ae^{A\theta u}|}{|f'(0)+\frac12 f''(\theta u)u|}.\tag{5.4}$$

Take $r>0$ sufficiently small such that

$$\frac12|f'(0)|\le|f'(0)+\frac12 f''(\theta u)u|\le\frac32|f'(0)|\quad for\quad u\in[-r,0],\tag{5.5}$$

then by combining (5.4), we derive

$$\frac{2Ae^{-Ar}}{3|f'(0)|}\le w(u)\le\frac{2A}{|f'(0)|}.\tag{5.6}$$

Next, we consider the interval $[u_-,-r]$. By the definition of the weight function $w(u)$, we have

$$\frac{-e^{Ar}+1}{\max_{u_-\le u\le -r}|f(u)|}\le w(u)\le\frac{-e^{-Au_-}+1}{\min_{u_-\le u\le -r}|f(u)|}.\tag{5.7}$$

This complete the proof of (i)

Proof of (iii). We also divide the interval $[u_-,0]$ into $[u_-,-r]$ and $[-r,0]$. We prove $(|f'(w)|^2/w^2)<1$ which is enough to derive the inequality (iii). We first consider the interval [-r,0]. Around the origin, $f(u)$ is written as $f(u)=f'(0)u^n+\frac12 f''(\theta u)u^2$. It follows from the definition of the weight function that

$$\begin{aligned}\left|\frac{(fw)'}{w}\right| = \left|\frac{-f(u)Ae^{Au}}{-e^{Au}+1}\right| &= \frac{|(f'(0)u+\frac{1}{2}f''(\theta u)u^2)Ae^{Au}|}{|Ae^{A\theta u}u|}\\ &= \left|f'(0)+\frac{f''(\theta u)}{2}u\right|e^{Au(1-\theta)} \\ &\le \left|f'(0)\right| + \left|\frac{f^{(n+1)}(\theta u)}{(n+1)!}\right|r.\end{aligned} \tag{5.8}$$

By the sub-characteristic condition $|f'(0)| < 1$ and $f'(0) > 0$, we can choose r sufficiently small such that

$$\left|f'(0)\right| + \left|\frac{f''(\theta u)}{2}\right|r < 1. \tag{5.9}$$

For $u \in [r, u_-]$, we have

$$\left|\frac{(fw)'}{w}\right| = \left|\frac{-f(u)Ae^{Au}}{-e^{Au}+1}\right| \le \frac{(\max_{u_-\le u\le r}|f(u)|)Ae^{Au}}{-e^{Au}+1} = \frac{MA}{-1+e^{-Au}}. \tag{5.10}$$

where $M := \max_{u_-\le u\le r}|f(u)|$. Hence, taking A sufficiently large, we can make

$$\frac{MA}{-1+e^{-Au}} < 1. \tag{5.11}$$

Thus the proof of (iii) and Lemma 5.3 is completed. □

Now, we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. First, put

$$N(T) = \sup_{0<t<T}\left(\|z(t)\|_{H^2} + \|z_t(t)\|_{H^1}\right),$$

and then we suppose $N(T) < 1$ throughout this section. The first equality of (2.6) is rewrited as

$$z_{tt} - z_{xx} + f'(\phi)z_x + z_t = F, \tag{5.12}$$

where

$$F = -(f(\phi+z_x) - f(\phi) - f'(\phi)z_x) = O(|z_x|^2).$$

Multiplying (5.12) by $w(\phi)z$, then we get

$$\begin{aligned}&(\frac{1}{2}w(\phi)z^2 + w(\phi)z_tz)_t - w(\phi)z_t^2 + (\frac{1}{2}f'(\phi)w(\phi)z^2 - w(\phi)zz_x + \frac{1}{2}w'(\phi)\phi_xz^2)_x \\ &-\frac{1}{2}(f''(\phi)w(\phi) + 2f'(\phi)w'(\phi) + f(\phi)w''(\phi))\phi_xz^2 + w(\phi)z_x^2 = O(|z_x^2|)w(\phi)z,\end{aligned} \tag{5.13}$$

