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A Simple Derivation of Hadamard's Variational Formula

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ここでは、$\mathbb{R}^n$ 内の有界領域上での Poisson 問題と、その問題の Green 関数を考える。約 100 年前、Hadamard は、領域の境界が摂動を受けた際に Green 関数がどのような影響を受けるかという問題を考え、領域の摂動に対する Green 関数の第一変分を求めた。それは、現在 Hadamard の変分公式と呼ばれている。この論文では、Hadamard の変分公式の別証明を与える。さらに、領域の摂動に対する Green 関数の第二変分も計算することができた。我々の公式は、Garabedian-Schiffer の公式の拡張になっている。

1 Introduction

Let $\mathbb{R}^n$ be n-dimensional Euclidean space ($n \geq 2$) and $\Omega \subset \mathbb{R}^n$ be a bounded domain. For a given function f, we consider Poisson's equation

$$-\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The Green function $G(x,y)$ is a function which provides the solution u of the Poisson equation by

$$u(x) = \int_\Omega G(x,y) f(y) \mathrm{d}y.$$

If the domain Ω is modified, then the Green function $G(x,y)$ would vary. Hadamard considered how $G(x,y)$ would vary and computed the first variation $\delta G(x,y)$ with respect to domain perturbation [3]. His result is now called **Hadamard's variational formula**. Hadamard showed his formula under the assumption that $\partial\Omega$ and the perturbation are analytic. Later, Garabedian and Schiffer gave a simpler and more rigorous proof of Hadamard's variational formula under the assumption that $\partial\Omega$ and the perturbation are of C^2 class (see [1]). Further, they obtained Hadamard's second variational formula [2], [4]. The main aim of this paper is to reconsider Hadamard's variational formula. In particular, we develop a methodology which provides us a much clearer understanding of Hadamard's variational formula. As a result, we obtain a very simple proof of Hadamard's variational formula (see Section 3.1). We also obtain Hadamard's second variational formula which is an extension of Grabedian-Schiffer's formula (Theorem 3.3).

Here, we briefly summarize the notation which we use in this paper. We denote the Euclidean inner product by $x \cdot y$ or $(x,y)_{\mathbb{R}^n}$ for $x, y \in \mathbb{R}^n$. When we do not specify, all vectors in $\mathbb{R}^n$ are regarded as column vectors. Transposing of vectors and matrices are denoted by $(\cdot)^T$. Let $f(x)$ be a smooth function defined in a domain of $\mathbb{R}^n$. The gradient of f is denoted by

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right).$$

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When we need specify the variable of a gradient, we denote such as $\nabla_x f(x)$, $\nabla_{x^*} f(x^*)$. We regard gradients of functions as row vectors. Hence, for a vector field $\mathbf{F}(x)$, $\nabla\mathbf{F}(x)$ is the Jacobi matrix $D\mathbf{F}(x)$. Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote by $L^2(\Omega)$, $H^1(\Omega)$, $H^s(\partial\Omega)$ the usual Lebesgue and Sobolev spaces. The inner product of $L^2(\Omega)$ is denoted by

$$(u,v)_\Omega := \int_\Omega uv\mathrm{d}x, \qquad u,v \in L^2(\Omega)$$

On a point $x \in \partial\Omega$, we denote the unit outer normal vector of $\partial\Omega$ by $\boldsymbol{\nu} = \boldsymbol{\nu}(x)$. For a subset $\Gamma \subset \partial\Omega$, we denote the duality pair of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ by $\langle\varphi, v\rangle_\Gamma$, $\varphi \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$.

2 Basic Definitions

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\widetilde{\Omega}$ be a sufficiently larger domain which satisfies $\overline{\Omega} \subset \operatorname{int}\widetilde{\Omega}$. For a parameter $t \geq 0$, we define transformation $\mathcal{T}_t : \Omega \to \mathcal{T}_t(\Omega) \subset \mathbb{R}^n$ of Ω with respect t in the following way. Let a $C^{0,1}$-class vector field $\mathsf{S}(x)$ be given. We suppose that $\operatorname{supp}\mathsf{S} \subset \widetilde{\Omega}$. Then, a transformation $\mathcal{T}_t(x)$ on $\widetilde{\Omega}$ is defined as a solution of the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t(x) = \mathsf{S}(\mathcal{T}_t(x)), \quad \mathcal{T}_0(x) = x. \tag{2.1}$$

That is, for each $x \in \widetilde{\Omega}$, $\mathcal{T}_t(x)$ is the integral curve generated by (2.1). This $\mathcal{T}_t(x)$ satisfies the following properties:

- For any $x \in \Omega$, $\mathcal{T}_0(x) = x$.
- For a sufficiently small t, $\Omega_t := \mathcal{T}_t(\Omega) \subset \widetilde{\Omega}$.
- $\mathcal{T}_t$ is a diffeomorphism for a sufficiently small $t \geq 0$.
- $\mathcal{T}_t$ is smooth with respect to t.

From the definition (2.1) we have $\mathsf{S}(x) = \frac{\partial}{\partial t}\mathcal{T}_t(x)\Big|_{t=0}$. Moreover, we define

$$\mathsf{T}(x) := \frac{\partial^2}{\partial t^2}\mathcal{T}_t(x)\Big|_{t=0}.$$

Then, the transformation has the Taylor expansion

$$\mathcal{T}_t(x) = x + t\mathsf{S}(x) + \tfrac{1}{2}t^2\mathsf{T}(x) + o(t^2)$$

with respect to t. Here, $o(t^2)$ denote a quantity which would be expressed by $t^2\omega(x,t)$, where $\omega(x,t)$ is a function which converges uniformly (with respect to x) to 0 as $t \to +0$. In the sequel, notations such as $o(t)$, $o(t^2)$ are understood in this way. Let $D\mathsf{S}(x)$ be the Jacobi matrix of S. From (2.1), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{T}_t(x) = \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}(\mathcal{T}_t(x)) = D\mathsf{S}(\mathcal{T}_t(x))\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t(x) = D\mathsf{S}(\mathcal{T}_t(x))\mathsf{S}(\mathcal{T}_t(x)),$$

which implies

$$\mathsf{T}(x) = (D\mathsf{S}(x))\mathsf{S}(x). \tag{2.2}$$

Let a function φ be defined on $\widetilde{\Omega}$ and $\varphi \in H^2(\widetilde{\Omega})$. Suppose that a function $u = u(x,t) \in H^1(\Omega_t)$ is a solution of the boundary value problem

$$\begin{cases} \Delta u(\cdot,t) = 0 & \text{in } \Omega_t, \\ u(\cdot,t) = \varphi & \text{on } \partial\Omega_t. \end{cases} \tag{2.3}$$

Here, $\Delta := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the usual Laplacian with respect to $x = (x_1, \cdots, x_n)^T$. In this section, we investigate differentiations of quantities which depend on $u(x,t)$. Such variations of quantities with respect to domain perturbation are called **Hadamard's variation**. To compute Hadamard's variation it is important to know **Lagrangian derivative** [4] $\dot{u}_{\mathcal{L}}$, $\ddot{u}_{\mathcal{L}}$ and **Eulerian derivative** [5] $\dot{u}_{\mathcal{E}}$, $\ddot{u}_{\mathcal{E}}$, defined by, for $x \in \Omega$,

$$\dot{u}_{\mathcal{L}}(x) := \frac{\mathrm{d}}{\mathrm{d}t}\left(u(\mathcal{T}_t(x),t)\right)\Big|_{t=0}, \quad \ddot{u}_{\mathcal{L}}(x) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(u(\mathcal{T}_t(x),t)\right)\Big|_{t=0},$$
$$\dot{u}_{\mathcal{E}}(x) := \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}, \quad \ddot{u}_{\mathcal{E}}(x) := \frac{\partial^2}{\partial t^2}u(x,t)\Big|_{t=0}.$$

