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ON PETRIE'S THEOREM FOR TORUS MANIFOLDS

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ABSTRACT. Petrie [9] has shown that all homotopy equivalence between homotopy projective spaces admitting effective smooth half-dimensional compact torus actions should preserve their Pontrjagin classes. In this article, we propose several problems about the invariance of Pontrjagin classes of torus manifolds, which can be regarded as a generalization of Petrie's theorem. In addition, we introduce known results [1] and their applications.

1. PETRIE'S THEOREM

A *torus manifold*, introduced by Hattori and Masuda [5], is a closed smooth manifold of dimension $2n$ admitting an effective smooth T^n-action with non-empty fixed point set. Let M be a torus manifold homotopy equivalent to the projective space $\mathbb{C}P^n$ of the same dimension. In other words, M is a homotopy projective space of dimension $2n$ admitting an effective smooth T^n-action because its fixed point set M^{T^n} is always non-empty; $\chi(M^{T^n}) = \chi(M) = n+1 \neq 0$, where $\chi(X)$ is the euler characteristic number of X.

Theorem 1.1 (Petrie [9]). *Let M_1 and M_2 be torus manifolds homotopy equivalent to $\mathbb{C}P^n$. Then, any homotopy equivalence between M_1 and M_2 preserves their Pontrjagin classes, namely, if $f\colon M_1 \to M_2$ is a homotopy equivalence, then $f^*(p(M_2)) = p(M_1)$, where $p(X)$ denotes the total Pontrjagin class of X and $f^*\colon H^*(M_2) \to H^*(M_1)$ is the induced map of f.*

More precisely, he has shown that the total Pontrjagin class of the *torus homotopy projective space* M of dimension $2n$ is $(1+x^2)^{n+1} \in H^*(M) = \mathbb{Z}[x]/x^{n+1}$, where $\deg x = 2$. Since any cohomology ring isomorphism between two homotopy projective spaces sends a generator to a generator up to sign, we can conclude that it should preserve their Pontrjagin classes. Surprisingly, using the theory of Masuda [6] without hard difficulties, one can show that Theorem 1.1 holds even if we replace the condition which M_i is homotopy equivalent to $\mathbb{C}P^n$ to the condition which $H^*(M_i)$ is isomorphic to $H^*(\mathbb{C}P^n)$ as a graded ring for $i = 1, 2$.

Theorem 1.2. *Let M be a torus manifold whose cohomology ring is isomorphic to $H^*(\mathbb{C}P^n)$ as a graded ring. Then, its total Pontrjagin class is*

$$(1+x^2)^{n+1} \in H^*(M) = \mathbb{Z}[x]/x^{n+1},$$

where $\deg x = 2$.

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Motivated by this, we may ask whether *any cohomology ring isomorphism preserves the Pontrjagin classes of torus manifolds or not.* The purpose of this article is to propose several problems related on this question and to introduce partial answers and their applications based on [1].

2. Torus manifolds

In this section, we briefly review torus manifolds following [7]. A *torus manifold* is a $2n$-dimensional closed connected manifold M with an effective smooth action of an n-dimensional torus $T = (S^1)^n$ such that the fixed point set M^T is non-empty. Since $\dim M = 2\dim T$ and M is compact, M^T is a finite set of isolated points. A codimension-two connected component of the set fixed pointwisely by a circle subgroup of T is called a *characteristic submanifold* of M. Since M is compact, there are only finitely many characteristic submanifolds, and we denote them by M_i, $i = 1, \dots, m$.

Example 2.1. Let n be a positive integer with $n \geq 2$. Let S^{2n} be the $2n$-dimensional sphere identified with the subset $\{(z_1, \dots, z_n, y) \in \mathbb{C}^n \times \mathbb{R} \mid |z_1|^2 + \cdots + |z_n|^2 + y^2 = 1\}$ of $\mathbb{C}^n \times \mathbb{R}$, and define an action of $T^n := (S^1)^n$ on S^{2n} by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n, y) = (t_1 z_1, \dots, t_n z_n, y),$$

where S^1 is the unit circle in $\mathbb{C}^1$. Then this action is effective and smooth, and the points $(0, \dots, 0, \pm 1)$ are fixed by T^n-action. Hence, S^{2n} is a torus manifold. A map

$$(z_1, \dots, z_n, y) \mapsto (|z_1|, \dots, |z_n|, y)$$

induces a homeomorphism from the orbit space S^{2n}/T^n onto the manifold with corners

$$\{(x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + y^2 = 1, x_1 \geq 0, \dots, x_n \geq 0\}.$$

Note that every face of the orbit space S^{2n}/T^n is contractible. The facets are images of characteristic submanifolds $\{z_i = 0\}$ of S^{2n} $(i = 1, \dots, n)$ under the quotient map above and the intersection of the n codimension-one faces, called *facets*, consists of two points $(0, \dots, 0, \pm 1)$.