where we use the fact that

$$f'(\phi)z_xw(\phi)z = (\frac{1}{2}f'(\phi)w(\phi)z^2)_x - \frac{1}{2}(f''(\phi)w(\phi) + f'(\phi)w'(\phi))\phi_xz^2, \tag{5.14}$$

and

$$\begin{aligned}&-z_{xx}w(\phi)z = -(z_xw(\phi)z)_x + w'(\phi)\phi_x(\frac{1}{2}z^2)_x + w(\phi)z_x^2 \\ &= -(z_xw(\phi)z - \frac{1}{2}w'(\phi)\phi_xz^2)_x - \frac{1}{2}(w''(\phi)f(\phi) + w'(\phi)f'(\phi))\phi_xz^2 + w(\phi)z_x^2.\end{aligned} \tag{5.15}$$

Next, multiplying (5.12) by $w(\phi)z_t$, then we have

$$\begin{aligned}&\frac{1}{2}(w(\phi)z_t^2 + w(\phi)z_x^2)_t - (z_x w(\phi)z_t)_x + w(\phi)z_t^2 \\ &\qquad\qquad +f'(\phi)w(\phi)z_x z_t + w'(\phi)\phi_x z_x z_t = O(|z_x^2|)w(\phi)z_t.\end{aligned} \tag{5.16}$$

We make a combination (5.13) $+2\times$(5.16), then we have the following equality

$$E_t + D + F_x = R, \tag{5.17}$$

where

$$\begin{aligned}&E = w(\phi)z_t^2 + w(\phi)z_x^2 + w(\phi)z_t z + \frac{1}{2}w(\phi)z^2, \\ &D = w(\phi)z_t^2 + 2(fw)'(\phi)z_t z_x + w(\phi)z_x^2 - \frac{1}{2}(fw)''(\phi)\phi_x z^2, \\ &F = \frac{1}{2}(fw)'(\phi)z^2 - w(\phi)zz_x - 2w(\phi)z_x z_t, \\ &R = O(|z| + |z_t|)z_x^2.\end{aligned} \tag{5.18}$$

Consider E as the quadratic form in terms of z, z_x and z_t, then calculating the discriminant of E and using Lemmas 5.3, then we have the condition

$$E \sim w(\phi)\cdot(z^2 + z_x^2 + z_t^2). \tag{5.19}$$

In a similar way, calculating the discriminant of E and using Lemmas 5.3, then we have

$$D \sim z_x^2 + z_t^2 + \phi_x z^2. \tag{5.20}$$

Therefore, taking $N(T)$ suitably small, putting (5.19) and (5.20) into (5.17), and integrating it over $(0,\infty)\times(0,t)$ with respect to x and t , we have

$$\begin{aligned}&\int_0^\infty w(\phi)\cdot(z^2 + z_t^2 + z_x^2)(t)\,dx + \int_0^t\int_0^\infty (z_x^2 + z_t^2 + \phi_x z^2)(\tau)\,dxd\tau \\ &\qquad + \int_0^t z(0,s)^2 ds \le C(\|z_0\|_{H_w^1}^2 + \|z_1\|_{L_w^2}^2),\end{aligned}$$

that is

$$\begin{aligned}&\|z\|_{H_w^1}^2 + \|z_t\|_{L_w^2}^2 + \int_0^t (\|\sqrt{\phi_x}z(\tau)\|_{L^2}^2 + \|z_x(\tau)\|_{L^2}^2 + \|z_t(\tau)\|_{L^2}^2)\,d\tau \\ &\le C(\|z_0\|_{H_w^1}^2 + \|z_1\|_{L_w^2}^2).\end{aligned} \tag{5.21}$$