For a function $f(x,t)$, let

$$\mathcal{H}_x f = \mathcal{H}_x f(x,t) := \left(\frac{\partial^2 f(x,t)}{\partial x_i \partial x_j}\right)_{i,j=1,\cdots,n}$$

be the Hesse matrix. We use the same notation $\mathcal{H}_x f$ for the second order tensor $\mathcal{H}_x f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $\mathcal{H}_x f(X,Y) := ((\mathcal{H}_x f)X, Y)_{\mathbb{R}^n}$ for X, $Y \in \mathbb{R}^n$. In particular, in the case of $X = Y$, we denote as $\mathcal{H}_x f(X,X) = \mathcal{H}_x f \cdot (X)^2$. A straightforward computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u(\mathcal{T}_t(x),t)\right) = \frac{\partial}{\partial t}u(\mathcal{T}_t(x),t) + \nabla u(\mathcal{T}_t(x),t)\cdot\left(\frac{\partial}{\partial t}\mathcal{T}_t(x)\right), \tag{2.4}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(u(\mathcal{T}_t(x),t)\right) =& \frac{\partial^2}{\partial t^2}u(\mathcal{T}_t(x),t) + 2\nabla\left(\frac{\partial}{\partial t}u(\mathcal{T}_t(x),t)\right)\cdot\left(\frac{\partial}{\partial t}\mathcal{T}_t(x)\right) \\ &+ \nabla u(\mathcal{T}_t(x),t)\cdot\left(\frac{\partial^2}{\partial t^2}\mathcal{T}_t(x)\right) + \mathcal{H}_x u(\mathcal{T}_t(x),t)\cdot\left(\frac{\partial}{\partial t}\mathcal{T}_t(x)\right)^2. \end{aligned} \tag{2.5}$$

2.1 Eulerian Derivatives $\dot{u}_{\mathcal{E}}$, $\ddot{u}_{\mathcal{E}}$

In this subsection, we check properties which Eulerian derivatives $\dot{u}_{\mathcal{E}}$, $\ddot{u}_{\mathcal{E}}$ should satisfy. At an inner point $x \in \Omega$ we have $\Delta u(\cdot,t) = 0$ for any t. Hence,

$$\Delta\dot{u}_{\mathcal{E}} = 0, \qquad \Delta\ddot{u}_{\mathcal{E}} = 0 \qquad \text{in } \Omega.$$

On the boundary $\partial\Omega$ we have $u(\mathcal{T}_t(x),t) = \varphi(\mathcal{T}_t(x))$. Differentiating the both side and letting $t \to +0$, we see $\dot{u}_{\mathcal{E}} + \mathsf{S}\cdot\nabla u = \mathsf{S}\cdot\nabla\varphi$. Therefore, we find that the Eulerian derivative $\dot{u}_{\mathcal{E}}$ is a solution of the following boundary value problem:

$$\Delta\dot{u}_{\mathcal{E}} = 0 \text{ in } \Omega, \qquad \dot{u}_{\mathcal{E}} = \mathsf{S}\cdot(\nabla\varphi - \nabla u) \text{ on } \partial\Omega. \tag{2.6}$$

In the same manner, we conclude that $\ddot{u}_{\mathcal{E}}$ is a solution of the boundary value problem

$$\begin{aligned} &\Delta\ddot{u}_{\mathcal{E}} = 0 \quad \text{in } \Omega, \\ &\ddot{u}_{\mathcal{E}} = -2\mathsf{S}\cdot\nabla\dot{u}_{\mathcal{E}} + \mathsf{T}\cdot(\nabla\varphi - \nabla u) + (\mathcal{H}_x\varphi - \mathcal{H}_x u)\cdot(\mathsf{S})^2 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.7}$$

[4] It is also called *material derivative* or *covariant derivative.*

[5] This is a usual partial derivative with respect to t which is also called *shape derivative.*

2.2 Lagrangian Derivatives $\dot{u}_{\mathcal{L}}$, $\ddot{u}_{\mathcal{L}}$

In this subsection, we check properties which Lagrangian derivatives $\dot{u}_{\mathcal{L}}$, $\ddot{u}_{\mathcal{L}}$ should satisfy. Here, variable on Ω_t is denoted as $x^* = \mathcal{T}_t(x)$. A function $\tilde{f}(x^*)$ defined on Ω_t is pulled back by $\mathcal{T}_t$ to a function $f(x)$ on Ω as

$$f(x) := \tilde{f}(\mathcal{T}_t(x)).$$

Note that we have

$$\nabla_{x^*}\tilde{f} = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) \begin{pmatrix} \frac{\partial x_1}{\partial x_1^*} & \cdots & \frac{\partial x_1}{\partial x_n^*} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1^*} & \cdots & \frac{\partial x_n}{\partial x_n^*} \end{pmatrix} = (\nabla_x f)(D\mathcal{T}_t^{-1}),$$

where $D\mathcal{T}_t^{-1}$ is the Jacobi matrix of $\mathcal{T}_t^{-1}$.

The weak form of the boundary value problem (2.3) is

$$\text{(2.8)} \qquad \begin{cases} (\nabla u(\cdot,t), \nabla\tilde{v})_{\Omega_t} = 0, & \forall \tilde{v} \in H_0^1(\Omega_t), \\ u(\cdot,t) = \varphi & \text{on } \partial\Omega_t. \end{cases}$$

Using the transformation $\mathcal{T}_t$, we pull back the problem (2.8) to a problem defined on Ω. Note that

$$\tilde{v} \in H_0^1(\Omega_t) \Longleftrightarrow v := \tilde{v} \circ \mathcal{T}_t \in H_0^1(\Omega).$$

Then, setting $u_t(x) := u(\mathcal{T}_t(x), t)$, we see that

$$\begin{aligned}(\nabla u(\cdot,t), \nabla\tilde{v})_{\Omega_t} &= \int_\Omega (\det D\mathcal{T}_t) \left(\nabla u_t \,(D\mathcal{T}_t^{-1} \circ \mathcal{T}_t)\,(D\mathcal{T}_t^{-1} \circ \mathcal{T}_t)^T\right) \cdot \nabla v \, dx \\ &= (\mathsf{A}(t)\nabla u_t, \nabla v)_\Omega, \qquad \forall v \in H_0^1(\Omega),\end{aligned}$$

where

$$\mathsf{A}(t) := (\det D\mathcal{T}_t)\,(D\mathcal{T}_t^{-1} \circ \mathcal{T}_t)\,(D\mathcal{T}_t^{-1} \circ \mathcal{T}_t)^T.$$

That is, the boundary value problem (2.8) on Ω_t is pulled back to the boundary value problem

$$\text{(2.9)} \qquad \begin{cases} (\mathsf{A}(t)\nabla u_t, \nabla v)_\Omega = 0, & \forall v \in H_0^1(\Omega), \\ u_t = \varphi \circ \mathcal{T}_t & \text{on } \partial\Omega \end{cases}$$

on Ω. If $u(x,t)$ is a solution of (2.8), then $u_t(x) = u(\mathcal{T}_t(x), t)$ is a solution of (2.9) and vice versa. We set

$$\text{(2.10)} \qquad \mathcal{A}' := \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{A}(t)\Big|_{t=0}, \qquad \mathcal{A}'' := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathsf{A}(t)\Big|_{t=0}.$$

Suppose that $\varphi \circ \mathcal{T}_t$ has the following Taylor expansion:

$$\varphi \circ \mathcal{T}_t = \varphi + t\dot{\varphi} + \tfrac{1}{2}t^2\ddot{\varphi} + o(t^2).$$

From the definition we find

$$\dot{\varphi} = \mathsf{S} \cdot \nabla\varphi, \qquad \ddot{\varphi} = \mathsf{T} \cdot \nabla\varphi + \mathcal{H}_x\varphi \cdot (\mathsf{S})^2.$$

Let u be a solution of (2.3). Then, we "differentiate" (2.9) and obtain the equation

$$(2.11)\qquad \begin{cases} (\nabla \dot{u}_{\mathcal{L}}, \nabla v)_\Omega = -(\mathcal{A}'\nabla u, \nabla v)_\Omega, & \forall v \in H_0^1(\Omega), \\ \dot{u}_{\mathcal{L}} = \dot{\varphi} \qquad \text{on } \partial\Omega. \end{cases}$$

One more "differentiation" yields the equation

$$(2.12)\qquad \begin{cases} (\nabla \ddot{u}_{\mathcal{L}}, \nabla v)_\Omega = -2(\mathcal{A}'\nabla \dot{u}_{\mathcal{L}}, \nabla v)_\Omega - (\mathcal{A}''\nabla u, \nabla v)_\Omega, & \forall v \in H_0^1(\Omega), \\ \ddot{u}_{\mathcal{L}} = \ddot{\varphi} \qquad \text{on } \partial\Omega. \end{cases}$$

For solutions of these equations, we have the following lemma.

