

# On the structure of the first homology of the group of equivariant diffeomorphisms of manifolds with smooth torus actions

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## §1. Preliminaries

In the previous paper [AF5] we calculate the first homology of the group of equivariant diffeomorphisms of representation spaces of finite groups, and apply to the case of the smooth orbifolds. In this talk we shall consider the case of smooth manifolds with smooth torus actions. First we describe the previous results.

For a finite group  $G$ , let  $M$  be a smooth connected  $G$ -manifold. Let  $\mathcal{D}_G(M)$  denote the group of  $G$ -equivariant smooth diffeomorphisms of  $M$  which are  $G$ -isotopic to the identity through the isotopies with compact support. We recall the result the case where  $M$  is a finite dimensional  $G$ -module  $V$ . Let  $V^G$  be the subspace of the fixed point set of  $V$ . Let  $Aut_G(V)$  denote the set of  $G$ -invariant automorphisms of  $V$  and  $Aut_G(V)_0$  the identity component of  $Aut_G(V)$ . Then we have the following.

### Theorem 1.1

- (1) If  $\dim V^G > 0$ , then  $\mathcal{D}_G(V)$  is perfect.
- (2) If  $\dim V^G = 0$ , then  $H_1(\mathcal{D}_G(V)) \cong H_1(Aut_G(V)_0)$ .

We can decompose  $V = \bigoplus_{i=1}^d k_i V_i$ , where  $V_i$  runs over the inequivalent irreducible representation space of  $G$  and  $k_i$  is a positive integer.

$End_G(V_i)$ : the set of  $G$ -invariant endomorphisms of  $V_i$

Then  $\dim End_G(V_i) = 1, 2$  or  $4$ .

**Corollary 1.2**

$$H_1(\mathcal{D}_G(V)) \cong \mathbf{R}^d \times \overbrace{U(1) \times \cdots \times U(1)}^{d_2},$$

where  $d_2$  is the number of  $V_i$  with  $\dim \text{End}_G(V_i) = 2$ .

If  $M$  is a smooth orbifold, then  $p \in M$  is said to be an isolated singular point of  $M$  if there exists a local chart  $(U_i, \phi_i)$  around  $p$  such that  $\tilde{p}$  is an isolated fixed point of  $(\Gamma_i, \tilde{U}_i)$  with  $\pi_i(\tilde{p}) = p$ . Let  $\phi_i : U_i \rightarrow \tilde{U}_i/\Gamma_i$ ,  $\pi_i : \tilde{U}_i \rightarrow U_i$  be the canonical projection. Then we have.

**Theorem 1.3** *If a smooth orbifold  $M$  has  $\{p_1, \dots, p_k\}$  as the isolated singular point set. Let  $(\Gamma_i, V_{p_i})$  ( $1 \leq i \leq k$ ) be the representation space associated to the isolated singular points. Then*

$$H_1(\mathcal{D}(M)) \cong H_1(\text{Aut}_{\Gamma_{p_1}}(V_{p_1})_0) \times \cdots \times H_1(\text{Aut}_{\Gamma_{p_k}}(V_{p_k})_0).$$

## §2. Orbit preserving $G$ -diffeomorphisms

Let  $M$  be a connected closed  $G$ -manifold and  $B$  be the orbit space. Then the natural projection  $\pi : M \rightarrow B$  induces the group homomorphism  $P : D_G(M) \rightarrow D(B)$ .

Let  $(H_0)$  be the principal orbit type of  $M$  and let  $\{(H_i) \mid 0 \leq i \leq \ell\}$  be the other  $G$ -orbit types of  $M$ . Put

$$M_i = \{p \in M \mid (G_p) = (H_i)\},$$

$$W_i = N(H_i)/H_i, \quad F_i = M_i^{H_i}, \quad B_i = F_i/W_i.$$

Then  $q_i : F_i \rightarrow B_i$  is the principal  $W_i$ -bundle and  $\pi_i : M_i \rightarrow B_i$  is the associated fiber bundle with the fiber  $G/H_i$ . Thus

$$M_i \cong G/H_i \times_{W_i} F_i.$$

Let  $\{U_{i,\alpha}\}_{(i,\alpha) \in \Lambda_i}$  be an open covering of  $B_i$  such that there exists a local section  $\sigma_{i,\alpha}$  on  $U_{i,\alpha}$  of  $q_i$ . Then we have the transition functions  $\{\varphi_{i,\alpha\beta}\}$  of the principal  $W_i$ -bundle  $q_i$  given by

$$\varphi_{i,\alpha\beta}(b)\sigma_{i,\beta}(b) = \sigma_{i,\alpha}(b) \quad (b \in U_{i,\alpha} \cap U_{i,\beta}).$$

We shall study the group  $\text{Ker } P$  which coincides with the group of orbit preserving equivariant diffeomorphisms of  $M$ .