A torus manifold M is said to be *locally standard* if every point in M has an invariant neighborhood U weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard T-action on $\mathbb{C}^n$, namely, there is an automorphism $\psi\colon T \to T$ and a diffeomorphism $f\colon U \to V$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T$ and $y \in U$. Let M be a locally standard torus manifold. Let $Q := M/T$ denote the orbit space of M and $\pi\colon M \to Q$ the quotient projection. Then, Q can be regarded as a manifold with corners, and faces of Q can be defined in a natural way. We note that the projection $\pi\colon M \to Q$ maps every k-dimensional orbit to a point in the interior of a codimension-k face of Q for all $k = 0, \dots, n$. We set $Q_i := \pi(M_i)$. Then, Q_i is a codimension-one face of Q, called a *facet* of Q. Since M is locally standard, any point in Q has a neighborhood diffeomorphic to an open subset in the positive cone $\mathbb{R}^n_{\geq 0} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$. Such manifolds are called *nice* manifolds with corners.

Let Q be a compact nice manifold with corners. Faces of Q are defined naturally. A nice manifold Q with corners is called a *homology cell* if all faces of Q, including Q itself, are acyclic, namely, their reduced cohomology rings vanish. We also say that Q is a *homology polytope* if it is a homology cell and any multiple intersection of faces is acyclic whenever it is non-empty. A simple polytope provides a typical example of homology polytope and the orbit space of S^{2n}/T^n in Example 2.1 is a homology cell but not a homology polytope.

It is shown in [7] that if $H^{odd}(M) = 0$ for a torus manifold M, then M is locally standard. In addition, they have shown the following theorem.

Theorem 2.2 (Masuda-Panov [7]). *Let M be a torus manifold and Q the orbit space of M. Then*

(1) *Q is a homology cell if and only if $H^{odd}(M) = 0$, and*
(2) *Q is a homology polytope if and only if $H^*(M)$ is generated by $H^2(M)$ as a ring.*

3. Invariance of Pontrjagin classes of torus manifolds

We recall a torus manifold homotopy equivalent to the projective space in Theorem 1.1. We note that the orbit space of the torus homotopy projective spaces is a homology polytope. It is natural to ask whether Theorem 1.1 still holds when we replace a torus homotopy projective space to a torus manifold whose orbit space is a homology polytope (or a homology cell), namely, we have the following problems.

Problem 3.1. *Let M_1 and M_2 be torus manifolds whose orbit spaces are homology polytopes or homology cells. Let $f\colon M_1 \to M_2$ be a homotopy equivalence. Then is it true that*

$$f^*(p(M_2)) = p(M_1)?$$

As we discussed in Section 1, we can also ask the following stronger question.

Problem 3.2. *Let M_1 and M_2 be torus manifolds whose orbit spaces are homology polytopes or homology cells. Let $\varphi\colon H^*(M_2) \to H^*(M_1)$ be a ring isomorphism. Then is it true that*

$$\varphi(p(M_2)) = p(M_1)?$$

Unfortunately, the answer of Problem 3.2 is negative in general. We have the following counter example.

Example 3.3. Let

$$M_1 = S^7 \times_{S^1} S(\mathbb{C}_2^1 \oplus \mathbb{R}^9) \quad \text{and} \quad M_2 = S^7 \times_{S^1} S(\mathbb{C}_2^4 \oplus \mathbb{R}^1),$$

where $S(\mathbb{C}^\ell \oplus \mathbb{R}^m) \subset \mathbb{C}^\ell \oplus \mathbb{R}^m$ stands the unit sphere, and $\mathbb{C}_\rho$ is $\mathbb{C}$ with S^1-action by $t \cdot z = t^\rho z$. The author and Kuroki [2] have shown that $H^*(M_1) \cong H^*(M_2) = \mathbb{Z}[x, y]/\langle x^4, z(z + 2x)^4\rangle$ with $\deg x = 2, \deg z = 8$, and $p_1(M_1) = 4x^2$ and $p_1(M_2) = 16x^2$. One can easily check that both M_1 and M_2 are torus manifolds, and since the generators have even degree, their orbit spaces are homology cells (but not homology polytopes). Because of the degree of generators, any cohomology ring isomorphism $\varphi\colon H^*(M_2) \to H^*(M_1)$ sends a degree-two generator to a degree-two generator up to sign. Hence, $\varphi(p_1(M_2)) \neq p_1(M_1)$. Hence, it gives us the negative answer to Problem 3.2. But we still do not know whether Problem 3.2 is negative on the case where orbit spaces are homology polytopes.