Lemma 2.1 *Suppose that u, u_t, $\dot{u}_{\mathcal{L}}$, $\ddot{u}_{\mathcal{L}} \in H^1(\Omega)$ are solutions of the equations* (2.3), (2.9), (2.11), (2.12), *respectively. Then, u_t has a Taylor expansion $u_t = u + t\dot{u}_{\mathcal{L}} + \frac{1}{2}t^2\ddot{u}_{\mathcal{L}} + o(t^2)$ in $H^1(\Omega)$. That is, the following is valid:*

$$\lim_{t\to 0+} \frac{\|\chi_t\|_{H^1(\Omega)}}{t^2} = 0, \qquad \chi_t := u_t - \left(u + t\dot{u}_{\mathcal{L}} + \tfrac{1}{2}t^2\ddot{u}_{\mathcal{L}}\right).$$

Proof Since $\mathsf{S} \in W^{1,\infty}(\widetilde{\Omega}; \mathbb{R}^n)$ and $\mathsf{T} \in L^\infty(\widetilde{\Omega}; \mathbb{R}^n)$, we see $\mathsf{A}(t) \in L^\infty(\Omega; \mathbb{R}^{n^2})$ and

$$(2.13)\qquad \lim_{t\to 0+} \frac{\|\alpha_t\|_{L^\infty}}{t^2} = 0, \quad \alpha_t := \mathsf{A}(t) - \left(I + t\mathcal{A}' + \tfrac{1}{2}t^2\mathcal{A}''\right).$$

Define z_t, $\dot{z}$, $\ddot{z}$ as solutions of the following boundary value problems:

$$\begin{aligned} (\nabla z_t, \nabla v)_\Omega &= 0, \quad \forall v \in H_0^1(\Omega), \qquad & z_t &= \varphi \circ T_t \text{ on } \partial\Omega, \\ (\nabla \dot{z}, \nabla v)_\Omega &= 0, \quad \forall v \in H_0^1(\Omega), & \dot{z} &= \dot{\varphi} \text{ on } \partial\Omega, \\ (\nabla \ddot{z}, \nabla v)_\Omega &= 0, \quad \forall v \in H_0^1(\Omega), & \ddot{z} &= \ddot{\varphi} \text{ on } \partial\Omega. \end{aligned}$$

Letting

$$\eta_t := z_t - \left(u + t\dot{z} + \tfrac{1}{2}t^2\ddot{z}\right), \quad \psi_t := \varphi \circ T_t - \left(\varphi + t\dot{\varphi} + \tfrac{1}{2}t^2\ddot{\varphi}\right),$$

we notice $\eta_t - \psi_t \in H_0^1(\Omega)$. Since $0 = (\nabla \eta_t, \nabla v)_\Omega$ for any $v \in H_0^1(\Omega)$, we set $v := \eta_t - \psi_t$ and obtain

$$\|\nabla \eta_t\|_{L^2(\Omega)}^2 = (\nabla \eta_t, \nabla \eta_t)_\Omega = (\nabla \eta_t, \nabla \psi_t)_\Omega \le \|\nabla \eta_t\|_{L^2(\Omega)} \|\nabla \psi_t\|_{L^2(\Omega)},$$
$$\lim_{t\to 0+} \frac{\|\nabla \eta_t\|_{L^2(\Omega)}}{t^2} \le \lim_{t\to 0+} \frac{\|\nabla \psi_t\|_{L^2(\Omega)}}{t^2} = 0.$$

Similarly, set

$$\beta_t := u_t - z_t - \left(t(\dot{u}_{\mathcal{L}} - \dot{z}) + \tfrac{1}{2}t^2(\ddot{u}_{\mathcal{L}} - \ddot{z})\right) \in H_0^1(\Omega).$$

Then, from (2.11), (2.12), we find that for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} (\mathsf{A}(t)\nabla \beta_t, \nabla v)_\Omega =& ((I - \mathsf{A}(t))\nabla z_t, \nabla v)_\Omega \\ & - t(\mathsf{A}(t)\nabla(\dot{u}_{\mathcal{L}} - \dot{z}), \nabla v)_\Omega - \tfrac{1}{2}t^2(\mathsf{A}(t)\nabla(\ddot{u}_{\mathcal{L}} - \ddot{z}), \nabla v)_\Omega \\ =& ((I - \mathsf{A}(t))\nabla(\eta_t + \tfrac{1}{2}t^2\ddot{u}_{\mathcal{L}}), \nabla v)_\Omega + t((I + t\mathcal{A}' - \mathsf{A}(t))\nabla \dot{u}_{\mathcal{L}}, \nabla v)_\Omega \\ & + ((I + t\mathcal{A}' + \tfrac{1}{2}t^2\mathcal{A}'' - \mathsf{A}(t))\nabla u, \nabla v)_\Omega. \end{aligned}$$

By (2.13) there exists a positive constant λ such that, for any sufficiently small $t > 0$,

$$\lambda\|\nabla v\|^2_{L^2(\Omega)} \leq (\mathsf{A}(t)\nabla v, \nabla v)_\Omega\,, \qquad \forall v \in H^1_0(\Omega).$$

Inserting $v = \beta_t$ into the above equation, we obtain

$$\begin{aligned}\frac{\lambda}{t^2}\|\nabla\beta_t\|_{L^2(\Omega)} \leq & \|I - \mathsf{A}(t)\|_{L^\infty(\Omega)}\left(\frac{\|\nabla\eta_t\|_{L^2(\Omega)}}{t^2} + \|\nabla\ddot{u}_{\mathcal{L}}\|_{L^2(\Omega)}\right)\\ &+ \frac{1}{t}\|I + t\mathcal{A}' - \mathsf{A}(t)\|_{L^\infty(\Omega)}\|\nabla\dot{u}_{\mathcal{L}}\|_{L^2(\Omega)} + \frac{1}{t^2}\|\alpha_t\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, we conclude $\lim_{t\to 0+}\|\nabla\beta_t\|_{L^2(\Omega)}/t^2 = 0$ and complete the proof since $\chi_t = \beta_t + \eta_t$. □

2.3 The Relationship between Eulerian and Lagrangian derivatives

In this subsection we consider the relationship between Eulerian and Lagrangian derivatives. From (2.4) we immediately notice

$$\dot{u}_{\mathcal{L}} = \dot{u}_{\mathcal{E}} + \mathsf{S}\cdot\nabla u \qquad \text{in } \Omega.$$

Since $(\nabla\dot{u}_{\mathcal{L}}, \nabla v) = -(\mathcal{A}'\nabla u, \nabla v)$ and $(\nabla\dot{u}_{\mathcal{E}}, \nabla v) = 0$ for any $v \in H^1_0(\Omega)$, we have

$$(\nabla(\mathsf{S}\cdot\nabla u), \nabla v)_\Omega = -(\mathcal{A}'\nabla u, \nabla v)_\Omega, \qquad \forall v \in H^1_0(\Omega).$$