Let  $h \in \text{Ker } P$ . Then  $h$  induces the bundle isomorphisms  $h_i$  of  $\pi_i$  ( $0 \leq i \leq \ell$ ). We have smooth maps

$$s_{i,\alpha} : U_{i,\alpha} \rightarrow W_i \quad ((i, \alpha) \in \Lambda_i) \quad \text{satisfying}$$

$$h_i(\sigma_{i,\alpha}(b)) = s_{i,\alpha}(b)\sigma_{i,\alpha}(b) \quad (b \in U_{i,\alpha}).$$

It follows that, for  $b \in U_{i,\alpha} \cap U_{i,\beta}$ , we have

$$s_{i,\alpha}(b)\varphi_{i,\alpha\beta}(b) = \varphi_{i,\alpha\beta}(b)s_{i,\beta}(b).$$

Then  $h_i$  corresponds to the collections  $S_i(h) = \{s_{i,\alpha}\}_{(i,\alpha) \in \Lambda_i}$  satisfying

$$(3.1) \quad s_{i,\alpha} \cdot \varphi_{i,\alpha\beta} = \varphi_{i,\alpha\beta} \cdot s_{i,\beta} \quad \text{on } U_{i,\alpha} \cap U_{i,\beta}.$$

Put  $S(h) = \{S_i(h) \mid 0 \leq i \leq \ell\}$ .

**Definition 2.1** *A cocycle of an orbit preserving  $G$ -diffeomorphism of  $M$  is a collection  $S = \{S_i \mid 1 \leq i \leq \ell\}$  such that*

(1)  $S_i$  is the set of smooth functions  $\{s_{i,\alpha}\}_{(i,\alpha) \in \Lambda_i}$  from  $U_{i,\alpha}$  to  $W_i$  satisfying the condition (3.1),

(2) Let  $V$  be a slice at  $p \in M$ . Then  $\psi_V : V \rightarrow G \cdot V$  is a smooth map. Here the map  $\psi_V$  is given by  $\psi_V(v) = s_{i,\alpha}(\pi(v)) \cdot v$  if  $\pi(v) \in U_{i,\alpha}$ .

By definition  $S(h)$  is a cocycle of an orbit preserving  $G$ -diffeomorphism of  $M$  for each  $h \in \text{Ker } P$ . Let  $S = \{S_i \mid 1 \leq i \leq \ell\}$  be a cocycle of an orbit preserving  $G$ -diffeomorphism of  $M$ . Then we have a map  $h : M \rightarrow M$  defined by

$$h([gH_i, x]) = [g \cdot s_{i,\alpha}(q_i(x)), x]$$

for  $g \in G$ ,  $x \in q_i^{-1}(U_{i,\alpha})$ .

**Lemma 2.2**  $h \in \text{Ker } P$ .

Let  $S(M)$  be the set of all cocycles of an orbit preserving  $G$ -diffeomorphism of  $M$ .

**Corollary 2.3** *Let  $S : \text{Ker } P \rightarrow S(M)$  be a map which assigns each  $h \in \text{Ker } P$  to  $S(h)$ . Then  $S$  is bijective.*

**Remark 2.4** *M. Davis [DA] introduced the  $G$ -normal system of smooth  $G$ -manifolds to classify the set of  $G$ -manifolds and suggested that the orbit preserving  $G$ -diffeomorphisms are expressed by using this system. We can express them more easy way by using the above cocycles.*

Let  $M(m, \mathbf{R})$  denote the set of all  $m \times m$ -matrices. Let  $f : \mathbf{R}^n \setminus \mathbf{R}^m \rightarrow M(m, \mathbf{R})$  be a smooth map. Define a map  $\hat{f} : \mathbf{R}^n \rightarrow M(m, \mathbf{R})$  by

$$\hat{f}(x, y) = \begin{cases} f(x, y)x & x \in \mathbf{R}^m, y \in \mathbf{R}^{n-m} (y \neq 0) \\ 0 & x \in \mathbf{R}^m, y = 0. \end{cases}$$

**Lemma 2.5** *If  $\hat{f}$  is a smooth map, then  $f$  can be extended to a smooth map  $F : \mathbf{R}^n \rightarrow M(m, \mathbf{R})$ .*

If  $H_i$  is a subgroup of  $H_j$ , let  $r_{i,j} : G/H_i \rightarrow G/H_j$  be the canonical projection.

