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The Impact of Callable Convertible Debt Financing on Investment Timing*

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1 Introduction

Many companies issue convertible debt as a means of debt financing since 1980's. There are already many studies on the valuation of convertible debt (e.g. Ayache et al. [1], Brennan and Schwartz [2], Ingersoll [5], Sirbu et al. [17], Takahashi et al. [20], Tsiveriotis and Fernandes [21], Yagi and Sawaki [22], etc.). However, in these studies the tradeoff between tax shield and bankruptcy cost, and firm value have not been argued. Koziol [8], Liao and Huang [11] and Sarkar [16],have presented the valuation of convertible debt by the framework of Leland [10], but the investment has not been taken into account and the optimal capital structure has not been analyzed. Egami [3] and Lyandres and Zhdanov [12] have investigated the interaction among firm's investment and convertible debt financing decision, but the optimal capital structure has not been analyzed. We examine the optimal strategy for the investment financed by issuing convertible debt on the optimal capital structure and investigate the consistency with empirical evidence in Korkeamaki and Moore [7].

Most convertible debt are callable. Liao and Huang [11] and Sarkar [16] have presented the valuation of callable convertible debt by the framework of Leland [10], but the investment and optimal capital structure have not been analyzed. Korkeamaki and Moore [6] and Mayers [15] are the empirical studies on callable convertible debt financing and investment. We suggest theoretical model about callable convertible debt financing and investment, and discuss the consistency of our model with empirical evidence.

In this paper we examine the optimal investment strategy of the firm financed by issuing callable convertible debt on the optimal capital structure. Especially, we investigate how the issue of callable convertible debt affect the optimal capital structure and the optimal investment strategies.

2 The Model

Consider a firm with an option to invest at any time by paying a fixed investment cost I. The firm decides whether to invest, observing a demand shock X_t for its product. We suppose the

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firm can observe the demand shock X_t, whereX_t is given by a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x, \tag{2.1}$$

where μ and σ are the risk-adjusted expected growth rate and the volatility of X_t, respectively, and W_t is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We consider a firm which has an option of the investment is financed with equity and convertible debt with coupon payment c and infinite maturity. Once the investment option is exercised, we assume that the firm can receive instantaneous profit

$$\pi(X_t) = (1-\tau)(QX_t - c), \tag{2.2}$$

where τ is a constant corporate tax rate and Q is the quantity produced from the asset in place.

Once the investment option has been exercised, the optimal default policy is established from the issue of debt. The optimal default policy of the equity holders selects the optimal default time, maximizing the equity value. On the other hand, the optimal conversion policy of the convertible debt holders selects the optimal conversion time, maximizing the value of convertible debt. In this case, the optimal problems for the holders of equity and convertible debt must be solved simultaneously. Here, we assume that the holders of convertible debt can convert the debt into a fraction η of the original equity. We follow Brennan and Schwartz [2] and assume block conversion, that is, all convertible debt holders exercise the conversion option at the same time. First, we present the formulations for the values of equity and convertible debt issued at investment time. After that, we consider the optimal investment strategies.

2.1 The Value of Convertible Debt

In this section we examine the values of equity and convertible debt issued at investment time. Let $\mathcal{T}_{t_1,t_2}$ be the set of stopping times with respect to the filtration as $\{\mathcal{F}_s; t_1 \le s \le t_2\}$ and $T_d \in \mathcal{T}_{0,\infty}$ and $T_c \in \mathcal{T}_{0,\infty}$ be the default and conversion times. Denoting $E(x,c)$ as the total value of equity issued at investment time and $D_c(x,c)$ as that of convertible debt with coupon payment of c, $E(x,c)$ and $D_c(x,c)$ are formulated as

$$\begin{aligned} E(x,c) &= \sup_{T_d \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[\int_0^{T_c^*(c)\wedge T_d} e^{-ru}(1-\tau)(QX_u - c)du \right. \\ &\quad \left. + 1_{\{T_c^*(c)<T_d\}} \frac{1}{1+\eta} \int_{T_c^*(c)}^{\infty} e^{-ru}(1-\tau)QX_u du \right], \qquad (2.3) \\ D_c(x,c) &= \sup_{T_c \in \mathcal{T}_{0,\infty}} E_0^x \left[\int_0^{T_c \wedge T_d^*(c)} e^{-ru} c du + 1_{\{T_d^*(c)<T_c\}} e^{-rT_d^*(c)}(1-\theta)\epsilon(X_{T_d^*(c)}) \right. \\ &\quad \left. + 1_{\{T_c<T_d^*(c)\}} \frac{\eta}{1+\eta} \int_{T_c}^{\infty} e^{-ru}(1-\tau)QX_u du \right], \qquad (2.4) \end{aligned}$$

where $\mathbb{E}_t^x$ is the conditional expectation operator given that X_t equals x, r is the risk-free interest rate with $r > \mu$, $1_{\{T_c^*(c)<T_d\}}$ is an indicator function that is equal to one if $T_c^*(c) < T_d$ and is equal to zero otherwise, θ is the proportional bankruptcy cost and $\epsilon(x)$ is the total post-investment profit in which the investment is financed entirely with equity,

$$\epsilon(x) = \frac{1-\tau}{r-\mu} Qx. \tag{2.5}$$

Also, the optimal default and conversion times for any c, $T_d^*(c)$ and $T_c^*(c)$, respectively, are given by