Similarly, from (2.5) we obtain

$$\ddot{u}_{\mathcal{L}} = \ddot{u}_{\mathcal{E}} + 2\mathsf{S}\cdot\nabla\dot{u}_{\mathcal{E}} + \mathsf{T}\cdot\nabla u + \mathcal{H}_x u\cdot(\mathsf{S})^2 \qquad \text{in } \Omega.$$

Since

$$(\nabla\ddot{u}_{\mathcal{L}}, \nabla v)_\Omega = -(2\mathcal{A}'\nabla\dot{u}_{\mathcal{L}} + \mathcal{A}''\nabla u, \nabla v)_\Omega, \quad (\nabla\ddot{u}_{\mathcal{E}}, \nabla v) = 0, \quad \forall v \in H^1_0(\Omega),$$

we have, for any $v \in H^1_0(\Omega)$,

$$(2\nabla(\mathsf{S}\cdot\nabla\dot{u}_{\mathcal{E}}) + \nabla(\mathsf{T}\cdot\nabla u) + \nabla(\mathcal{H}_x u\cdot(\mathsf{S})^2), \nabla v)_\Omega = -(2\mathcal{A}'\nabla\dot{u}_{\mathcal{L}} + \mathcal{A}''\nabla u, \nabla v)_\Omega.$$

2.4 Liouville's Theorem

In this section we prepare Liouville's theorem which plays an important role in calculus of Hadamard's variation. Following Garabedian [1] and (2.2), we denote normal components of S and T by $\delta\rho$ and $\delta^2\rho$, respectively:

$$\delta\rho := \mathsf{S}\cdot\boldsymbol{\nu}, \qquad \delta^2\rho := \mathsf{T}\cdot\boldsymbol{\nu} = \boldsymbol{\nu}^t D\mathsf{S}(x)\mathsf{S}(x). \tag{2.14}$$

Theorem 2.2 (Liouville's Theorem) *Let a sufficiently smooth function $c(x,t)$ be defined on the domain $\Omega_t := T_t(\Omega)$ for each $t \geq 0$. Suppose also that $c(x,t)$, $c_t(x,t) := \frac{\partial c}{\partial t}(x,t)$ are measurable on Ω_t. Then, the following holds:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right)\Big|_{t=0} &= \int_\Omega (c_t(x,0) + \nabla\cdot(c(x,0)\mathsf{S}(x)))\,\mathrm{d}x\\ &= \int_\Omega c_t(x,0)\mathrm{d}x + \langle c(\cdot,0), \delta\rho\rangle_{\partial\Omega}\,.\end{aligned} \tag{2.15}$$

Proof. We may suppose without loss of generality that $\partial\Omega$, c, c_t are all sufficiently smooth. The proof for general cases follows from the density property of $C^\infty(\tilde{\Omega})$ in $H^1(\tilde{\Omega})$. Let $J\mathcal{T}_t(x)$ be the Jacobi matrix of $\mathcal{T}_t(x)$. Differentiating the both sides of

$$\int_{\Omega_t} c(x,t)\mathrm{d}x = \int_\Omega c(\mathcal{T}_t(x),t)\det(J\mathcal{T}_t(x))\mathrm{d}x \tag{2.16}$$

with respect to t, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right) &= \int_\Omega\left(c_t(\mathcal{T}_t(x),t) + \nabla c(\mathcal{T}_t(x),t)\cdot\frac{\partial}{\partial t}\mathcal{T}_t(x)\right)\det(J\mathcal{T}_t(x))\mathrm{d}x \\ &\quad + \int_\Omega c(\mathcal{T}_t(x),t)\frac{\partial}{\partial t}\det(J\mathcal{T}_t(x))\mathrm{d}x.\end{aligned} \tag{2.17}$$

Then, letting $t\to 0+$, we obtain the first equality of (2.15). Here, we use

$$\left.\frac{\partial}{\partial t}\det(J\mathcal{T}_t(x))\right|_{t=0} = \nabla\cdot\mathsf{S}(x). \tag{2.18}$$

The second equality immediately follows from the divergence theorem. □

Corollary 2.3 *Suppose that a function $f(x,t)$ is in $H^1(\Omega_t)$ for each $t\geq 0$ and harmonic on Ω_t with respect to $x\in\mathbb{R}^n$. Then, we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t}|\nabla_x f(x,t)|^2\mathrm{d}x\right)\right|_{t=0} = 2\left\langle\frac{\partial f}{\partial\boldsymbol{\nu}},\dot{f}_\mathcal{E}\right\rangle_{\partial\Omega} + \langle|\nabla f|^2,\delta\rho\rangle_{\partial\Omega}.$$

where $f(x) := f(x,0)$, $\dot{f}_\mathcal{E}(x) := \frac{\partial}{\partial t}f(x,t)|_{t=0}$.

Proof: Set $c(x,t) := |\nabla_x f(x,t)|^2$ and apply Theorem 2.2. □

We now try to obtain a second order Liouville's theorem. Assume that $\partial\Omega$, c, S are sufficiently smooth. We have obtained (2.17) by differentiating the both side of (2.16) with respect to t. One more differentiation of the both side of (2.17) and letting $t\to 0$ yield

$$\begin{aligned}&\int_\Omega\left[c_{tt}(x,0) + 2\nabla_x c_t(x,0)\cdot\mathsf{S} + \nabla_x c(x,0)\cdot\mathsf{T} + \mathcal{H}_x c(x,0)\cdot(\mathsf{S})^2\right]\mathrm{d}x \\ &+ 2\int_\Omega\left[c_t(x,0) + \nabla_x c(x,0)\cdot\mathsf{S}\right](\nabla\cdot\mathsf{S})\mathrm{d}x \\ &+ \int_\Omega c(x,0)\left[\nabla\cdot\mathsf{T} + 2\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i})\right]\mathrm{d}x.\end{aligned}$$

Here, we used (2.18) and, for $\mathsf{S} = (S_1,\cdots,S_n)^T$,

$$\left.\frac{\partial^2}{\partial t^2}\det(D\mathcal{T}_t(x))\right|_{t=0} = \nabla\cdot\mathsf{T}(x) + 2\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}),\quad S_{kx_l} := \frac{\partial S_k}{\partial x_l}.$$

Hence, we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right)\right|_{t=0} &= \int_\Omega c_{tt}(x,0)\mathrm{d}x + \int_\Omega\mathcal{H}_x c(x,0)\cdot(\mathsf{S})^2\mathrm{d}x \\ &+ 2\int_{\partial\Omega} c_t(x,0)\delta\rho\mathrm{d}s + \int_{\partial\Omega} c(x,0)\delta^2\rho\mathrm{d}s \\ &+ 2\int_\Omega(\nabla_x c(x,0)\cdot\mathsf{S})(\nabla\cdot\mathsf{S})\mathrm{d}x + 2\int_\Omega c(x,0)\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i})\mathrm{d}x.\end{aligned}$$

We try to simplify this formula. Recall that DS is the Jacobi matrix of the vector field S. Since it follows from the divergence theorem that

$$
\begin{aligned}
2\int_\Omega (\nabla_x c(x,0)\cdot \mathsf{S})(\nabla\cdot \mathsf{S})\mathrm{d}x =& 2\int_{\partial\Omega} (\nabla_x c(x,0)\cdot \mathsf{S})\delta\rho \mathrm{d}s \\
&- 2\int_\Omega \mathcal{H}_x c(x,0)\cdot (\mathsf{S})^2 \mathrm{d}x - 2\int_\Omega ((D\mathsf{S})\mathsf{S})\cdot \nabla_x c(x,0)\mathrm{d}x,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right)\bigg|_{t=0} =& \int_\Omega c_{tt}(x,0)\mathrm{d}x + 2\int_{\partial\Omega} c_t(x,0)\delta\rho\mathrm{d}s \\
&+ \int_{\partial\Omega} c(x,0)\delta^2\rho \mathrm{d}s + 2\int_{\partial\Omega}(\nabla_x c(x,0)\cdot \mathsf{S})\delta\rho\mathrm{d}s \\
&- \int_\Omega \mathcal{H}_x c(x,0)\cdot(\mathsf{S})^2\mathrm{d}x - 2\int_\Omega((D\mathsf{S})\mathsf{S})\cdot\nabla_x c(x,0)\mathrm{d}x \\
&+ 2\int_\Omega c(x,0)\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i})\mathrm{d}x.
\end{aligned}
$$