**Corollary 2.6** *Let  $h \in \text{Ker } P$ . Assume that  $H_i$  is a subgroup of  $H_j$  and  $U_{j,\beta}$  is contained in the closure  $\overline{U_{i,\alpha}}$  of  $U_{i,\alpha}$ . Then  $s_{i,\alpha}$  is extended to a map  $\bar{s}_{i,\alpha}$  on  $\overline{U_{i,\alpha}}$  satisfying*

$$r_{i,j}(\bar{s}_{i,\alpha}(b)) = s_{j,\beta}(b) \quad \text{for } b \in U_{j,\beta}.$$

**Example 2.7**

(1) *Assume that  $q_0 : F_0 \rightarrow B_0$  is a trivial  $W_0$ -bundle. It follows from Corollary 2.6 that each  $h \in \text{Ker } P$  corresponds to a smooth map  $s : B \rightarrow W_0$  satisfying the following condition. If  $b \in B_i$  ( $1 \leq i \leq \ell$ ), then*

$$s(b) \in r_{0i}^{-1}(W_i) = (N(H_0) \cap N(H_i))/H_0.$$

(2) *If  $M$  is a  $(2n - 1)$ -dimensional homotopy sphere with a smooth  $O(n)$ -action with  $B = D^2$ . Then*

$$H_0 = O(n - 2), \quad H_1 = O(n - 1), \quad W_0 = O(2), \quad W_1 = O(1).$$

*Thus  $\text{Ker } P$  is one to one correspondence with the smooth maps from  $s : D^2 \rightarrow SO(2)$  such that  $s(\partial D^2) = 1$ .*

**Example 2.8** *Let  $M$  be a  $2n$ -dimensional torus manifold with the local standard action. Note that  $N(H)/H = T^n/H \cong H^c$  for each toral subgroup  $H$  in  $T^n$ , where  $H^c$  is the complementary torus subgroup of  $H$ . Let  $\mathcal{F}(M)$  be the set of smooth maps  $s : B \rightarrow T^n$  such that  $s(\pi(p)) \in (T_p^n)^c$  for each  $p \in M$ . Then  $\text{Ker } P$  is isomorphic to  $\mathcal{F}(M)$  as a group.*

Let  $V$  be a slice at  $p \in M$  with  $\pi(p) \in U_{i,\alpha}$ . Let  $P_V : D_G(G \cdot V) \rightarrow D(G \cdot V/G)$  be the natural group homomorphism. Note that  $\dim U_{i,\alpha} = \dim F_i/W_i$ .

**Proposition 2.9**

*If  $\dim U_{i,\alpha} > 0$ , then*

$$\text{Ker } P_V \subset [\text{Ker } P_V, D_G(G \cdot V)].$$

**§3. Application to torus actions**

$M$ :  $2n$ -dimensional torus manifold with local standard action  
Let  $p$  be a fixed point of  $M$ . Let  $\text{Aut}_{T^n}(T_p(M))$  denote the set of  $T^n$ -equivariant linear automorphisms of  $T_p(M)$ . We have a group homomorphism  $\Phi_p : D_G(M) \rightarrow \text{Aut}_{T^n}(T_p(M))_0$  assigning each  $h \in D_G(M)$  to the differential  $dh_p$  at  $p$ . Set the homomorphism

$$\Phi = \{\Phi_p\} : D_G(M) \rightarrow \prod_{p \in F(M)} \text{Aut}_{T^n}(T_p(M))_0.$$

Here  $F(M)$  is the fixed point set of  $M$ .

Since  $T_p(M)$  is the standard representation space of  $T^n$ ,  $\text{Aut}_{T^n}(T_p(M))_0$  is isomorphic to  $(\mathbf{C}^*)^n$ . Define the group homomorphism

$$\Theta = (P, \Phi) : D_{T^n}(M) \rightarrow D(M/T^n) \times \prod_{p \in F(M)} (\mathbf{C}^*)^n.$$

Since  $M/T^n$  has a structure of an orbifold and is locally diffeomorphic to  $\tilde{\mathbf{R}}^n/\mathbf{Z}_2^n$  around the isolated singular point of  $M/T^n$ , where  $\tilde{\mathbf{R}}$  is the non-trivial 1-dimensional  $\mathbf{Z}_2$ -module. By Corollary 1.2 we have.