$$\begin{aligned} T_d^*(c) &= \inf\{T_d \in [0,\infty) \mid X_{T_d} \leq x_d(c)\}, & (2.6) \\ T_c^*(c) &= \inf\{T_c \in [0,\infty) \mid X_{T_c} \geq x_c(c)\}, & (2.7) \end{aligned}$$

where $x_d(c)$ and $x_c(c)$ are the optimal default and conversion thresholds for any c. Eq. (2.3) means that the equity holders can receive the tax-deductible earning after paying coupon until conversion or default and that the equity value is diluted by converting, that is, the dilution factor is one over one plus eta. On the other hand, Eq. (2.4) implies that the convertible debt holders can receive the coupon payment until conversion or default and a fraction eta of the original equity on conversion, and are entitled to $(1-\theta)\epsilon(X_{T_d^*(c)})$ at bankruptcy.

Once the convertible debt has been converted, the firm becomes an all-equity entity. It follows from the optimal problems of the equity holders and convertible debt holders in (2.3) and (2.4), respectively, that the general solutions for the values of equity and convertible debt prior to default and conversion are given by

$$\begin{aligned} E(x,c) &= a_1 x^{\beta_1} + a_2 x^{\beta_2} + (1-\tau)\left(\frac{Qx}{r-\mu} - \frac{c}{r}\right), & (2.8) \\ D_c(x,c) &= a_3 x^{\beta_1} + a_4 x^{\beta_2} + \frac{c}{r}, & (2.9) \end{aligned}$$

where a_i, $i = 1, \cdots, 4$ are determined by boundary conditions, $\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ and $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$. The upper boundary conditions which come from the conversion policy of convertible debt holders are given by

$$\begin{aligned} E(x_c(c),c) &= \frac{1}{1+\eta}\frac{1-\tau}{r-\mu} Q x_c(c), & (2.10) \\ D_c(x_c(c),c) &= \frac{\eta}{1+\eta}\frac{1-\tau}{r-\mu} Q x_c(c). & (2.11) \end{aligned}$$

The lower boundary conditions which relate to the default threshold are given by

$$\begin{aligned} E(x_d(c),c) &= 0, & (2.12) \\ D_c(x_d(c),c) &= (1-\theta)\frac{1-\tau}{r-\mu} Q x_d(c). & (2.13) \end{aligned}$$

Substituting Eqs. (2.8) and (2.9) into Eqs. (2.10)–(2.13), we may determine that

$$
\begin{aligned}
E(x,c) &= (1-\tau)\left(\frac{Qx}{r-\mu}-\frac{c}{r}\right)-(1-\tau)\left(\frac{Qx_d(c)}{r-\mu}-\frac{c}{r}\right)p_d(x,c;x_c(c)) \\
&\quad -(1-\tau)\left(\frac{\eta}{1+\eta}\frac{Qx_c(c)}{r-\mu}-\frac{c}{r}\right)p_c(x,c;x_d(c)), \qquad (2.14)\\
D_c(x,c) &= \frac{c}{r}+\left((1-\theta)\frac{1-\tau}{r-\mu}Qx_d(c)-\frac{c}{r}\right)p_d(x,c;x_c(c)) \\
&\quad +(1-\tau)\left(\frac{\eta}{1+\eta}\frac{Qx_c(c)}{r-\mu}-\frac{c}{r}\right)p_c(x,c;x_d(c)), \qquad (2.15)
\end{aligned}
$$

where $p_d(x,c;x_c(c))$ is the expected present values of \$1 contingent on X_t first reaching the default threshold $x_d(c)$ from above before reaching the conversion threshold $x_c(c)$ and $p_c(x,c;x_d(c))$ is that of \$1 contingent on X_t first reaching the conversion threshold $x_c(c)$ from bellow before reaching the default threshold $x_d(c)$, that is,

$$
\begin{aligned}
p_d(x,c;x_c(c)) &= \frac{x_c(c)^{\beta_1}x^{\beta_2}-x_c(c)^{\beta_2}x^{\beta_1}}{x_c(c)^{\beta_1}x_d(c)^{\beta_2}-x_c(c)^{\beta_2}x_d(c)^{\beta_1}}, \qquad (2.16)\\
p_c(x,c;x_d(c)) &= \frac{x^{\beta_1}x_d(c)^{\beta_2}-x^{\beta_2}x_d(c)^{\beta_1}}{x_c(c)^{\beta_1}x_d(c)^{\beta_2}-x_c(c)^{\beta_2}x_d(c)^{\beta_1}}. \qquad (2.17)
\end{aligned}
$$

Then, summing the values of equity and convertible debt, the firm value $V(x,c)$ is represented by

$$
\begin{aligned}
V(x,c) &= E(x,c)+D_c(x,c) \\
&= \epsilon(x)+\frac{\tau c}{r}(1-p_d(x,c;x_c(c)))-\theta\epsilon(x_d(c))p_d(x,c;x_c(c)). \qquad (2.18)
\end{aligned}
$$

Eq. (2.18) equals the unlevered firm value plus the expected present value of debt tax shields minus the expected present value of bankruptcy cost†.