We gather the last three terms on the right-hand side of the above equality. Set

$$
\begin{aligned}
W &:= 2\int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathsf{S})\delta\rho\mathrm{d}s, & X &:= -\int_\Omega \mathcal{H}_x c(x,0)\cdot(\mathsf{S})^2\mathrm{d}x, \\
Y &:= -2\int_\Omega((D\mathsf{S})\mathsf{S})\cdot\nabla_x c(x,0)\mathrm{d}x, & Z &:= 2\int_\Omega c(x,0)\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i})\mathrm{d}x.
\end{aligned}
$$

We omit "dx", "ds" from the notation of integrals for a while. A straightforward computation using the divergence theorem yields

$$
\begin{aligned}
X + Y &= -\sum_{i,j}\int_\Omega c_{x_ix_j}S_iS_j - 2\sum_{i,j}\int_\Omega c_{x_i}S_{ix_j}S_j \\
&= -\int_{\partial\Omega}(\nabla c\cdot\mathsf{S})\delta\rho + \sum_{i,j}\int_\Omega c_{x_i}S_iS_{jx_j} - \sum_{i,j}\int_\Omega c_{x_i}S_{ix_j}S_j \\
Z &= 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}\nu_i - 2\sum_{i<j}\int_\Omega c_{x_i}S_iS_{jx_j} - 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}\nu_j + 2\sum_{i<j}\int_\Omega c_{x_j}S_iS_{jx_i}.
\end{aligned}
$$

Thus, "$X+Y+Z$" becomes

$$
\begin{aligned}
X+Y+Z =& -\int_{\partial\Omega}(\nabla_x c\cdot\mathsf{S})\delta\rho + 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}n_i - 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}n_j \\
&- \sum_{i<j}\int_\Omega c_{x_i}S_iS_{jx_j} + \sum_{i>j}\int_\Omega c_{x_i}S_iS_{jx_j} + \sum_{i<j}\int_\Omega c_{x_j}S_iS_{jx_i} - \sum_{i>j}\int_\Omega c_{x_j}S_iS_{jx_i}.
\end{aligned}
$$

With appropriate exchanges of i and j, the last four terms of the right-hand side turn to

$$
\sum_{i<j}\int_\Omega c_{x_j}(S_iS_j)_{x_i} - \sum_{i<j}\int_\Omega c_{x_i}(S_iS_j)_{x_j} = \sum_{i<j}\int_{\partial\Omega} c(S_iS_j)_{x_i}\nu_j - \sum_{i<j}\int_{\partial\Omega} c(S_iS_j)_{x_j}\nu_i.
$$

Therefore, we have

$$\begin{aligned}X+Y+Z &= -\int_{\partial\Omega}(\nabla_x c\cdot \mathsf{S})\delta\rho + \sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}\nu_i - \sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}\nu_j \\ &\quad + \sum_{i<j}\int_{\partial\Omega} cS_{ix_i}S_j\nu_j - \sum_{i<j}\int_{\partial\Omega} cS_{ix_j}S_j\nu_i \\ &= -\int_{\partial\Omega}(\nabla_x c\cdot \mathsf{S})\delta\rho + \int_{\partial\Omega} c(\nabla\cdot\mathsf{S})\delta\rho - \int_{\partial\Omega} c\delta^2\rho.\end{aligned}$$

Here, we used (2.14). Hence, "$W+X+Y+Z$" becomes

$$\begin{aligned}W+X+Y+Z &= \int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathsf{S})\delta\rho \mathrm{d}s \\ &\quad + \int_{\partial\Omega} c(x,0)(\nabla\cdot\mathsf{S})\delta\rho\mathrm{d}s - \int_{\partial\Omega} c(x,0)\delta^2\rho\mathrm{d}s\end{aligned}$$

Gathering all terms, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right)\bigg|_{t=0} &= \int_\Omega c_{tt}(x,0)\mathrm{d}x + 2\int_{\partial\Omega} c_t(x,0)\delta\rho\mathrm{d}s \\ &\quad + \int_{\partial\Omega}\nabla\cdot(c(x,0)\mathsf{S})\delta\rho\mathrm{d}s.\end{aligned}$$

We have done these computation under the assumption that c, $\partial\Omega$ are sufficiently smooth. With a usual density argument we obtain the following theorem for general cases.

Theorem 2.4 (Extended Liouville's Theorem) *Suppose that a C^2-class function $c(x,t)$ is given on $\widetilde{\Omega}$ and, for each $t \geq 0$, $c(x,t)$, $c_t(x,t) := \frac{\partial c}{\partial t}(x,t)$, $c_{tt}(x,t) := \frac{\partial^2 c}{\partial t^2}(x,t)$ are integrable on $\Omega_t := \mathcal{T}_t(\Omega)$. Then, we have the following equality:*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_{\Omega_t} c(x,t)\mathrm{d}x\right)\bigg|_{t=0} = \int_\Omega c_{tt}(x,0)\mathrm{d}x + \langle 2c_t(\cdot,0) + \nabla\cdot(c(\cdot,0)\mathsf{S}), \delta\rho\rangle_{\partial\Omega}.$$

Here, $\boldsymbol{\nu}$ is unit outer normal and $\delta\rho := \mathsf{S}\cdot\boldsymbol{\nu}$.

3 Hadamard's Variational Formula

Let $\Omega \subset \mathbb{R}^n$ be a domain and $G(x,y)$ be the Green function of the Laplacian Δ on Ω. If Ω is perturbed, $G(x,y)$ also varies. Hadamard presented the first variation $\delta G(x,y)$ of $G(x,y)$ with respect to domain perturbation. His formula is called **Hadamard's variational formula**. Later, Garabedian and Schiffer gave an rigorous alternative proof of Hadamard's variational formula. They also computed the second variation of the Green function. In this section, using the result obtained in the previous section, we give an alternative and much simpler proof of Hadamard's variation formula, the both first and second variations. In the sequel, the boundary $\partial\Omega$ is assumed to be sufficiently smooth. The fundamental solution $\Gamma(x)$ of the Laplacian Δ is defined by

$$\Gamma(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2, \\ \frac{1}{(n-2)\omega_n}|x|^{2-n}, & n\geq 3.\end{cases}$$

Here, ω_n is the measure of $(n-1)$-dimensional sphere S^{n-1}. Then, for sufficiently smooth function f we have Green's formula

$$(3.1)\qquad -\int_\Omega \Gamma(x-y)\Delta f(x)\mathrm{d}x + \int_{\partial\Omega} \frac{\partial f}{\partial \boldsymbol{\nu}}(x)\Gamma(x-y)\mathrm{d}s_x = \int_{\partial\Omega} f(x)\frac{\partial}{\partial \boldsymbol{\nu}_x}\Gamma(x-y)\mathrm{d}s_x + f(y).$$

For the fundamental solution $\Gamma(x-y)$, define u as a solution of the following boundary value problem:

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u(x) = -\Gamma(x-y), \quad x \in \partial\Omega.$$

Then,

$$G(x,y) := \Gamma(x-y) + u(x)$$

is the Green function of Δ on Ω. It follows from the definition that $G(x,y) = 0$ for $x \in \partial\Omega$ and $y \in \Omega$. Adding the following Green's formula with respect to f and u

$$-\int_\Omega u(x)\Delta f(x)\mathrm{d}x + \int_{\partial\Omega} \frac{\partial f}{\partial \boldsymbol{\nu}}(x)u(x)\mathrm{d}s = \int_{\partial\Omega} f(x)\frac{\partial u}{\partial \boldsymbol{\nu}}(x)\mathrm{d}s$$

to (3.1), we obtain Green's second formula

$$(3.2)\qquad f(y) = -\int_\Omega G(x,y)\Delta f(x)\mathrm{d}x - \int_{\partial\Omega} f(x)\frac{\partial}{\partial \boldsymbol{\nu}_x}G(x,y)\mathrm{d}s_x.$$

