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# Optimal Reinsurance and Investment in a Point Process Market Model *

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## 1 Introduction

We consider an insurance model that allows for reinsurance and investment in the financial market. The goal is to choose the level of reinsurance and the investment so as to maximize expected utility of terminal wealth and at the same time minimize the probability of ruin up to a given horizon.

According to our model not only claims arrive at discrete random time points triggered by a Poisson process, but also asset prices change only at such random time points. This can be justified by the fact that, in reality, prices do not change continuously but rather at random discrete points in time, when market makers update their quotes. By allowing the claim sizes and the amounts of price changes to take the value zero with positive probability, one can use the same Poisson process to model the discrete random time points when either a claim arrives or a price change takes place. We shall call these random times *event times*. As a consequence of this modeling approach it is natural to allow decisions on the level of reinsurance and on the amount of investment to be made only at event times. These decisions will also be called *controls*. For a given horizon the number of event times is random and this makes our problem nonstandard. In the next Section 2 we describe more precisely our model, in Section 3 we define the risk process and state our objective on a more formal basis. The solution approach is then described for a general setup in Section 4 and in Section 5 we consider a specific model that allows for a semianalytic solution in the sense that the value function can be determined by an analytic formula while the controls have to be determined numerically.

## 2 The model

On a given horizon $[0,T]$ consider a Poisson process $N_t$ with known intensity $\lambda_t = \lambda$, of which the jump times $T_i$ are the random times determining the events (arrivals of claims and/or price changes). The interarrival times $T_{i+1} - T_i$ are the i.i.d. distributed according to a negative exponential random variable $Z$ with parameter $\lambda$. Claims arrive and prices may change only at the event times $T_i$. We shall assume the claim sizes $\{Y_{T_i}\}_{i=1,\ldots,N_T}$ to be i.i.d. and, to keep the model as simple as possible, without loss of generality we assume them to take only two values, namely $Y_{T_i} \in \{0, \overline{y}\}$ and let the probability $p := \mathbb{P}\{Y_{T_i} = \overline{y}\}$ be given. Again, to keep the model as simple as possible and without loss of generality we assume that there is only a single risky

*This paper is an abbreviated version of Edoli and Runggaldier [3]. Further details, proofs and numerical results can be found there.

asset to invest in (according to a self-financing portfolio) and we let its price evolve according to

$$S_{T_i} = S_{T_i^-} e^{W_{T_i}} \tag{2.1}$$

or, equivalently,

$$\frac{S_{T_i} - S_{T_i^-}}{S_{T_i^-}} = e^{W_{T_i}} - 1 \tag{2.2}$$

where $W_{T_i} \in [\underline{w}, \overline{w}]$ are i.i.d. with $\underline{w} < 0 < \overline{w}$ having a point mass at zero. Notice, in fact, that if at an event time $T_i$ we have $W_{T_i} = 0$, this means that at this event time only the arrival of a claim may occur and viceversa. Always for simplicity we let $W_{T_i} \in \{-d, 0, u\}$ with $d, u > 0$ and assume that $N_t$ and the distributions of $Y_{T_i}$ and $W_{T_i}$ are independent.

The controls (decision variables) are given by the level of reinsurance and by the monetary amount invested in the risky asset at the various event times. We consider a proportional reinsurance scheme, namely the part of the claim paid by the company is $h(b, Y) = bY$ with $b_{T_i} \in [0, 1]$. The amount invested in the risky asset is denoted by $\delta_{T_i}$ and we let $\delta_{T_i} \in [-C_1, C_2]$ (the investor cannot get indebted beyond a certain level, nor invest arbitrarily large amounts) so that we obtain a compact control space $U = [0, 1] \times [-C_1, C_2]$. Denoting by $c$ the premium rate collected by the company (we suppose it to be given), the net premium rate of the company is then

$$c(b) := c - (1 + \theta)\frac{\mathbb{E}\left[Y - h(b, Y)\right]}{\mathbb{E}\left[Z \wedge T\right]} \quad \text{(recall that } T_{i+1} - T_i \sim Z) \tag{2.3}$$

where we have used the so-called *expected value principle with safety loading of the insurer* (see e.g. [4]). Choosing $c \geq (1 + \theta)\frac{\mathbb{E}[Y]}{\mathbb{E}[Z \wedge T]}$ guarantees that $c(b) \geq 0$ for all $b \in [0, 1]$ and notice that, for $c = (1 + \theta)\frac{\mathbb{E}[Y]}{\mathbb{E}[Z \wedge T]}$, one has $c(b) = 0$ for $b = 0$.