Here, we determine the optimal default and conversion thresholds. The optimal default threshold is determined by the smooth-pasting condition which equals the partial derivation of $E(x,c)$ with respect to x with the deviation of payoff for the equity holders at the default threshold $x_d(c)$. On the other hand, the optimal conversion threshold derived from the smooth-pasting condition which the partial derivation of $D_c(x,c)$ with respect to x with the deviation of payoff for the holders of convertible debt at the conversion threshold $x_c(c)$. Hence,

$$
\begin{aligned}
\frac{\partial E}{\partial x}(x_d(c),c) &= 0, \qquad (2.19)\\
\frac{\partial D_c}{\partial x}(x_c(c),c) &= \frac{\eta}{1+\eta}\frac{1-\tau}{r-\mu}Q. \qquad (2.20)
\end{aligned}
$$

†From Mauer and Sarkar [13], in the case of straight debt financing the firm value for any coupon payment s is given by

$$
V(x,s)=\epsilon(x)+\frac{\tau s}{r}\left(1-\left(\frac{x}{x_d(s)}\right)^{\beta_2}\right)-\theta\epsilon(x_d(s))\left(\frac{x}{x_d(s)}\right)^{\beta_2}.
$$

Even if substituting Eqs. (2.14) and (2.15) into Eqs. (2.18) and (2.19), the optimal default and conversion thresholds cannot be solved analytically. Hence, the optimal threshold must be solved numerically.

2.2 The Investment Strategies

Next, we consider the optimal investment strategy. The optimal capital structure, that is, the optimal coupon payment is determined from maximizing the firm value given by equation (2.18) on investment. On the otner hand, the equity holders of the firm which invests selects the optimal investment time, maximizing the equity value. Letting $T \in \mathcal{T}_{0,\infty}$ be the investment time, the value of the investment partially financed with convertible debt $F(x)$ is formulated as

$$\begin{aligned} F(x) &= \sup_{T \in \mathcal{T}_{0,\infty}, c>0} \mathbb{E}_0^x \left[e^{-rT} \left(E(X_T, c) - (I - D_c(X_T, c)) \right) \right] \\ &= \sup_{T \in \mathcal{T}_{0,\infty}, c>0} \mathbb{E}_0^x \left[e^{-rT} \left(V(X_T, c) - I \right) \right]. \end{aligned} \tag{2.21}$$

The optimal investment time T^* is given by

$$T^* = \inf\{T \in [0, \infty) \mid X_T \geq x^*\}, \tag{2.22}$$

where x^* is the optimal investment thresholds.

From Sundaresan and Wang [19], the optimal coupon payment for any x is given by maximizing the firm value

$$c^*(x) = \arg\max_{c>0} V(x, c). \tag{2.23}$$

From the boundary condition at the investment threshold, the investment value is given by

$$F(x) = (V(x^*, c^*(x^*)) - I) \left(\frac{x}{x^*} \right)^{\beta_1} \tag{2.24}$$

and the optimal investment threshold is given by the smooth-pasting condition

$$\frac{dF}{dx}(x^*) = \frac{\partial V}{\partial x}(x^*, c^*(x^*)). \tag{2.25}$$

Hence, the optimal capital structure and the optimal investment strategies are determined by solving nonlinear simultaneous equations (2.23) and (2.25).

3 Numerical Analysis

In this section, the calculation results of the values of equity, convertible debt and the investment option, the optimal thresholds for default, conversion and investment, the optimal coupon payments and the optimal leverage ratio are presented in order to investigate how the issue of convertible debt affects the optimal investment strategies and the optimal capital structure. We use the following base case parameters: $Q = 1,\ \mu = 0,\ \sigma = 0.2,\ r = 0.05,\ \theta = 0.3,\ \tau = 0.3,\ \eta = 0.4,\ I = 5.0$.

Fig. 1 shows the values of equity and convertible debt as a function of demand shock x and Fig. 2 shows the investment value. Under the base parameters, the optimal coupon payment is $c^* = c^*(x^*) = 0.420$, the optimal default threshold is $x_d = x_d(c^*) = 0.229$, the optimal conversion threshold is $x_c = x_c(c^*) = 3.181$ and the optimal investment threshold is $x^* = 0.575$.

Tabs. 1 and 2 represent the optimal coupon payments, the optimal investment threshold, optimal default threshold, the optimal conversion threshold, the equity value, the debt value and the optimal leverage ratio. In Tab. 1 we derive the optimal capital structure, that is, the optimal coupon payment is determined from maximizing the firm value. In Tab. 2 the coupon payment is determined under the financing constraint, that is, the condition that the investment cost I is equal to the issue value of convertible debt $D_c(x^*, c)$[‡].

First, we compare the results in the optimal capital structure case in Tab. 1 with that in the financing constraint case in Tab. 2. Coupon payments, investment threshold, default threshold, conversion threshold, debt value and optimal leverage ratio in the optimal capital structure case is larger than that in the financing constraint case. On the other hand, equity value in the optimal capital structure case is smaller than that in the financing constraint case. Since the issuance of debt on the optimal capital structure has no restriction relative to the financing constraint case, the firm sets a higher coupon payment, delaying the investment. Being leveraged, the firm issues more convertible debt, so the equity value decreases and then the default occurs earlier. Also, since the value of conversion option decreases, the conversion occurs later.

