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Evaluating the occurrence and disappearance of real options[1]

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1 Introduction

The real options approach, in which option pricing theory is applied to capital budgeting decisions, better enables us to find the optimal investment strategy and undertake project valuation under uncertainty than is possible under more classical methods. The early literature has investigated a real option that has a rather simple payoff structure, assuming that the dynamics of project value follow a one-dimensional stochastic process (see [9]). Naturally, the studies have developed into a more complicated analysis on the basis of a multidimensional process (e.g., [10, 15, 16, 1, 20]). For example, [10] investigates land development timing with an alternative land use choice. In [10], the option's payoff depends on the maximum of several underlying asset prices. This type of option is called a max-option, which has been also investigated in [22, 13, 5, 23, 8].[2]

This paper extends the previous max-option literature to a model that allows the Poisson arrival or death of an alternative project in which to invest. The uncertain occurrence or disappearance of the investment opportunity will be caused by changes in regulation, the exit and entry of rival firms[3], technological innovation, political risk, catastrophes, etc. The model captures these risks in addition to uncertainty about the future market values of projects. For example, the model applies to land development with an alternative land use choice under uncertainty about changes in zoning and development regulations. Especially in an emerging country, regulatory and political changes frequently happen, and hence, a firm is required to evaluate the option value and optimize the investment strategy taking account of these risks.

Conventionally, numerous studies on financial derivatives have modeled the catastrophic risks as a stock price following a discontinuous stochastic process with jumps. Most of the real options studies have followed this convention. For example, [17] investigates both a growth option and an extension option involving various types of rare events, assuming the underlying asset value follows a jump diffusion process. [3] presents the option values and the optimal investment strategies for both one-shot investment with fixed costs and incremental capacity expansion, assuming the underlying asset value follows a geometric Lévy process.

A distinction between this paper and the previous works is that this paper directly models the possibility that an opportunity in which to invest occurs or disappears. This direct approach

[1]This paper is an abbreviated version of [19]. For all proofs, refer to [19]. This work was supported by KAKENHI 22710142.

[2][20] reveals the nature of a combination of a max-option and a spread option.

[3]An alternative approach is the game-theoretic approach. Strategic interactions among several firms are investigated in [11, 12, 24, 14] among others.

can provide a simpler and more appropriate valuation of the real options than the previous approach. The model directly captures the effects of the uncertain disappearance or occurrence of an alternative. To my knowledge, this is the first paper that reveals the interactions among the random disappearance or occurrence of an alternative project, investment timing, and project choice. Technically, this study links the standard option to the max-option via the Poisson arrival or death. In other words, I investigate an option that may change to the max-option by the Poisson arrival and a max-option that may change to the standard option by the Poisson death.

In the model, I reveal how the possibility of the occurrence or disappearance influences the optimal exercise policy and the option value. Naturally, a higher intensity of the occurrence (disappearance) plays the role in increasing (decreasing) the value of the option to defer the investment timing and discouraging (encouraging) investment. In addition, the properties, such as the monotonicity and convexity shown by the max-option literature (e.g., [10, 5, 8]), hold true even if the occurrence or disappearance of the alternative are taken into consideration. The result ensures the robustness of these properties.

Furthermore, the numerical analysis reveals the following characteristics. The possibility of uncertain changes influences the option value and the optimal investment policy for the option that may change to the max-option greater than the max-option that may change to the standard option. Indeed, for the rational parameter values, the prospective future occurrence of an alternative has the potential to enhance the option value by almost 50%. The effect becomes larger, especially for a weaker or negative correlation between the project values, because a weaker or negative correlation increases the value of the max-option that may appear in future.

This paper also entails real-world implications. For example, the results offer rational explanations for the behavior of an owner of farmland which has not been cultivated in many years. Recently, an increase in idled farmland has been a serious issue in Japan. Typically, idled farmland is restricted within the agricultural use because the zoning and development ordinances prohibit nonagricultural uses. However, prospective future regulatory and environmental changes may enable an owner to develop land for residential or commercial uses. That is, an owner of idled farmland has the option that may change to the max-option. In addition to the relatively high value of residential or commercial land, the weak correlation between the alternative use and the agricultural use increases the option value and deters an owner from cultivating farmland.

2 Preliminaries

Consider a firm that has an option to invest in a project. There are two exclusive projects $i = 1$ and 2. The risk-adjusted values of the projects, $X(t) = (X_1(t), X_2(t))$, are random and follow GBMs (Geometric Brownian Motion)

$$\mathrm{d}X_i(t) = \mu_i X_i(t)\mathrm{d}t + \sigma_i X_i(t)\mathrm{d}B_i(t), \tag{1}$$

where $B_1(t), B_2(t)$ are Brownian Motions (BM) with correlation coefficient ρ satisfying $|\rho| < 1$. Constants μ_i and $\sigma_i(> 0)$ denote the risk-adjusted growth rate and volatility of the project value, respectively. Investing in project i requires an irreversible capital expenditure of $I_i(>0)$. Mathematically, the model is built on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ generated by $(B_1(t), B_2(t))$. The set $\mathcal{F}_t$ represents the set of available information at time t, and the firm optimizes the investment policy under this information. The risk-free rate is a constant $r(>0)$. For convergence, I assume that $r > \mu$.[4] The maturity of the options is $T(>0)$.