3.1 First Variation

Now, we consider domain perturbation $\Omega_t = \mathcal{T}_t(\Omega)$ defined in the previous section. The Green function $G(x,y,t)$ on Ω_t is written as

$$G(x,y,t) = \Gamma(x-y) + u(x,t),$$

where $u(x,t)$ is the harmonic function which satisfies

$$(3.3)\qquad \Delta_x u(x,t) = 0 \quad \text{in } \Omega_t, \qquad u(x,t) = -\Gamma(x-y), \quad x \in \partial\Omega_t.$$

Obviously, we have $G(x,y,0) = G(x,y)$ and $u(x,0) = u(x)$. For two inner points $x,y \in \Omega$ and sufficiently small $t > 0$, we have $x,y \in \Omega_t$. The first variation $\delta G(x,y)$ with respect to domain perturbation is defined by

$$\delta G(x,y) := \lim_{t\to 0+} \frac{G(x,y,t) - G(x,y,0)}{t} = \lim_{t\to 0+} \frac{u(x,t) - u(x,0)}{t} = \dot{u}_{\mathcal{E}}(x),$$

and is equal to the Eulerian derivative $\dot{u}_{\mathcal{E}}$ of u.

By (2.6), we confirm that $\dot{u}_{\mathcal{E}}$ is a solution of the boundary value problem

$$\begin{aligned}
&\Delta \dot{u}_{\mathcal{E}} = 0 \quad \text{in } \Omega,\\
&\dot{u}_{\mathcal{E}} = \mathsf{S}\cdot(-\nabla_x\Gamma(x-y) - \nabla u) = -\mathsf{S}\cdot\nabla_x G(x,y) = -\delta\rho\frac{\partial}{\partial \boldsymbol{\nu}_x}G(x,y) \quad \text{on } \partial\Omega.
\end{aligned}$$

Here, we use the fact that $\mathsf{S}\cdot\nabla_x G(x,y) = (\mathsf{S}\cdot\boldsymbol{\nu})\frac{\partial}{\partial \boldsymbol{\nu}_x}G(x,y)$ on $\partial\Omega$. Applying the formula (3.2) to $\dot{u}_{\mathcal{E}}$, we obtain Hadamard's variational formula.

Theorem 3.1 (Hadamard's variational formula) *The first variation $\delta G(w,y)$ of the Green function $G(w,y)$ of Δ with respect to domain perturbation is given by*

$$\delta G(w,y) = \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{\nu}_x}G(x,y)\frac{\partial}{\partial \boldsymbol{\nu}_x}G(x,w)\delta\rho\mathrm{d}s_x, \qquad \delta\rho := \mathsf{S}\cdot\boldsymbol{\nu}.$$

3.2 Second Variation

In this subsection, we compute the second variation of the Green function with respect to domain perturbation. We prepare a lemma. Let a harmonic function $u(x,t)$ be a solution of the Dirichlet problem (3.3). Since $\delta G(x,y) = \dot{u}_{\mathcal{E}}(x)$, we recall that

$$\delta G(x,y) = -\delta\rho \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,y), \quad x \in \partial\Omega. \tag{3.4}$$

Hence, for a harmonic function $g(x)$, we find

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left(\delta G(x,y) + \delta\rho \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,y) \right) \frac{\partial g}{\partial \boldsymbol{\nu}}(x) \mathrm{d}s_x \\ &= \int_{\Omega} \nabla_x \delta G(x,y) \cdot \nabla g(x) \mathrm{d}x + \int_{\partial\Omega} \delta\rho \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,y) \frac{\partial g}{\partial \boldsymbol{\nu}}(x) \mathrm{d}s_x, \end{aligned}$$

and obtain the following lemma.

Lemma 3.2 *For a harmonic function g on Ω, the following equality holds:*

$$\int_{\Omega} \nabla_x \delta G(x,y) \cdot \nabla g(x) \mathrm{d}x = -\int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,y) \frac{\partial g}{\partial \boldsymbol{\nu}}(x) \delta\rho \mathrm{d}s_x.$$

The second variation $\delta^2 G(x,y)$ of the Green function $G(x,y)$ is defined by

$$\delta^2 G(x,y) := \frac{\partial^2}{\partial t^2} G(x,y,t) \Big|_{t=0} = \ddot{u}_{\mathcal{E}}(x),$$

and, therefore, we only need to compute $\ddot{u}_{\mathcal{E}}$. Recall that the harmonic function is a solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \qquad u = -\Gamma(\cdot - y) \text{ on } \partial\Omega.$$

By (2.7), the boundary value of $\ddot{u}_{\mathcal{E}}$ on $\partial\Omega$ is

$$\ddot{u}_{\mathcal{E}} = -2\mathsf{S} \cdot \nabla_x \delta G(x,y) - \mathsf{T} \cdot \nabla_x G(x,y) - \mathcal{H}_x G(x,y) \cdot (\mathsf{S})^2.$$

Here, we use $G(x,y) = \Gamma(x-y) + u(x)$ and $\dot{u}_{\mathcal{E}}(x) = \delta G(x,y)$. From (3.2), we find

$$\begin{aligned} \delta^2 G(x,y) = \ddot{u}_{\mathcal{E}}(x) =& 2\int_{\partial\Omega} \mathsf{S} \cdot \nabla_w \delta G(w,y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w) \mathrm{d}s_w \\ &+ \int_{\partial\Omega} \mathsf{T} \cdot \nabla_w G(w,y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w) \mathrm{d}s_w \\ &+ \int_{\partial\Omega} \mathcal{H}_w G(w,y) \cdot (\mathsf{S})^2 \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w) \mathrm{d}s_w. \end{aligned} \tag{3.5}$$

We denote the first, second and third terms of the right-hand side of (3.5) by X, Y, Z, respectively. As before, the term Y can be written as

$$Y = \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w,y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w) \delta^2\rho \mathrm{d}s_w. \tag{3.6}$$

To understand the terms X and Z, we consider the $(n-1)$-dimensional tangent space $T_x\partial\Omega$ of $\partial\Omega$ at $x \in \partial\Omega$. Let $\{\mathbf{s}_1, \cdots, \mathbf{s}_{n-1}\}$ be the orthonormal basis of $T_x\partial\Omega$. Then, $\{\mathbf{s}_1, \cdots, \mathbf{s}_{n-1}, \boldsymbol{\nu}\}$

is an orthonormal basis of the tangent space $T_x\mathbb{R}^n$ at $x \in \mathbb{R}^n$. For a generic function f, directional derivatives are defined by

$$\frac{\partial f}{\partial \boldsymbol{\nu}} = \nabla f \cdot \boldsymbol{\nu}, \qquad \frac{\partial f}{\partial \mathbf{s}_i} = \nabla f \cdot \mathbf{s}_i, \quad i = 1, \cdots, n-1.$$

Thus, defining the orthogonal matrix P by $P := (\mathbf{s}_1, \cdots, \mathbf{s}_{n-1}, \boldsymbol{\nu})$, we may write

$$\left(\frac{\partial f}{\partial \mathbf{s}_1}, \cdots, \frac{\partial f}{\partial \mathbf{s}_{n-1}}, \frac{\partial f}{\partial \boldsymbol{\nu}}\right) = (\nabla f)P, \quad \text{or}$$

$$(\nabla f)^T = \sum_{i=1}^{n-1} \mathbf{s}_i \frac{\partial f}{\partial \mathbf{s}_i} + \boldsymbol{\nu}\frac{\partial f}{\partial \boldsymbol{\nu}} \quad \text{and} \quad \nabla = \sum_{i=1}^{n-1} \mathbf{s}_i^T \frac{\partial}{\partial \mathbf{s}_i} + \boldsymbol{\nu}^T \frac{\partial}{\partial \boldsymbol{\nu}}. \tag{3.7}$$