We focus on the coupon payments and the investment, default and conversion thresholds with respect to volatility σ when the conversion ratio η equals 0.4. When volatility increases, the investment and conversion threshold in both cases also increases and the default threshold in the financing constraint case decreases. In standard real options model, it's noted that increase in volatility leads to delaying decision-making. On the other hand, the default threshold in the optimal capital structure case increases when volatility increases. On the optimal capital structure, since the firm finances with higher coupon payment in higher volatility, the default occurs earlier. Hence, the possibility of default increases. Also, when volatility increases, the optimal coupon payment increases, while the coupon payment in the financing constraint case decreases. Since the value of conversion option increases in volatility and the issue of debt for the coupon payment in the financing constraint case is limited, the firm must set lower coupon payments.

Next, we analyze the coupon payments, the investment and default thresholds and equity value with respect to conversion ratio η when the volatility is equal to 0.2. When conversion ratio increases, the default threshold in the optimal capital structure case increases, while threshold in the financing constraint case decreases. As the conversion ratio is higher, the equity value is more diluted. On the optimal capital structure, since the issue of debt has no restriction, the decrease in the equity value becomes apparent and the default occurs earlier. This result on the optimal capital structure is consistent with the results in Koziol [8]. Also, in the case of higher volatility($\sigma = 0.3$, $\sigma = 0.4$), the coupon payment in the both cases decreases when the conversion ratio η increases. In the case of lower volatility($\sigma = 0.2$), when conversion ratio

[‡]Tab. 2 uses the same constraint as Egami [3]: $I = D_c(x^*, c)$.

increases, the optimal coupon payment increases, while the coupon payment in the financing constraint case decreases. On the optimal capital structure, the equity value decreases and the value of conversion option also decreases in lower volatility. Hence, the firm sets higher coupon payment, and issues more convertible debt.

Tab. 3 represents the coupon payments and the investment threshold in the cases of straight and convertible debt financing and the difference of the investment threshold in both financing cases when $r = 0.03, 0, 05, 0.07$, $\mu = -0.12, 0, 0.12$ and $\sigma = 0.2, 0.3, 0.4$. Korkeamaki and Moore [7] show that the firm with high-growth prospection, high volatility, and low capital costs issuing convertible debt tends to defer investment longer. These results on Tab. 3 are consistent with that in Korkeamaki and Moore [7].

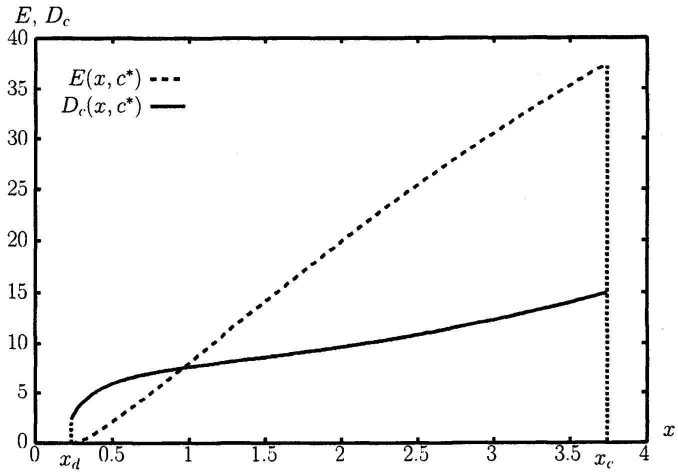

Figure 1: The values of equity and convertible debt

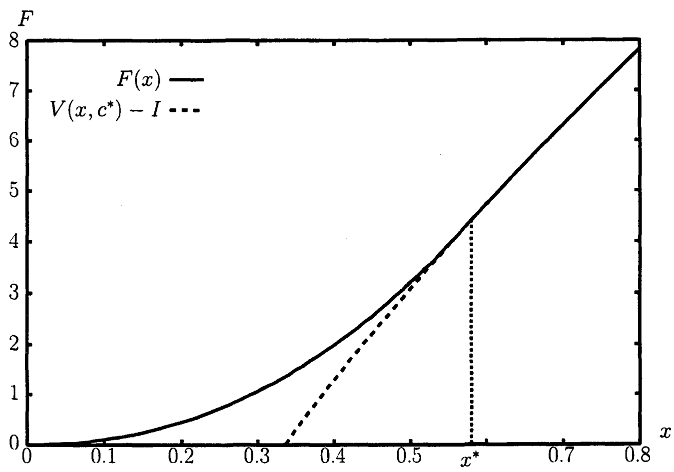

Figure 2: The value of investment option

Table 1: The effects for investment decision and financing for volatility and conversion ratio : the optimal capital structure