**Corollary 3.1** *If  $M$  has  $m$  fixed points, then  $H_1(D(M/T^n)) \cong \mathbf{R}^{mn}$ .*

**Proposition 3.2**  *$\text{Ker } \Theta$  is contained in the commutator subgroup of  $D_{T^n}(M)$ .*

There exists the following group exact sequence.

$$\text{Ker } \Theta / [\text{Ker } \Theta, D_{T^n}(M)] \xrightarrow{\iota_*} H_1(D_{T^n}(M)) \xrightarrow{\Theta_*} H_1(D_{T^n}(M) \times \prod_{p \in F_n} (\mathbf{C}^*)^n) \rightarrow 1.$$

By Proposition 3.2  $\iota_* = 0$  and  $\Theta_*$  is isomorphic. From Corollary 3.1 we have.

**Theorem 3.3** *Let  $M$  be a  $2n$ -dimensional torus manifold with local standard action. If  $M$  has  $m$  fixed points, then*

$$H_1(D_{T^n}(M)) \cong (\mathbf{R} \times \mathbf{C}^*)^{mn}.$$

In order to prove Proposition 3.2, we need the following lemmas.

**Lemma 3.4** (*Fragmentation lemma*)

*Let  $M$  be a smooth  $G$ -manifold and  $\{V_i \mid 1 \leq i \leq n\}$  be a  $G$ -invariant finite open covering of  $M$ . Let  $N$  be an open neighborhood of the identity in  $D_G(M)$ . Then there exists an open neighborhood  $N_0 \subset N$  of the identity with the following properties: For any  $f \in N_0$ , there exist  $f_i \in N$ ,  $1 \leq i \leq n$ , such that*

a)  $f_i$  is  $G$ -isotopic to the identity through an equivariant  $C^\infty$  isotopy whose support is contained in  $V_i$ , and

b)  $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ .

**Theorem 3.5** (Bierstone [BI1], Schwarz [SC2])

*Let  $N$  be a smooth  $G$ -manifold. Then each smooth isotopy on  $N/G$  with compact support lifts to a smooth  $G$ -equivariant isotopy on  $N$ .*

*Proof of Proposition 3.2 :*

Combining Fragmentation lemma and Theorem 3.5, the proof of Proposition 3.2 is reduced to the case  $M = T^n \cdot V$ , where  $V$  is a slice of a point  $p \in M$ . Then  $M = T^n \cdot V \cong T^n \times_{H_p} V$ . Let  $P_V : T^n \cdot V \rightarrow V/H_p$  be the natural projection. Then it is enough to prove that

$$\text{Ker } \Theta \subset [\text{Ker } P_V, D_{T^n}(T^n \cdot V)].$$

By Proposition 2.9, if  $p$  is not a isolated fixed point of  $T^n$ , then Proposition 3.2 is valid. Assume that  $p$  is a isolated fixed point of  $M$ .

Let  $T^n \cdot V = V$  and  $\Theta$  be the composition

$$\Theta = (P_V, \Phi_p) : D_{T^n}(V) \rightarrow D(V/T^n) \times \text{Aut}_{T^n}(T_p(M))_0 \cong D(V/T^n) \times (\mathbf{C}^*)^n.$$

Let  $h \in \text{Ker } \Theta$ . Then  $h$  is an orbit preserving equivariant diffeomorphism of  $V$  with compact support and  $dh_p = 0$ . From the linearization theorem by Sternberg we can prove that  $h \in [\text{Ker } \Phi_p, D_{T^n}(V)]$  by using the contraction map.

#### §4. $S^1$ -action on 3-manifolds

Let  $M$  be a smooth closed 3-manifold with a smooth  $U(1)$ -action. Let  $n_1$  and  $n_2 = m - n_1$  be the numbers of the exceptional orbits  $U(1) \cdot p$  with  $U(1)_p = \mathbf{Z}_2$  and  $U(1)_p = \mathbf{Z}_k$  ( $k \geq 3$ ), respectively. Then we have the following.

##### Proposition 4.1

$$H_1(\mathcal{D}(M/U(1))) \cong \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{n_1+n_2} \times \overbrace{U(1) \times \cdots \times U(1)}^{n_2}.$$

##### Theorem 4.2

$$H_1(\mathcal{D}_{U(1)}(M)) \cong \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{n_1+n_2} \times \overbrace{U(1) \times \cdots \times U(1)}^{n_1+2n_2}.$$

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