## 3 Risk process and objective

According to the model of the previous section the *risk process* (wealth process of the insurance company) satisfies

$$X_t = X_0 + \int_0^t c(b_t)dt + \sum_{n=1}^{N_t} \left[\delta_{T_n}\left(e^{W_{T_n}} - 1\right) - h\left(b_{T_n}, Y_{T_n}\right)\right] \tag{3.1}$$

where we have used the fact that, according to the self-financing investment in the financial market,

$$\frac{\delta_{T_n}}{S_{T_n^-}}\left(S_{T_n} - S_{T_n^-}\right) = \delta_{T_n}\left(e^{W_{T_i}} - 1\right) \tag{3.2}$$

Allowing only for wealth levels $x$ in the set

$$\mathcal{S} = \{x \in \mathbb{R} \quad s.t. \quad x \geqslant -K\} \tag{3.3}$$

the set of admissible controls is given by

$$\Phi = \left\{\phi := (b, \delta) \in U \mid \text{ constant on } (T_{i-1}, T_i],\ i = 1, \cdots, N_T,\ X_t \in \mathcal{S}\right\} \tag{3.4}$$

and notice that $\Phi \neq \emptyset$ as it contains $\phi = (0,0)$ (recall that $c(b) \geq 0$). With the constraint (3.3) the dynamics of $X_t$ can, according to (3.1), be written in differential form as

$$dX_t = \left[c(b_t)dt + \left[\delta_t\left(e^{W_t}-1\right) - h\left(b_t, Y_t\right)\right]dN_t\right]\mathbf{1}_{\{X_t \geqslant -K\}} \tag{3.5}$$

Inspired by [6], we have in mind as general objective the maximization of expected utility of terminal wealth and the simultaneous minimization of the probability of ruin over the given horizon $[0,T]$. To this effect we proceed as follows. Let $g(x), G(x) : \mathcal{S} \to \mathbb{R}$ be two (continuous) utility functions that are bounded, namely $g(x), G(x) \leq \mathcal{G}$. An example of such functions that we shall use later is

$$g(x) = 1 - \gamma e^{-\beta x} \quad , \quad G(x) = 1 - \mu e^{-\beta x}. \tag{3.6}$$

Denoting by $V^{\phi}(t,x)$ the expected total utility when the level of wealth at time $t$ is $V_t = v$ and a strategy $\phi$ is being used (with some abuse of notation we shall indicate by the same $\phi$ the entire strategy over the generic $[t,T]$ as well as its individual values at the various time points) we have

$$V^{\phi}(t,x) := \mathbf{1}_{\{t \leqslant T\}}\mathbb{E}^{\phi}_{t,x}\left[\sum_{k=N_t+1}^{N_T} g\left(X^{\phi}_{T_k}\right) + G\left(X^{\phi}_{T}\right)\right] \tag{3.7}$$

The problem is then to determine $\phi^* \in \Phi$ s.t.:

$$V^{\phi^*}(T_n,x) := V^*(T_n,x) = \sup_{\phi\in\Phi} V^{\phi}(T_n,x) \tag{3.8}$$

Notice that, with utility functions as in (3.6), by maximizing the expected total running utility $\mathbb{E}^{\phi}_{t,x}\left[\sum_{k=N_t+1}^{N_T} g\left(X^{\phi}_{T_k}\right)\right]$ one implicitly also minimizes the probability of ruin up to the terminal time $T$ (see e.g. also [6]).

## 4 Solution approach (Value iteration)

To solve the problem (3.8) we shall use an approach based on the Dynamic Programming Principle. Since the problem is formulated in continuous time, we could use an approach based on the HJB equation (see e.g. [4], [7]), but in our case this turns out to be difficult to implement and it requires regularity properties of the value functions that are not easy to be satisfied. Since, on the other hand, the problem can also be seen in discrete time (even if with random discrete time points), we may use *Value iteration*, for which we base ourselves on results in [5] (see also [1] and [2] in the context of portfolio optimization).