	Straight debt	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$	$\eta = 0.5$
	$s^* = 0.415$	$c^* = 0.415$	$c^* = 0.417$	$c^* = 0.418$	$c^* = 0.420$	$c^* = 0.421$
	$x^* = 0.570$	$x^* = 0.570$	$x^* = 0.571$	$x^* = 0.572$	$x^* = 0.574$	$x^* = 0.575$
	$x_d = 0.223$	$x_d = 0.223$	$x_d = 0.225$	$x_d = 0.226$	$x_d = 0.228$	$x_d = 0.229$
$\sigma = 0.2$		$x_c = 12.016$	$x_c = 6.507$	$x_c = 4.658$	$x_c = 3.740$	$x_c = 3.181$
	$E(x^*) = 3.076$	$E(x^*) = 3.066$	$E(x^*) = 3.026$	$E(x^*) = 2.980$	$E(x^*) = 2.921$	$E(x^*) = 2.864$
	$D_s(x^*) = 6.241$	$D_c(x^*) = 6.249$	$D_c(x^*) = 6.293$	$D_c(x^*) = 6.337$	$D_c(x^*) = 6.397$	$D_c(x^*) = 6.450$
	$\frac{D_s(x^*)}{V(x^*)} = 0.670$	$\frac{D_c(x^*)}{V(x^*)} = 0.671$	$\frac{D_c(x^*)}{V(x^*)} = 0.675$	$\frac{D_c(x^*)}{V(x^*)} = 0.680$	$\frac{D_c(x^*)}{V(x^*)} = 0.687$	$\frac{D_c(x^*)}{V(x^*)} = 0.692$
	$s^* = 0.630$	$c^* = 0.629$	$c^* = 0.629$	$c^* = 0.628$	$c^* = 0.628$	$c^* = 0.627$
	$x^* = 0.785$	$x^* = 0.786$	$x^* = 0.789$	$x^* = 0.791$	$x^* = 0.794$	$x^* = 0.797$
	$x_d = 0.252$	$x_d = 0.253$	$x_d = 0.255$	$x_d = 0.257$	$x_d = 0.259$	$x_d = 0.262$
$\sigma = 0.3$		$x_c = 23.350$	$x_c = 12.337$	$x_c = 8.673$	$x_c = 6.855$	$x_c = 5.756$
	$E(x^*) = 4.652$	$E(x^*) = 4.602$	$E(x^*) = 4.507$	$E(x^*) = 4.400$	$E(x^*) = 4.285$	$E(x^*) = 4.173$
	$D_s(x^*) = 7.848$	$D_c(x^*) = 7.896$	$D_c(x^*) = 7.993$	$D_c(x^*) = 8.098$	$D_c(x^*) = 8.217$	$D_c(x^*) = 8.329$
	$\frac{D_s(x^*)}{V(x^*)} = 0.628$	$\frac{D_c(x^*)}{V(x^*)} = 0.632$	$\frac{D_c(x^*)}{V(x^*)} = 0.639$	$\frac{D_c(x^*)}{V(x^*)} = 0.648$	$\frac{D_c(x^*)}{V(x^*)} = 0.657$	$\frac{D_c(x^*)}{V(x^*)} = 0.666$
	$s^* = 0.971$	$c^* = 0.965$	$c^* = 0.955$	$c^* = 0.946$	$c^* = 0.936$	$c^* = 0.926$
	$x^* = 1.051$	$x^* = 1.054$	$x^* = 1.058$	$x^* = 1.063$	$x^* = 1.068$	$x^* = 1.073$
	$x_d = 0.295$	$x_d = 0.296$	$x_d = 0.297$	$x_d = 0.299$	$x_d = 0.301$	$x_d = 0.303$
$\sigma = 0.4$		$x_c = 43.690$	$x_c = 22.428$	$x_c = 15.461$	$x_c = 11.987$	$x_c = 9.905$
	$E(x^*) = 6.565$	$E(x^*) = 6.451$	$E(x^*) = 6.288$	$E(x^*) = 6.111$	$E(x^*) = 5.937$	$E(x^*) = 5.769$
	$D_s(x^*) = 9.918$	$D_c(x^*) = 10.034$	$D_c(x^*) = 10.193$	$D_c(x^*) = 10.374$	$D_c(x^*) = 10.547$	$D_c(x^*) = 10.714$
	$\frac{D_s(x^*)}{V(x^*)} = 0.602$	$\frac{D_c(x^*)}{V(x^*)} = 0.609$	$\frac{D_c(x^*)}{V(x^*)} = 0.618$	$\frac{D_c(x^*)}{V(x^*)} = 0.629$	$\frac{D_c(x^*)}{V(x^*)} = 0.640$	$\frac{D_c(x^*)}{V(x^*)} = 0.650$

Table 2: The effects for investment decision and financing for volatility and conversion ratio : the financing constraint