We also have some partial answers to the problems. We note that the product of projective spaces $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$ is also torus manifold whose orbit space is a product of simplices $\prod_{i=1}^{h} \Delta^{n_i}$. It is easy to show that $H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$ is generated by degree two elements.

Theorem 3.4 (Choi [1]). *Let M_1 and M_2 be torus manifolds whose cohomology rings are isomorphic to $H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$. Then, any cohomology ring isomorphism between them preserves their Pontrjagin classes.*

We remark that Theorem 3.4 generalizes Theorem 1.1 strictly.

For a complex vector bundle E, we denote the total space of its projectivization by $P(E)$. A *generalized Bott tower* of height h is a sequence of projective bundles

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{3.1}$$

where each $\pi_i\colon B_i = P(\mathbb{C}\oplus\xi_i) \to B_{i-1}$ and ξ_i is the Whitney sum of n_i (≥ 1) complex line bundles over B_{i-1} for $i = 1, \dots, h$. We call B_h an h-stage *generalized Bott manifold*. Obviously, a complex projective space $\mathbb{C}P^n$ is an one-stage generalized Bott manifold. When all fibers in (3.1) are $\mathbb{C}P^1$, namely, $n_i = 1$ for all i, B_h is called a *Bott manifold*. A closed smooth manifold is called a *cohomology Bott manifold* if its cohomology ring is isomorphic to that of some Bott manifolds, and is called a *torus cohomology Bott manifold* if it is both a cohomology Bott manifold and a torus manifold. We note that a Bott manifold itself is a torus cohomology Bott manifold. In addition, all manifolds homotopy equivalent to Bott manifolds are cohomology Bott manifolds.

Theorem 3.5 (Choi [1]). *Let M_1 and M_2 be torus cohomology Bott manifolds. Then any ring isomorphism $\varphi\colon H^*(M_1) \to H^*(M_2)$ preserves their Pontrjagin classes, namely, $\varphi(p(M_1)) = p(M_2)$.*

We remark that both Theorems 3.4 and 3.5 provide affirmative evidences to Problems 3.1 and 3.2.

4. Applications

One of the most interesting problems in Toric topology is the topological classification of toric manifolds. A *toric manifold* is a non-singular compact complex algebraic variety with an algebraic torus action having a dense orbit. Clearly, a toric manifold is a torus manifold. Interestingly, many recent research provide evidences for toric manifolds to be classified by their cohomology rings. In general, the cohomology ring as an invariant is too weak to determine the topological type. However, in the category of toric manifolds, we do not know any examples of two distinct toric manifold having the same cohomology rings because of their tori-symmetries. Hence, it raises the following problem, called the *cohomological rigidity problem* for toric manifolds.

Problem 4.1 (Cohomological rigidity problem for toric manifolds). *Let M_1 and M_2 be two toric manifolds such that $H^*(M_1) \cong H^*(M_2)$ as graded rings. Then are they diffeomorphic (or homeomorphic)?*

See [8] for more details. By the classical theory on the low dimensional manifolds such as [4], the cohomological rigidity holds for all toric manifolds up to 4 dimension since toric manifolds are simply connected. In high dimensional case, this problem is still open.

We note that both a product of projective spaces and a Bott manifold are not only torus manifolds but also toric manifolds. Hence, by combining Theorems 3.4 and 3.5 with the result of Sullivan [10], we can say the finiteness of such manifolds having the isomorphic cohomology rings;

Corollary 4.2. *There are at most a finite number of torus manifolds homotopy equivalent to the given Bott manifold or the given product of projective spaces.*

So the corollary also provides affirmative evidences of cohomological rigidity of toric manifolds.

We remark that any diffeomorphism between two closed smooth manifolds preserves their Pontrjagin classes. Hence, we can ask the following problem, too.

Problem 4.3 (Strong cohomological rigidity problem for Bott manifolds). *Let B_n and B'_n be two Bott manifolds, and $\phi\colon H^*(B_n) \to H^*(B'_n)$ an isomorphism as a graded ring. Then, there is a diffeomorphism $f\colon B'_n \to B_n$ such that $f^* = \phi$.*

On the other hand, it is well-known that the invariance of Stifel-Whitney classes for a closed manifold whose cohomology ring is generated by the same degree elements.

Theorem 4.4 (Choi-Masuda-Suh [3]). *Suppose that $H^*(M)$ is generated by $H^r(M)$ for some r as a ring and let M' be another connected closed manifold of the same dimension such that $H^*(M')$ is isomorphic to $H^*(M)$ as a ring. Then $\varphi(w(M')) = w(M)$ for any ring isomorphism $\varphi\colon H^*(M') \to H^*(M)$, where $w(X)$ denotes for the total Stifel-Whitney class of X.*

Since any diffeomorphism preserves the total Stiefel-Whintey class, the above theorem also supports to Problem 4.3 affirmatively.

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