If we write S as

$$\mathsf{S} = \sum_{i=1}^{n-1} \mu_i \mathbf{s}_i + \delta\rho\boldsymbol{\nu}, \quad \delta\rho = \mathsf{S} \cdot \boldsymbol{\nu}, \quad \mu_i = \mathsf{S} \cdot \mathbf{s}_i, \ i = 1, \cdots, n-1 \tag{3.8}$$

on $\partial\Omega$, we obtain

$$\mathsf{S} \cdot \nabla_w \delta G(w, y) = \sum_{i=1}^{n-1} \mu_i \frac{\partial}{\partial \mathbf{s}_i} \delta G(w, y) + \delta\rho \frac{\partial}{\partial \boldsymbol{\nu}_w} \delta G(w, y).$$

Using Lemma 3.2 with $g := \delta G$, the term X (the first term of the right-hand side of (3.5)) is written as

$$\begin{aligned} X =& 2\sum_{i=1}^{n-1} \int_{\partial\Omega} \mu_i \frac{\partial}{\partial \mathbf{s}_i} \delta G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w + 2\int_{\partial\Omega} \delta\rho \frac{\partial}{\partial \boldsymbol{\nu}_w} \delta G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w \\ =& 2\sum_{i=1}^{n-1} \int_{\partial\Omega} \mu_i \frac{\partial}{\partial \mathbf{s}_i} \delta G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w - 2\int_{\Omega} \nabla_w \delta G(w, y) \cdot \nabla_x \delta G(x, w) \mathrm{d}s_w. \end{aligned}$$

Next, we try to rewrite the third term Z of the right-hand side of (3.5). To this end, we consider the curved coordinate defined by $\{\mathbf{s}_1, \cdots, \mathbf{s}_{n-1}, \boldsymbol{\nu}\}$ in the neighborhood of $x \in \partial\Omega$ and second order differentiation on the coordinate. For a C^2 class generic function f, the Hesse matrix $\mathcal{H}f$ is written by $\mathcal{H}f = \nabla(\nabla f)^T$ and

$$\begin{aligned} \mathcal{H}f =& \nabla\left(\sum_{i=1}^{n-1} \mathbf{s}_i \frac{\partial f}{\partial \mathbf{s}_i} + \boldsymbol{\nu}\frac{\partial f}{\partial \boldsymbol{\nu}}\right) \\ =& \sum_{i=1}^{n-1} \frac{\partial f}{\partial \mathbf{s}_i} D\mathbf{s}_i + \frac{\partial f}{\partial \boldsymbol{\nu}} D\boldsymbol{\nu} + \sum_{i=1}^{n-1} \mathbf{s}_i \nabla\left(\frac{\partial f}{\partial \mathbf{s}_i}\right) + \boldsymbol{\nu}\nabla\left(\frac{\partial f}{\partial \boldsymbol{\nu}}\right) \\ =& \sum_{i=1}^{n-1} \frac{\partial f}{\partial \mathbf{s}_i} D\mathbf{s}_i + \frac{\partial f}{\partial \boldsymbol{\nu}} D\boldsymbol{\nu} + \sum_{i,j=1}^{n-1} \frac{\partial^2 f}{\partial \mathbf{s}_j \partial \mathbf{s}_i} \mathbf{s}_i \mathbf{s}_j^T + \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial \mathbf{s}_i \partial \boldsymbol{\nu}} (\mathbf{s}_i \boldsymbol{\nu}^T + \boldsymbol{\nu}\mathbf{s}_i^T) + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \boldsymbol{\nu}\boldsymbol{\nu}^T. \end{aligned} \tag{3.9}$$

Similarly, Δf is computed as

$$
\begin{aligned}
\Delta f =& \nabla \cdot \nabla f = \nabla \cdot \left(\sum_{i=1}^{n-1} \mathbf{s}_i \frac{\partial f}{\partial \mathbf{s}_i} + \boldsymbol{\nu} \frac{\partial f}{\partial \boldsymbol{\nu}} \right) \\
=& (\nabla \cdot \boldsymbol{\nu}) \frac{\partial f}{\partial \boldsymbol{\nu}} + \boldsymbol{\nu} \cdot \nabla \frac{\partial f}{\partial \boldsymbol{\nu}} + \sum_{i=1}^{n-1} \left((\nabla \cdot \mathbf{s}_i) \frac{\partial f}{\partial \mathbf{s}_i} + \mathbf{s}_i \cdot \nabla \frac{\partial f}{\partial \mathbf{s}_i} \right) \\
=& (\nabla \cdot \boldsymbol{\nu}) \frac{\partial f}{\partial \boldsymbol{\nu}} + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} + \sum_{i=1}^{n-1} \left((\nabla \cdot \mathbf{s}_i) \frac{\partial f}{\partial \mathbf{s}_i} + \frac{\partial^2 f}{\partial \mathbf{s}_i^2} \right). \qquad (3.10)
\end{aligned}
$$

Since the Green function G satisfies $G(x, y) = 0$ on $\partial\Omega$, we have

$$
\frac{\partial}{\partial \mathbf{s}_i} G(x, y) = \frac{\partial^2}{\partial \mathbf{s}_i \partial \mathbf{s}_j} G(x, y) = 0, \qquad i, j = 1, \cdots, n-1.
$$

and, thus,

$$
0 = \Delta_w G(w, y) = (\nabla \cdot \boldsymbol{\nu}) \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y) + \frac{\partial^2}{\partial \boldsymbol{\nu}_w^2} G(w, y), \quad w \in \partial\Omega.
$$

Applying these results to computation of $\mathcal{H}_w G(w, y) \cdot \mathsf{S}^2$ with $\mathsf{S} = \sum_{i=1}^{n-1} \mu_i \mathbf{s}_i + \delta\rho\boldsymbol{\nu}$, we obtain

$$
\begin{aligned}
\mathcal{H}_w G(w, y) \cdot (\mathsf{S})^2 =& 2\delta\rho \sum_{i=1}^{n-1} \mu_i \frac{\partial^2}{\partial \mathbf{s}_i \partial \boldsymbol{\nu}_w} G(w, y) - (\nabla \cdot \boldsymbol{\nu})(\delta\rho)^2 \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y) \\
&+ \left(\sum_{i,j=1}^{n-1} \mu_i \mu_j \mathbf{s}_i^T (D\boldsymbol{\nu}) \mathbf{s}_j + 2 \sum_{i=1}^{n-1} \mu_i \delta\rho \mathbf{s}_i^T (D\boldsymbol{\nu}) \boldsymbol{\nu} + (\delta\rho)^2 \boldsymbol{\nu}^T (D\boldsymbol{\nu}) \boldsymbol{\nu} \right) \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y) \\
=& 2\delta\rho \sum_{i=1}^{n-1} \mu_i \frac{\partial^2}{\partial \mathbf{s}_i \partial \boldsymbol{\nu}_w} G(w, y) + \left(\sum_{i=1}^{n-1} \kappa_i \left(\mu_i^2 - (\delta\rho)^2 \right) \right) \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y).
\end{aligned}
$$

Here, we use the fact $\nabla \boldsymbol{\nu} = \sum_{i=1}^{n-1} \kappa_i \mathbf{s}_i \mathbf{s}_i^t$, $\nabla \cdot \boldsymbol{\nu} = \mathrm{tr}(\nabla \boldsymbol{\nu}) = \sum_{i=1}^{n-1} \kappa_i$, where κ_i is the curvature of the cross-section of $(n-1)$-dimensional surface $\partial\Omega$ by a two-dimensional plane defined by $\mathbf{s}_i$ and $\boldsymbol{\nu}$. Therefore, the third term Z of the right-hand side of (3.5) is written by

$$
\begin{aligned}
Z =& \int_{\partial\Omega} \sum_{i=1}^{n-1} \kappa_i \left(\mu_i^2 - (\delta\rho)^2 \right) \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w \\
&+ 2 \int_{\partial\Omega} \sum_{i=1}^{n-1} \mu_i \delta\rho \frac{\partial^2}{\partial \mathbf{s}_i \partial \boldsymbol{\nu}_w} G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w.
\end{aligned}
$$