	Straight debt	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$	$\eta = 0.5$
	$s = 0.306$	$c = 0.305$	$c = 0.302$	$c = 0.298$	$c = 0.293$	$c = 0.288$
	$x^* = 0.549$	$x^* = 0.549$	$x^* = 0.549$	$x^* = 0.549$	$x^* = 0.549$	$x^* = 0.550$
	$x_d = 0.164$	$x_d = 0.164$	$x_d = 0.163$	$x_d = 0.161$	$x_d = 0.159$	$x_d = 0.157$
$\sigma = 0.2$		$x_c = 8.820$	$x_c = 4.711$	$x_c = 3.320$	$x_c = 2.610$	$x_c = 2.174$
	$E(x^*) = 3.894$	$E(x^*) = 3.888$	$E(x^*) = 3.870$	$E(x^*) = 3.845$	$E(x^*) = 3.814$	$E(x^*) = 3.780$
	$D_s(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$
	$\frac{D_s(x^*)}{V(x^*)} = 0.562$	$\frac{D_c(x^*)}{V(x^*)} = 0.563$	$\frac{D_c(x^*)}{V(x^*)} = 0.564$	$\frac{D_c(x^*)}{V(x^*)} = 0.565$	$\frac{D_c(x^*)}{V(x^*)} = 0.567$	$\frac{D_c(x^*)}{V(x^*)} = 0.569$
	$s = 0.337$	$c = 0.330$	$c = 0.316$	$c = 0.300$	$c = 0.281$	$c = 0.260$
	$x^* = 0.740$	$x^* = 0.741$	$x^* = 0.742$	$x^* = 0.745$	$x^* = 0.748$	$x^* = 0.752$
	$x_d = 0.135$	$x_d = 0.133$	$x_d = 0.128$	$x_d = 0.123$	$x_d = 0.116$	$x_d = 0.109$
$\sigma = 0.3$		$x_c = 12.247$	$x_c = 6.207$	$x_c = 4.141$	$x_c = 3.069$	$x_c = 2.390$
	$E(x^*) = 6.547$	$E(x^*) = 6.526$	$E(x^*) = 6.489$	$E(x^*) = 6.445$	$E(x^*) = 6.398$	$E(x^*) = 6.350$
	$D_s(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$
	$\frac{D_s(x^*)}{V(x^*)} = 0.433$	$\frac{D_c(x^*)}{V(x^*)} = 0.434$	$\frac{D_c(x^*)}{V(x^*)} = 0.435$	$\frac{D_c(x^*)}{V(x^*)} = 0.437$	$\frac{D_c(x^*)}{V(x^*)} = 0.439$	$\frac{D_c(x^*)}{V(x^*)} = 0.441$
	$s = 0.375$	$c = 0.352$	$c = 0.315$	$c = 0.272$	$c = 0.219$	
	$x^* = 0.976$	$x^* = 0.981$	$x^* = 0.990$	$x^* = 1.003$	$x^* = 1.022$	
	$x_d = 0.114$	$x_d = 0.108$	$x_d = 0.098$	$x_d = 0.086$	$x_d = 0.071$	
$\sigma = 0.4$		$x_c = 15.926$	$x_c = 7.405$	$x_c = 4.445$	$x_c = 2.809$	
	$E(x^*) = 9.843$	$E(x^*) = 9.832$	$E(x^*) = 9.825$	$E(x^*) = 9.836$	$E(x^*) = 9.884$	
	$D_s(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	$D_c(x^*) = 5.0$	
	$\frac{D_s(x^*)}{V(x^*)} = 0.337$	$\frac{D_c(x^*)}{V(x^*)} = 0.337$	$\frac{D_c(x^*)}{V(x^*)} = 0.337$	$\frac{D_c(x^*)}{V(x^*)} = 0.337$	$\frac{D_c(x^*)}{V(x^*)} = 0.336$	

Table 3: The effect of early and late investment on straight dent and convertible debt financing

r	μ	σ	s^*	$x^*(s^*)$	c^*	$x^*(c^*)$	$x^*(c^*) - x^*(s^*)$
0.03	0	0.2	0.316	0.414	0.317	0.415	0.0014
0.05	0	0.2	0.415	0.570	0.417	0.571	0.0012
0.07	0	0.2	0.515	0.718	0.516	0.719	0.0007
0.05	-0.02	0.2	0.360	0.667	0.362	0.667	0.0004
0.05	0	0.2	0.415	0.570	0.417	0.571	0.0012
0.05	0.02	0.2	0.585	0.491	0.579	0.494	0.0031
0.05	0	0.2	0.415	0.570	0.417	0.571	0.0012
0.05	0	0.3	0.630	0.785	0.629	0.789	0.0036
0.05	0	0.4	0.971	1.051	0.955	1.058	0.0073

4 Call Provisions

In this section we consider the firm which is financed with callable convertible debt. The formulation of the value of investment option and optimal capital structure are the same as in the case of non-callable convertible debt financing in Sec. 2. We reformulate the values of equity and callable convertible debt. If the convertible debt has the call provision, the equity holders can redeem (buy back) the debt. When the equity holders call the debt, the convertible debt holders can select either to receive a call price or to convert the debt into the equity. Let $\gamma c/r$ be the call price. Denoting $T_l \in \mathcal{T}_{0,\infty}$ as the call time, the equity value $E(x,c)$ and the value of callable convertible debt $D_c(x,c)$ are formulated as

$$\begin{aligned} E(x,c) = & \sup_{T_d, T_l \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[\int_0^{T_c^*(c)\wedge T_d \wedge T_l} e^{-ru}(1-\tau)(QX_u - c)du \right. \\ + & 1_{\{T_c^*(c) < (T_d \wedge T_l)\}} \frac{1}{1+\eta} \int_{T_c^*(c)}^{\infty} e^{-ru}(1-\tau)QX_u du \\ + & 1_{\{T_l < (T_c^*(c)\wedge T_d)\}} \left\{ \int_{T_l}^{\infty} e^{-ru}(1-\tau)QX_u du \right. \\ & \left.\left. -e^{-rT_l} \max\left(\gamma\frac{c}{r}, \frac{\eta}{1+\eta}\int_{T_l}^{\infty} e^{-ru}(1-\tau)QX_u du\right)\right\}\right], \end{aligned} \tag{4.1}$$

$$\begin{aligned} D_c(x,c) = \sup_{T_c \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \Bigg[& \int_0^{T_c \wedge T_l^*(c) \wedge T_d^*(c)} e^{-ru} c du \\ + & 1_{\{T_c < (T_l^*(c) \wedge T_d^*(c))\}} \frac{\eta}{1+\eta} \int_{T_c}^{\infty} e^{-ru}(1-\tau) Q X_u du \\ + & 1_{\{T_d^*(c) < (T_c \wedge T_l^*(c))\}} e^{-rT_d^*(c)} (1-\theta) \epsilon(X_{T_d^*(c)}) \\ + & 1_{\{T_l^*(c) < (T_c \wedge T_d^*(c))\}} e^{-rT_l^*(c)} \max\left(\gamma \frac{c}{r}, \frac{\eta}{1+\eta} \int_{T_l^*(c)}^{\infty} e^{-ru}(1-\tau) Q X_u du\right) \Bigg] \end{aligned} \tag{4.2}$$