Recalling the value iteration in discrete time, namely

$$(T^{\phi}v)(n,x) = g(X^{\phi}_n) + E_{n,x}\{v(X_{n+1})\}, \tag{4.1}$$

for $v \in \mathcal{B}(\mathbb{R}^+ \times \mathcal{S}) = \mathcal{B}$, the space of bounded functions, and for a given $\phi \in \Phi$ define the operator $T^{\phi} : \mathcal{B} \to \mathcal{B}$ as

$$\left(T^{\phi}v\right)(t,x) := \mathbb{E}^{\phi}_{t,x}\left[\mathbf{1}_{\{t+Z\leqslant T\}}\, g\left(X^{\phi}_{t+Z}\right) + v\left(t+Z, X^{\phi}_{t+Z}\right) + \mathbf{1}_{\{t\leqslant T<t+Z\}}G\left(X^{\phi}_{T}\right)\right] \tag{4.2}$$

where, we recall, $Z$ has a negative-exponential distribution that has the property of being memory-less (recall also that in the definition of the value function in (3.7) we had the indicator function, namely $V^\phi(t,x) := \mathbf{1}_{\{t\leqslant T\}}\mathbb{E}^{\phi}_{t,x}[\cdots]$.

Notice that, given $t$ and $X_t^\phi = x$, the dependence on $\phi$ of $X_{t+Z}^\phi$ (and of $X_T^\phi$ for the event $t \leq T < t+Z$) in the definition of $T^\phi$ is only through its value at the jump time $N_t$. This justifies us to define also the following operator

$$(T^* v)(t,x) := \sup_{\phi\in\Phi}\left(T^\phi v\right)(t,x) \tag{4.3}$$

with the meaning *(we use the shorthand $\phi_{N_t}$ for $\phi_{T_{N_t}}$)*

$$\sup_{\phi\in\Phi}\left(T^\phi v\right)(t,x) = \sup_{\phi_{N_t}=(b_{N_t},\delta_{N_t})\in U}\left(T^\phi v\right)(t,x) \tag{4.4}$$

so that the maximizing $\phi^*$, if it exists, is a function of $(t,x)$.

We have now the following propositions (their proofs can be found in [3])

**Proposition 1 (Contraction property on the space $\mathcal{B}(\mathbb{R}^+\times\mathcal{S})$)**
*Let $v, v' \in \mathcal{B}(\mathbb{R}^+\times\mathcal{S})$ then*

- $\left\|T^\phi v - T^\phi v'\right\|_\infty \leqslant \left(1-e^{-\lambda T}\right)\|v-v'\|_\infty \quad , \quad \forall\phi\in\Phi$
- $\|T^* v - T^* v'\|_\infty \leqslant \left(1-e^{-\lambda T}\right)\|v-v'\|_\infty$

**Proposition 2 (Existence of a fixed point in $\mathcal{B}$)**
*Let $\phi\in\Phi$ be a strategy. There exist $V^\phi(t,x)$ and $\bar{V}^*(t,x)$ such that $\forall(t,x)\in[0,T]\times\mathcal{S}$ the following equalities hold:*

- $\left(T^\phi V^\phi\right)(t,x) = V^\phi(t,x)$
- $\left(T^*\bar{V}^*\right)(t,x) = \bar{V}^*(t,x)$

Furthermore, recalling that $\phi^*$ is a function $\phi^*(t,x)$,

**Proposition 3 (Existence of an optimal control)**
*Let $v\in\mathcal{C}_\mathcal{B}(\mathbb{R}^+,\mathcal{S})$ the space of continuous and bounded functions. Then*

*i.* $\phi^*(t,x) = \arg\left(\sup_{\phi\in\Phi}(T^\phi v)\right)(t,x)$ *exists and belongs to $\mathcal{C}_\mathcal{B}$ (Continuous selection theorem)*

*ii.* $T^* : \mathcal{C}_\mathcal{B} \to \mathcal{C}_\mathcal{B}$

This allows us now to obtain the following theorem (it can be considered as a form of *verification theorem*)

**Theorem 1** *Under the assumption that $g(\cdot)$ and $G(\cdot)$ in the definition of $V^\phi(\cdot)$ in (3.7) are bounded and continuous we have*