Next, we proceed to the estimates of z_x. Noting that $z_x = v$, we calculate L^2-estimate of $v(x,t)$ for the estimates of z_x. Multiplying (2.6) by $w(\phi)v$, then making use of the equality

$$\begin{aligned}&\{f(\phi+v) - f(\phi)\}_x w(\phi)v \\ =&\{(f(\phi+v) - f(\phi))w(\phi)v\}_x - (f(\phi+v) - f(\phi))(wv)_x \\ =&\left\{(f(\phi+v) - f(\phi))w(\phi)v - w(\phi)\int_0^v f(\phi+\eta) - f(\phi)d\eta\right\}_x \\ &+ \phi_x\left\{w'(\phi)\int_0^v f(\phi+\eta) - f(\phi)d\eta + w(\phi)\int_0^v f'(\phi+\eta) - f'(\phi)d\eta\right\} \\ &- (f(\phi+v) - f(\phi))w(\phi)_x v,\end{aligned}$$

we obtain

$$
\begin{aligned}
&(w(\phi)vv_t+\frac{1}{2}w(\phi)v^2)_t-w(\phi)v_t^2-\frac{1}{2}\big(w''(\phi)f+w'(\phi)f'(\phi)\big)\phi_x v^2+w(\phi)v_x^2\\
&+\phi_x\left\{w'\int_0^v f(\phi+\eta)-f(\phi)d\eta+w\int_0^v f'(\phi+\eta)-f'(\phi)d\eta\right\}\\
&\qquad\qquad\qquad+\mathcal{F}_x-(f(\phi+v)-f(\phi))w(\phi)_x v,
\end{aligned}
\tag{5.22}
$$

where

$$
\mathcal{F}=-w(\phi)vv_x+\frac{1}{2}w(\phi)_x v^2+(f(\phi+v)-f(\phi))w(\phi)v-w(\phi)\int_0^v f(\phi+\eta)-f(\phi)d\eta.
$$

Next, multiplying (2.6) by $w(\phi)v_t$ and using the equality

$$
\begin{aligned}
&\{f(\phi+v)-f(\phi)\}_x w(\phi)v_t=(f'(\phi+v)-f'(\phi))\phi_x w(\phi)v_t+f'(\phi+v)v_x wv_t\\
&=\left\{w(\phi)\phi_x\int_0^v f'(\phi+\eta)-f'(\phi)d\eta\right\}_t+f'(\phi+v)v_x w(\phi)v_t,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}(w(\phi)v_t^2+w(\phi)v_x^2)_t-(w(\phi)v_tv_x)_x+w(\phi)v_t^2+w(\phi)f'(\phi+v)v_tv_x\\
&+w'(\phi)f(\phi)v_xv_t+\left\{w(\phi)\phi_x\int_0^v f'(\phi+\eta)-f'(\phi)d\eta\right\}_t=0.
\end{aligned}
\tag{5.23}
$$

We make a combination (5.22)+2×(5.23). This yields the differential equality

$$
(\bar{E}+\bar{R}_1)_t+\bar{D}+\bar{F}_x=\bar{R}_1+\bar{R}_2+\bar{R}_3,
\tag{5.24}
$$

where $\bar{E}$, $\bar{D}$, $\bar{F}$, $\bar{R}_1$ and $\bar{R}_2$ are defined by

$$
\begin{aligned}
\bar{E}&=w(\phi)(vv_t+\frac{1}{2}v^2+v_t^2+v_x^2),\\
\bar{D}&=w(\phi)(v_x^2+v_t^2)+2(fw)'(\phi)v_tv_x,\\
\bar{\mathcal{F}}&=-\frac{1}{2}w(\phi)_x v^2-w(\phi)(vv_x+2v_tv_x)\\
&\qquad+w(\phi)\big(f(\phi+v)-f(\phi)\big)v-w(\phi)\int_0^v f(\phi+\eta)-f(\phi)d\eta,\\
\bar{R}_1&=w(\phi)\phi_x\int_0^v f'(\phi+\eta)-f'(\phi)d\eta,\\
\bar{R}_2&=O(1)\phi_x\tilde{v}^2,\\
\bar{R}_3&=O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x.
\end{aligned}
$$

We consider $\bar{E}$ and $\bar{D}$ as quadratic forms in terms of v, v_x and v_t. Then calculating the discriminant of $\bar{E}$ and $\bar{D}$, and using Lemma 5.3, we have the condition

$$
\bar{E}\sim w(\phi)\cdot(v^2+v_x^2+v_t^2),\quad \bar{D}\sim v_x^2+v_t^2.
\tag{5.25}
$$