The optimal call time for any c, $T_l^*(c)$ is given by

$$T_l^*(c) = \inf\{T_l \in [0,\infty) \mid X_{T_l} \geq x_l(c)\}, \tag{4.3}$$

where $x_l(c)$ is the optimal call threshold for any c. Similar to the case of non-callable convertible debt financing, the general solutions for the values of equity and callable convertible debt prior to default, conversion and call are given by Eqs. (2.8) and (2.9), respectively.

When the demand level X_t is lower, the default occurs. On the other hand, when the demand level X_t is higher, the either conversion or call occurs. The conversion occurs before the call, that is, if thresholds for conversion and call satisfies $x_c(c) < x_l(c)$, the values of equity and convertible debt are equal to that in the non-callable convertible debt financing case in Eqs. (2.14) and (2.15). Here, we consider the case that the call occurs before the conversion, that is, $x_l(c) < x_c(c)$. Once the equity holders decides to call, the convertible debt holders convert the debt into equity when the conversion value is higher than call price. In other words, the equity holders force the convertible debt holders to convert into equity. Since the forcing-conversion occurs when the conversion value is equal to the call price, the upper boundary conditions in the case that the convertible debt is redeemed at the call price, which come from the call policy of equity holders are given by

$$E(x_l(c),c) = \frac{1-\tau}{r-\mu} Q x_l(c) - \gamma \frac{c}{r}, \tag{4.4}$$

$$D_c(x_l(c),c) = \gamma \frac{c}{r}. \tag{4.5}$$

Since the lower boundary conditions are given by Eqs. (2.12) and (2.13) as in the case of non-callable convertible debt financing, the values of equity and callable convertible debt are given by

$$\begin{aligned} E(x,c) &= (1-\tau)\left(\frac{Qx}{r-\mu} - \frac{c}{r}\right) - (1-\tau)\left(\frac{Qx_d(c)}{r-\mu} - \frac{c}{r}\right) p_d(x,c;x_l(c)) \\ &+ (1-\tau-\gamma)\frac{c}{r} p_l(x,c;x_d(c)), \end{aligned} \tag{4.6}$$

$$\begin{aligned} D_c(x,c) &= \frac{c}{r} + \left((1-\theta)\frac{1-\tau}{r-\mu} Q x_d(c) - \frac{c}{r}\right) p_d(x,c;x_l(c)) \\ &+ (\gamma - 1)\frac{c}{r} p_l(x,c;x_d(c)), \end{aligned} \tag{4.7}$$

where $p_l(x, c; x_d(c))$ is that of \$1 contingent on X_t first reaching the call threshold $x_l(c)$ from bellow before reaching the default threshold $x_d(c)$, that is,

$$p_l(x, c; x_d(c)) = \frac{x^{\beta_1} x_d(c)^{\beta_2} - x^{\beta_2} x_d(c)^{\beta_1}}{x_l(c)^{\beta_1} x_d(c)^{\beta_2} - x_l(c)^{\beta_2} x_d(c)^{\beta_1}}. \tag{4.8}$$

Then, the firm value is represented by

$$V(x, c) = \epsilon(x) + \frac{\tau c}{r}(1 - p_d(x, c; x_l(c)) - p_l(x, c; x_d(c))) - \theta\epsilon(x_d(c)) p_d(x, c; x_l(c)). \tag{4.9}$$

In order to determine the optimal default and call thresholds, we derive the smooth-pasting condition on the default and call thresholds. Both thresholds are determined by the conditions which equal the partial derivation of $E(x, c)$ with respect to x with the deviation of payoff for the equity holders at the default threshold $x_d(c)$ and the call threshold $x_l(c)$. Hence,

$$\frac{\partial E}{\partial x}(x_d(c), c) = 0, \tag{4.10}$$

$$\frac{\partial E}{\partial x}(x_l(c), c) = \frac{1 - \tau}{r - \mu} Q. \tag{4.11}$$

Since the optimal default and call thresholds cannot be also solved analytically, the optimal threshold must be solved numerically.

4.1 Numerical Analysis

Here, to investigate how the issue of callable convertible debt affects the optimal investment strategy and the optimal capital structure, we present the calculation results. We use the base case parameters as in the case of non-callable convertible debt financing : $Q = 1$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $\theta = 0.3$, $\tau = 0.3$, $\eta = 0.4$, $I = 5.0$.