*i. The optimal $V^*$ coincides with $\bar{V}^*$ which is the unique fixed point of $T^*$ in $\mathcal{C}_\mathcal{B}(\mathbb{R}^+,\mathcal{S})$;*

*ii. the (stationary) strategy* $\tilde{\phi} = \arg\left(\sup_{\phi\in\Phi} T^{\phi}\bar{V}^*\right)$ *is optimal.*

In view of this theorem, in order to solve our problem we would have to iterate the operator $T^*$ infinitely often. In practice we shall iterate it only a finite number $m$ of times. Starting from

$$v_0^*(T_{m-1}, x) = \mathbf{1}_{\{T_{m-1}\leq T\}} \sup_{b\in[0,1]} G\left(x + c(b)\left(T - T_{m-1}\right)\right) \tag{4.5}$$

where $v_0^*$ represents the maximum expected utility when there are no more jumps before $T$, this then leads to a strategy in $\Phi^m := \{\phi_1^m, \phi_2^m, \cdots, \phi_m^m\} \subset \Phi$ given by

$$\phi^{*,m} := \left(\phi_1^{*,m}(0,x), \cdots, \phi_m^{*,m}\left(T_{m-1}, X_{m-1}^{(\phi_1^m,\cdots,\phi_{m-1}^m)}\right)\right) \tag{4.6}$$

where $\phi_1^{*,m}(T_0, X_{T_0})$ results from the last iteration of $T^*$ with $T_0 = 0$, $X_0 = x$ and, generically, for $i \in \{1, \cdots, m\}$ the $\phi_i^{*,m}\left(T_{i-1}, X_{T_{i-1}}\right)$results from the $(m-i)-th$ iteration of $T^*$.

Notice now that the strategy $\phi^{*,m} \in \Phi^m$ can be extended into a strategy $\phi \in \Phi$ by adding arbitrary, e.g. zero, components. On the other hand, given a strategy for $N_T + 1$ periods, namely $\phi = \left(\phi_{T_0}, \phi_{T_1}, \cdots, \phi_{T_m-1}, \phi_{T_m}, \cdots, \phi_{T_{N_T}}\right)$, denote by $\phi^{|m}$ its restriction to the first $m$ components.

The following convergence results show that, iterating $T^*$ a finite but sufficiently large number of times, leads to a strategy for which one obtains a value that is arbitrarily close to the actual optimal value. We have in fact (proofs are again in [3])

**Proposition 4 (Fixed point estimates)** *Given* $\varepsilon > 0$, $\forall m > m_\varepsilon$ *with* $m_\varepsilon = \dfrac{\log\left(\frac{\varepsilon}{4G}\right) - \lambda T}{\log\left(1 - e^{-\lambda T}\right)}$ *and* $\forall \phi \in \Phi$ *we have*

- $\left\|V^{\phi} - v_m^{\phi^{|m}}\right\|_\infty < \varepsilon,$
- $\left\|V^* - v_m^{\phi^{*,m}}\right\|_\infty < \varepsilon$

As a corollary we then obtain

**Theorem 2** *Completing arbitrarily the strategy* $\phi^{*,m}$ *to become a strategy* $\hat{\phi} \in \Phi$ *one has*

$$\left\|V^* - V^{\hat{\phi}}\right\|_\infty \leqslant \left\|V^* - v_m^{\phi^{*,m}}\right\|_\infty + \left\|v_m^{\phi^{*,m}} - V^{\hat{\phi}}\right\|_\infty \leqslant 2\varepsilon \tag{4.7}$$

Truncating the value iteration according to the previous results is a standard approach to compute numerically a nearly optimal value and control. Another way to obtain a solution, this time an analytic solution, is to see whether there exists a (finitely parametrized) class of functions that is closed under the operator $T^*$. In this latter case the optimal value can then be found within this class. This is the subject of the next section.