Noting the fact that $z_x=v$, we see from (5.21) that

$$
\int_0^t\int_0^\infty|\bar{R}_1+\bar{R}_2|dxd\tau\le C(\|z_0\|_{H_w^1}^2+\|z_1\|_{L_w^2}^2).
\tag{5.26}
$$

Then, integrating (5.24) with respect to x and t over $(0,\infty)\times(0,t)$, and substituting

(5.25) and (5.26), we have

$$\begin{aligned}&\|v(t)\|^2_{L^2_w}+\|v_t(t)\|^2_{L^2_w}+\|v_x(t)\|^2_{L^2_w}+\int_0^t(\|v_x\|^2_{L^2}+\|v_t\|^2_{L^2})(\tau)\,d\tau\\&\le C(\|v_0\|^2_{H^1_w}+\|v_1\|^2_{L^2_w}+\|z_0\|^2_{H^1_w}+\|z_1\|^2_{L^2_w}),\end{aligned}\tag{5.27}$$

for suitably small $N(T)$. By the definition of $z(x,t)$, (5.27) means

$$\begin{aligned}&\|z_x(t)\|^2_{H^1_w}+\|z_{xt}(t)\|^2_{L^2_w}+\int_0^t(\|z_{xx}\|^2_{L^2}+\|z_{x,t}\|^2_{L^2})(\tau)\,d\tau\\&\le C(\|z_0\|^2_{H^2_w}+\|z_1\|^2_{H^1_w}).\end{aligned}\tag{5.28}$$

Combining (5.28) with (5.21), we finally have (5.2). Thus the proof of the Proposition 5.2 is completed. □

Proof of Theorem 2.4-(i). The global existence of solutions to the initial-boundary value problem (1.1) can be proved by the continuation argument based on a local existence result in Proposition 5.1 combined with the corresponding *a priori* estimate in Proposition 5.2. We omit the detail. □

5.2 Convergence rates of stationary solutions

In this section, we prove Theorem 2.4-(ii), (iii). The main idea of the proofs are due to Ueda [7]. We use the space-time weighted energy method used in Section 3 and 4.

Proof of Theorem 2.4-(ii). We start with the energy equality (5.17). Applying Lemma 5.3 to $\bar{F}$, we calculate $\bar{F}$ as

$$\begin{aligned}-\bar{F}&=-\frac{1}{2}(fw)'(\phi)z^2-w(\phi)(zz_x+2z_tz_x)\\&\ge cz^2-C(z_x^2+z_t^2),\end{aligned}\tag{5.29}$$

where c and C are positive constants.

Let γ and β be any positive constants satisfying $0\le\gamma,\beta\le\alpha$. We multiply the equality (5.17) by $(1+t)^\gamma(1+x)^\beta$, obtaining

$$\begin{aligned}&\{(1+t)^\gamma(1+x)^\beta E\}_t-\gamma(1+t)^{\gamma-1}(1+x)^\beta E+(1+t)^\gamma(1+x)^\beta D\\&+\{(1+t)^\gamma(1+x)^\beta F\}_x-\beta(1+t)^\gamma(1+x)^{\beta-1}F=(1+t)^\gamma(1+x)^\beta R.\end{aligned}\tag{5.30}$$

Substituting (5.29) into (5.30), integrating the resultant inequality over $\mathbb{R}_+\times(0,t)$, we have

$$\begin{aligned}&(1+t)^\gamma\|(z,z_t,z_x)(t)\|^2_{L^2_\beta}+\int_0^t(1+\tau)^\gamma\big(\|(z_t,z_x,\sqrt{\phi_x}z)(\tau)\|^2_{L^2_\beta}+\beta\|z(\tau)\|^2_{L^2_{\beta-1}}\big)d\tau\\&\le CE_\beta^2+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|(z,z_t,z_x)(\tau)\|^2_{L^2_\beta}d\tau,\end{aligned}\tag{5.31}$$

for an arbitrary γ and β with $0\le\gamma$, $\beta\le\alpha$, where C is a constant independent of γ and β. For the above estimate, applying the induction argument, we can obtain the following estimate.