Fig. 3 shows the optimal investment thresholds on the optimal coupon payment and on the constant coupon payment which uses the coupon payment in the non-callable convertible debt financing case when the size of call price γ changes. In the optimal coupon payment case, the investment threshold decreases when γ increases. On the other hand, the investment threshold increases in the constant coupon payment case. Korkeamaki and Moore [6] describe that the firm financing with callable convertible debt invests earlier than that with non-callable convertible debt. As Fischer, et al. [4], Leary and Roberts [9], Mauer and Triantis [14] and Strebulaev [18] show that firms significantly deviate from target optimal capital structures for extended periods of time even when there are small adjustment costs, the results in Korkeamaki and Moore [6] are consistent with the results in the case of constant coupon payment.

In the optimal coupon payment case, when $\gamma \geq 1.82$, the conversion occurs before the forcing-conversion or call, that is, $x_c(c) < (x_f(c) \wedge x_l(c))$. Also, when $0.67 \leq \gamma < 1.82$, the forcing conversion occurs before the conversion or call, that is, $x_f(c) < (x_c(c) \wedge x_l(c))$. If $\gamma < 0.67$, the call occurs before the conversion or forcing-conversion, that is, $x_l(c) < (x_c(c) \wedge x_f(c))$. We can find the boundaries for the size of call price γ between conversion and forcing-conversion and between forcing-conversion and call. Let $\overline{\gamma}$ be the boundary for γ between conversion and forcing-conversion and $\underline{\gamma}$ be the boundary between forcing-conversion and call.

Fig. 4 shows the two boundaries $\overline{\gamma}$ and $\underline{\gamma}$ for volatility σ. Sarkar [16] shows that the band for coupon payment, (which leads to the forcing-conversion threshold) widens as the volatility increases. The high(low) coupon payment in Sarkar [16] has the same meaning of the low(high) call price in this paper. When the call price is lower, the firm must give up the tax shield by calling early. On the other hand, when the call price is higher, the value of conversion option becomes more valuable with higher σ. Then, it becomes more relevant in the call decision. This result is also consistent with the results in Sarkar [16].

5 Conclusions

In this paper, we have investigated the optimal investment strategies of the firm financed by issuing callable convertible debt. On the optimal capital structure, we found that the firm with higher volatility finances with higher coupon payment, delaying investment. Hence, the possibility of default is higher. Also, the default of the firm financing with convertible debt occurs earlier than that with straight debt (Koziol [8]). Furthermore, on the optimal capital structure the firm with high-growth prospection, high volatility, and low capital costs issuing convertible debt tends to defer investment longer (Korkeamaki and Moore [7]). The firm financing with callable convertible debt invests earlier than that with non-callable convertible debt (Koreamaki and Moore [6]). Using non-optimal coupon payment, the range leading to the forcing-conversion threshold widens as the volatility increases (Sarkar [16]). In the future, we will examine the effect of outstanding convertible debt on investment decisions.

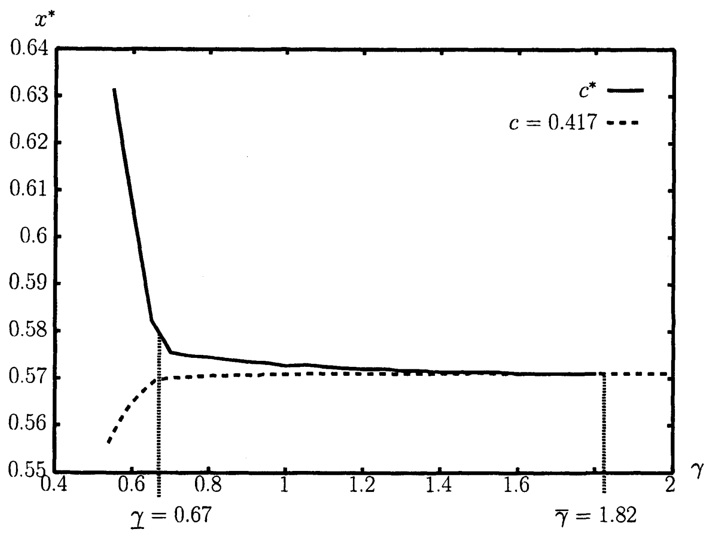

Figure 3: The investment threshold for the size of call price

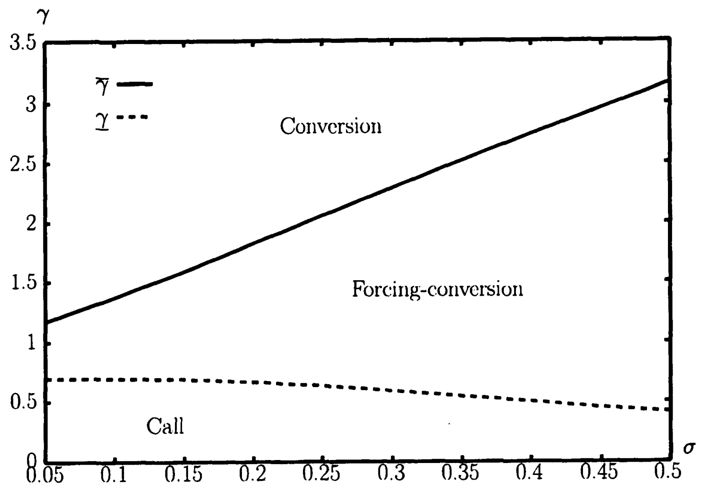

Figure 4: The optimal regions of the size of call price for volatility

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