# 5 A specific case with a semianalytic solution

Let

$$g(x) = 1 - \gamma e^{-\beta x} \quad , \quad G(x) = 1 - \mu e^{-\beta x} \tag{5.1}$$

for constants $\gamma, \mu, \beta \in \mathbb{R}^+, \beta \neq 0$ and define the set of functions

$$\mathcal{V} = \left\{ v : \mathbb{R}^+ \times \mathcal{S} \longrightarrow \mathbb{R} \quad s.t. \quad v(t,x) = \mathbf{1}_{\{t \leqslant T\}} \left( M(t) - e^{-\beta x} \nu(t) \right), \quad \nu(t) > 0, \beta > 0 \right\} \tag{5.2}$$

We have the following theorems (with proof in [3])

**Theorem 3 (Closedness of $\mathcal{V}$ under $T^\phi$ and $T^*$)**
*Given $\phi = (b, \delta) \in \Phi$, let*

$$v(t,x) = \mathbf{1}_{\{t \leqslant T\}} \left( M(t) - e^{-\beta x} \nu(t) \right) \in \mathcal{V}$$

*Then there exists $\tilde{M}(t)$ and $\tilde{\nu}(t, b, \delta) > 0$ s.t.*

$$\left(T^\phi v\right)(t,x) = \mathbf{1}_{\{t \leqslant T\}} \left[ \tilde{M}_M(t) - e^{-\beta x} \tilde{\nu}_\nu(t, b, \delta) \right] \in \mathcal{V}$$

$$(T^* v)(t,x) = \mathbf{1}_{\{t \leqslant T\}} \left[ \tilde{M}_M(t) - e^{-\beta x} \tilde{\nu}_\nu(t, b^*, \delta^*) \right] \in \mathcal{V}$$

*whereby*

$$\begin{aligned} \tilde{M}_M(t) &= 1 + \mathbb{E}\left[ \mathbf{1}_{\{t+Z \leqslant T\}} M(t+Z) \right] \\ \tilde{\nu}_\nu(t,b,\delta) &= \mathbb{E}^\phi \Big[ \mathbf{1}_{\{t+Z \leqslant T\}} e^{-\beta\left(c(b)t - bY + \delta\left(e^W - 1\right)\right)} \left(\gamma + \nu(t+Z, b, \delta)\right) \\ &\qquad + \mathbf{1}_{\{t+Z > T\}} \mu e^{-\beta c(b)(T-t)} \Big] > 0 \\ (b^*, \delta^*) &= \arg\inf_{(b,\delta)} \tilde{\nu}_\nu(t, b, \delta) \end{aligned}$$

**Theorem 4 (Characterization of $V^*$)**
*Let*

$$V^*(t,x) = \mathbf{1}_{\{t \leqslant T\}} \left( M^*(t) - e^{-\beta x} \nu^*(t, b^*, \delta^*) \right)$$

*for $M^*(t)$ and $\nu^*(t,b,\delta)$ satisfying the following Volterra-type integral equations*

$$\begin{aligned} M^*(t) &= 1 + \lambda \int_0^{T-t} M^*(t+\xi) e^{-\lambda\xi} d\xi \\ \nu^*(t,b,\delta) &= \mathbb{E}^\phi \left[ e^{-\beta\left(\delta\left(e^W - 1\right) - bY\right)} \right] \\ &\quad \cdot \mathbb{E}^\phi \left[ \mathbf{1}_{\{t+Z \leqslant T\}} e^{-\beta c(b) Z} \left(\gamma + \nu^*(t+Z, b, \delta)\right) \right] \\ &\quad + \mu e^{-\left(\lambda + \beta c(b)\right)(T-t)} \end{aligned}$$

*Then $V^*(t,x) \in \mathcal{V}$ and it is the unique fixed point of $T^*$; furthermore,*

$$\phi^* = (b^*, \delta^*) = \arg\inf_{(b,\delta)} \nu^*(t, b, \delta)$$

*is the optimal strategy (independent of $x$).*

As a result of the last theorem we see that $M^*(t)$ and $\nu^*(t,b,\delta)$ can be given explicit analytic expressions. On the other hand, the $\arg\inf_{(b,\delta)} \nu^*(t,b,\delta)$ has to be computed numerically. This is the sense in which we intended the *semianalytic solution*. Since the optimal control $(b^*, \delta^*)$ is determined by the behavior of $\nu(t,b,\delta)$, it depends at most on time. From the numerical calculations it turned out, see [3], that the optimal control is most sensitive to the distributional characteristics of the claim size $Y$ and of $W$, which drives the prices, with a behavior that corresponds to intuition: large and frequent claims lead to larger levels of reinsurance; furthermore, a favorable market situation leads to higher levels of investment.

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