Noticing (3.4), we see the equality

$$
\frac{\partial}{\partial \mathbf{s}_i} \delta G(w, y) + \delta\rho \frac{\partial^2}{\partial \mathbf{s}_i \partial \boldsymbol{\nu}_w} G(w, y) = -\frac{\partial(\delta\rho)}{\partial \mathbf{s}_i} \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y).
$$

Computing $X + Z$ using this equality, we find

$$
\begin{aligned}
X + Z =& -2 \int_\Omega \nabla_w \delta G(w, y) \nabla_w \delta G(x, w) \mathrm{d}s_w \\
&+ \int_{\partial\Omega} \sum_{i=1}^{n-1} \left[\kappa_i \left(\mu_i^2 - (\delta\rho)^2 \right) - 2\mu_i \frac{\partial(\delta\rho)}{\partial \mathbf{s}_i} \right] \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w, y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x, w) \mathrm{d}s_w. \qquad (3.11)
\end{aligned}
$$

Gathering (3.5), (3.6), and (3.11) we obtain

$$\delta^2 G(x,y) = -2\int_\Omega \nabla_w \delta G(w,y)\nabla_w \delta G(x,w)\mathrm{d}s_w + \int_{\partial\Omega} \chi \frac{\partial}{\partial \boldsymbol{\nu}_w} G(w,y)\frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w)\mathrm{d}s_w,$$
$$\chi := \delta^2\rho + \sum_{i=1}^{n-1}\left[\kappa_i\left(\mu_i^2 - (\delta\rho)^2\right) - 2\mu_i \frac{\partial(\delta\rho)}{\partial \mathbf{s}_i}\right].$$

We further try to simplify χ. At first, since $\frac{\partial \boldsymbol{\nu}}{\partial \mathbf{s}_i} = \kappa_i \mathbf{s}_i$, we notice

$$\sum_{i=1}^{n-1} \kappa_i \mu_i^2 = \sum_{i=1}^{n-1}\kappa_i (\mathsf{S}\cdot \mathbf{s}_i)^2 = \sum_{i=1}^{n-1} \mathsf{S}\cdot\mathbf{s}_i\left(\frac{\partial(\delta\rho)}{\partial \mathbf{s}_i} - \left(\frac{\partial \mathsf{S}}{\partial \mathbf{s}_i}\cdot\boldsymbol{\nu}\right)\right).$$

Thus, recalling from (3.7) that $D\mathsf{S} = \sum_{n=1}^{n-1}\frac{\partial \mathsf{S}}{\partial \mathbf{s}_i}\mathbf{s}_i^T + \frac{\partial \mathsf{S}}{\partial \boldsymbol{\nu}}\boldsymbol{\nu}^T$, we see

$$\begin{aligned}\sum_{i=1}^{n-1}\mathsf{S}\cdot\mathbf{s}_i\left(\frac{\partial \mathsf{S}}{\partial \mathbf{s}_i}\cdot\boldsymbol{\nu}\right) &= \sum_{i=1}^{n-1}\mathsf{S}\cdot\mathbf{s}_i\left(\boldsymbol{\nu}^T(D\mathsf{S})\mathbf{s}_i\right) = \boldsymbol{\nu}^T(D\mathsf{S})\left(\textstyle\sum_{i=1}^{n-1}(\mathsf{S}\cdot\mathbf{s}_i)\mathbf{s}_i\right)\\ &= \boldsymbol{\nu}^T(D\mathsf{S})(\mathsf{S} - \delta\rho\boldsymbol{\nu}) = \delta^2\rho - \delta\rho\left(\frac{\partial \mathsf{S}}{\partial \boldsymbol{\nu}}\cdot\boldsymbol{\nu}\right) = \delta^2\rho - \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}.\end{aligned} \tag{3.12}$$

Here, we use the facts $\delta^2\rho = \boldsymbol{\nu}^T(D\mathsf{S})\mathsf{S}$ by (2.14) and $\frac{\partial \boldsymbol{\nu}}{\partial \boldsymbol{\nu}} = 0$. Similarly, since $\mathsf{S}\cdot\nabla = \sum_{n=1}^{n-1}(\mathsf{S}\cdot\mathbf{s}_i)\frac{\partial}{\partial \mathbf{s}_i} + (\mathsf{S}\cdot\boldsymbol{\nu})\frac{\partial}{\partial\boldsymbol{\nu}}$ by (3.7), we have

$$\sum_{i=1}^{n-1}\mathsf{S}\cdot\mathbf{s}_i\frac{\partial(\delta\rho)}{\partial \mathbf{s}_i} = (\mathsf{S}\cdot\nabla)\delta\rho - \delta\rho\frac{\partial(\delta\rho)}{\partial\boldsymbol{\nu}} = (\mathsf{S}\cdot\nabla)\delta\rho - \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}. \tag{3.13}$$

Letting $\tilde{\kappa} := \sum_{i=1}^{n-1}\kappa_i$, χ is rewritten as

$$\chi = -\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}.$$

If the domain perturbation $\mathcal{T}_t(x)$ satisfies $\mathsf{S}\cdot\mathbf{s}_i = 0$, $i = 1,\cdots,n-1$, it follows from (3.12), (3.13) that

$$\delta^2\rho = (\mathsf{S}\cdot\nabla)\delta\rho = \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}. \tag{3.14}$$

Therefore, in this case, we find

$$\chi = -\tilde{\kappa}(\delta\rho)^2 + \delta^2\rho.$$

So far, computation has been done under smoothness assumptions. A usual density argument yields the following theorem:

Theorem 3.3 (Hadamard's second variational formula) *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\mathcal{T}_t(x)$ be a $W^{1,\infty}$ class domain perturbation on Ω. Then, the second variation $\delta^2 G(w,y)$ of the Green function $G(w,y)$ of Laplacian on Ω is written by*

$$\delta^2 G(x,y) = \left\langle \chi\frac{\partial}{\partial\boldsymbol{\nu}}G(x,\cdot), \frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,y)\right\rangle_{\partial\Omega} - 2\left(\nabla\delta G(x,\cdot), \nabla\delta G(\cdot,y)\right)_\Omega,$$
$$\chi = -\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}, \quad \delta\rho := \mathsf{S}\cdot\boldsymbol{\nu}, \quad \tilde{\kappa} := \sum_{i=1}^{n-1}\kappa_i,$$

where $\{\mathbf{s}_i\}_{i=1}^{n-1}$ *is an orthonormal basis of the tangent space of* $\partial\Omega$ *and* κ_i *is the curvature of* $\partial\Omega$ *along* $\mathbf{s}_i$. *In particular, if the perturbation satisfies* $\mathsf{S}\cdot\mathbf{s}_i = 0$, $i = 1,\cdots,n-1$, *we have*

$$\delta^2 G(x,y) = \left\langle (\delta^2\rho - \tilde{\kappa}(\delta\rho)^2)\frac{\partial}{\partial\boldsymbol{\nu}}G(x,\cdot), \frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,y)\right\rangle_{\partial\Omega} - 2\,(\nabla\delta G(x,\cdot), \nabla\delta G(\cdot,y))_{\Omega}\,. \tag{3.15}$$

Remark: Garabedian and Schiffer [2] dealt with domain perturbation such as

$$\mathcal{T}_t(x) = x + t\,h(x)\boldsymbol{\nu}(x), \qquad t \geq 0,$$

where $h(x)$ is a scalar function defined on $\partial\Omega$. In this case, Hadamard's second variational formula is (3.15) with $\delta\rho = h$ and $\delta^2\rho(x) = 0$ which is exactly same to Garabedian-Schiffer's formula [2]. Therefore, Theorem 3.3 is an extension of Garabedian-Schiffer's formula.

References

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