$$
\begin{aligned}
&(1+t)^l\|(z,z_t,z_x)(t)\|^2_{L^2_{\alpha-l}}\\
&+\int_0^t(1+\tau)^l\left(\|(z_t,z_x,\sqrt{\phi_x}z)(\tau)\|^2_{L^2_{\alpha-l}}+(\alpha-l)\|z(\tau)\|^2_{L^2_{\alpha-l-1}}\right)d\tau\le CE^2_\alpha,
\end{aligned}
\tag{5.32}
$$

for any integer l with $0\le l\le\alpha$. Then, we have

$$
\|(z,z_x,z_t)(t)\|^2_{L^2}\le CE^2_\alpha(1+t)^{\gamma-\alpha}. \tag{5.33}
$$

For detail of the proof, we refer the readers to [6, 7].

Next, we proceed to the higher order estimate. We multiply the equality (5.24) by $(1+t)^\gamma(1+x)^\beta$ and integrating the resultant equality over $\mathbb{R}_+\times(0,t)$, obtaining

$$
\begin{aligned}
&(1+t)^\gamma\|(v,v_t,v_x)(t)\|^2_{L^2_\beta}+\int_0^t(1+\tau)^\gamma\left(\|(v_t,v_x,\sqrt{\phi_x}v)(\tau)\|^2_{L^2_\beta}\right)d\tau\\
&\le CE^2_\beta+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|(v,v_t,v_x)(\tau)\|^2_{L^2_\beta}d\tau+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|(z,z_t,z_x)(\tau)\|^2_{L^2_\beta}d\tau,
\end{aligned}
\tag{5.34}
$$

here we use (5.31), that is

$$
\begin{aligned}
&(1+t)^\gamma\int_0^\infty(1+x)^\beta\phi_x v^2dx+\int_0^t(1+\tau)^\gamma\int_0^\infty(1+x)^\beta\phi_x v^2dxd\tau\\
&\le CE_\beta+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|(z,z_t,z_x)(\tau)\|^2_{L^2_\beta}d\tau.
\end{aligned}
$$

Applying the induction argument and combining the estimate (5.32), we obtain the following estimate.

$$
(1+t)^l\|(v,v_t,v_x)(t)\|^2_{L^2_{\alpha-l}}+\int_0^t(1+\tau)^l\|(v_t,v_x)(\tau)\|^2_{L^2_{\alpha-l}}d\tau\le CE^2_\alpha,
$$

for any integer l with $0\le l\le\alpha$. Then, using the same strategy as in [6, 7], we have

$$
\|(v,v_x,v_t)(t)\|^2_{L^2}\le CE^2_\alpha(1+t)^{\gamma-\alpha}. \tag{5.35}
$$

Combining (5.33) and (5.35), we have the desired estimate in Theorem 2.4-(ii), and complete the proof. □

Finally, we prove Theorem 2.4-(iii) by using the space-time weighted energy method.

Proof of Theorem 2.4-(iii). Let α, $\beta>0$. Multiplying (5.17) by $e^{\beta t}e^{\alpha x}$, we obtain

$$
\begin{aligned}
&\{e^{\beta t}e^{\alpha x}(\bar{E}+\bar{R}_1)\}_t-\beta e^{\beta t}e^{\alpha x}(\bar{E}+\bar{R}_1)+e^{\beta t}e^{\alpha x}\bar{D}\\
&\qquad+\{e^{\beta t}e^{\alpha x}\bar{F}\}_x-\alpha e^{\beta t}e^{\alpha x}\bar{F}=e^{\beta t}e^{\alpha x}(\bar{R}_1+\bar{R}_2).
\end{aligned}
\tag{5.36}
$$

Substituting (5.29) into (5.36), integrating the resultant inequality over $\mathbb{R}_+\times(0,t)$, we get

$$
\begin{aligned}
&e^{\beta t}\|(z_t,z_x,z)(t)\|^2_{L^2_{\alpha,exp}}+\int_0^t e^{\beta\tau}\|(z_t,z_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau+\alpha\int_0^t e^{\beta\tau}\|z(\tau)\|^2_{L^2_{\alpha,exp}}d\tau\\
&\le CE^2_{\alpha,exp}+\beta C_0\int_0^t e^{\beta\tau}\|z(\tau)\|^2_{L^2_{\alpha,exp}}d\tau+(\alpha+\beta)C_1\int_0^t e^{\beta\tau}\|(z_t,z_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau,
\end{aligned}
$$

where C_0, C_1 and C are positive constants independent of α and β. Taking $\alpha>0$ and

$\beta > 0$ suitably small such that $\beta C_0 \leq \alpha$ and $(\alpha+\beta)C_1 \leq 1$, we obtain

$$e^{\beta t}\|(z_t, z_x, z)(t)\|^2_{L^2_{\alpha,exp}} + \int_0^t e^{\beta\tau}\|(z_t, z_x)(\tau)\|^2_{L^2_{\alpha,exp}} d\tau + \alpha\int_0^t e^{\beta\tau}\|z(\tau)\|^2_{L^2_{\alpha,exp}} d\tau \leq CE^2_{\alpha,exp} \tag{5.37}$$

Next, we proceed to the higher order estimate. We multiply the equality (5.24) by $e^{\beta t}e^{\alpha x}$ and integrating the resultant equality over $\mathbb{R}_+ \times (0,t)$, obtaining

$$\begin{aligned} &e^{\beta t}\|(v_t, v_x, v)(t)\|^2_{L^2_{\alpha,exp}} + \int_0^t e^{\beta\tau}\|(v_t, v_x)(\tau)\|^2_{L^2_{\alpha,exp}} d\tau \\ &\leq C(E^2_{\alpha,exp} + \bar{E}^2_{\alpha,exp}) + (\alpha+\beta)C_1\int_0^t e^{\beta\tau}\|(v_t, v_x)(\tau)\|^2_{L^2_{\alpha,exp}} d\tau, \end{aligned} \tag{5.38}$$

here we use the fact that $v = z_x$ and (5.37), that is

$$e^{\beta t}\|v(t)\|_{L^2_{\alpha,exp}} + \int_0^t e^{\beta\tau}\|(v)(\tau)\|^2_{L^2_{\alpha,exp}} d\tau \leq CE^2_{\alpha,exp}. \tag{5.39}$$

Taking $\alpha > 0$ and $\beta > 0$ suitably small such that $\beta C_0 \leq \alpha$ and $(\alpha+\beta)C_1 \leq 1$ in (5.38), and combining with (5.37), then we have the desired estimate in Theorem 2.4-(iii), and complete the proof. □

References

[1] I. Hashimoto and A. Matsumura: *Large time behavior of solutions to an initial boundary value problem on the half line for scalar viscous conservation law*, Methods Appl. Anal. **14** (2007), 45-60.

[2] S. Kawashima and A. Matsumura: *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys. **101**(1985), 97-127.

[3] T.-P. Liu, A. Matsumura and K. Nishihara: *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal. **29** (1998), 293-308.

[4] T.-P. Liu and K. Nishihara: *Asymptotic behavior for scalar viscous conservation laws with boundary effect*, J. Differential Equations **133** (1997), 296-320.

[5] A. Matsumura and M. Mei: *Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity*, Osaka J. Math. **34**(1997), 589-603.

[6] M. Nishikawa: *Convergence rate to the traveling wave for viscous conservation laws*, Funkcial. Ekvac. **41** (1998), 107-132.

[7] Y. Ueda: *Asymptotic stability of stationary waves for damped wave equations with a nonlinear convection term*, Adv. Math. Sci. Appl. **18** (2008), 329-343.

[8] Y. Ueda, S. Kawashima: *Large time behavior of solutions to a semilinear hyperbolic system with relaxation*, J. Hyperbolic Differ Equ. **4** (2007), 147-179.

[9] Y. Ueda, N. Tohru, S. Kawashima: *Stability of degenerate stationary waves for viscous gases*, to appear in Archive for Rational Mechanics and Analysis.