2.1 Standard option

As a benchmark, I consider a firm that has an option to invest in a given project i. A firm cannot invest in project $j(\neq i)$. For $X_i(t) = x_i$, the option value is equal to the value function of the optimal stopping problem as follows:[5]

$$V_i(x_i, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i}[1_{\{\tau \leq T\}} e^{-r(\tau - t)}(X_i(\tau) - I_i)], \tag{2}$$

where $\mathcal{T}_t$ denotes the set of all stopping times $\tau \geq t$ and $E_t^{x_i}[\cdot]$ denotes the expectation conditional on $X_i(t) = x_i$. The subscript i denotes the option to invest in project i. Note that problem (2) is analogous to an American call option written on a stock with a dividend. The optimal stopping time for problem (2) becomes $\tau_i(t) := \inf\{s \geq t \mid X(s) \in S_i^i(s)\}$, where the stopping region $S_i^i(s)$ is defined by

$$S_i^i(s) := \{x \in \mathbb{R}_{++}^2 \mid V_i(x_i, s) = x_i - I_i\}. \tag{3}$$

The superscript i means the immediate exercise region for project i. The optimal policy is that a firm makes investment in project i as soon as $X(s)$ hits $S_i^i(s)$. The following properties are well known (e.g., [8]):

(Convexity of the value function) $V_i(x_i, t)$ is convex with respect to x_i.

(Monotonicity of the stopping region) $x \in S_i^i(t) \Rightarrow x' \in S_i^i(t)$ $(\forall x_i' \geq x_i, \forall x_j' \in \mathbb{R}_{++})$, where $i \neq j$.

The monotonicity implies that $S_i^i(t)$ can be expressed as $S_i^i(t) = \{x \in \mathbb{R}_{++}^2 \mid x_i \geq x_i^*(t)\}$, where $x_i^*(t)$ denotes the threshold. This type of optimal policy is called the threshold policy.

Next, consider a case in which the investment opportunity for project i is killed at an instantaneous rate λdt, where a positive constant λ denotes the intensity of the Poisson death. I assume that the disappearance is independent of $X(t)$. The disappearance of an opportunity in which to invest will be caused by the enforcement of new regulations, preemption by rival firms, political changes, natural disasters, etc. For $X_i(t) = x_i$, the option value prior to the death is equal to the value function of the optimal stopping problem as follows:

$$V_{i \to \emptyset}(x_i, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i}[\int_0^\infty 1_{\{\tau \leq T\}} 1_{\{\tau < t+y\}} e^{-r(\tau - t)}(X_i(\tau) - I_i)\lambda e^{-\lambda y} dy], \tag{4}$$

[4] Refer to [9] for the economic rationale for this assumption.

[5] When the maturity is infinite, I have only to replace $1_{\{\tau \leq T\}}$ with $1_{\{\tau < \infty\}}$.

where the subscript $i \to \emptyset$ means that project i may be killed. I have

$$\begin{aligned}
& \mathbb{E}_t^{x_i}[\int_0^\infty 1_{\{\tau \le T\}} 1_{\{\tau < t+y\}} e^{-r(\tau-t)}(X_i(\tau) - I_i)\lambda e^{-\lambda y} dy] \\
= \;& \mathbb{E}_t^{x_i}[1_{\{\tau \le T\}} e^{-r(\tau-t)}(X_i(\tau) - I_i) \int_{\tau-t}^\infty \lambda e^{-\lambda y} dy] \\
= \;& \mathbb{E}_t^{x_i}[1_{\{\tau \le T\}} e^{-(r+\lambda)(\tau-t)}(X_i(\tau) - I_i) \int_0^\infty \lambda e^{-\lambda y} dy] \\
= \;& \mathbb{E}_t^{x_i}[1_{\{\tau \le T\}} e^{-(r+\lambda)(\tau-t)}(X_i(\tau) - I_i)]. \qquad (5)
\end{aligned}$$

Then, problem (4) can be rewritten as follows:

$$V_{i\to\emptyset}(x_i, t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i}[1_{\{\tau \le T\}} e^{-(r+\lambda)(\tau-t)}(X_i(\tau) - I_i)]. \qquad (6)$$

That is, the problem with the random disappearance is equivalent to the standard problem (2) with the augmented discount rate $r + \lambda$. This has also been shown in [24]. Then, problem (6) satisfies the same properties, such as the convexity of the value function and the monotonicity of the stopping region, as problem (2). Until the death the option value follows the continuous process $V_{i\to\emptyset}(X_i(t), t)$, and it jumps downward to zero at the random death. In addition, because of $V_{i\to\emptyset}(x_i, t) \le V_i(x_i, t)$, the stopping region $S^i_{i\to\emptyset}(t)$ for problem (6) is larger than $S^i_i(t)$. Note that $V_{i\to\emptyset}(x_i, t) \downarrow \max\{x_i - I_i, 0\}, S^i_{i\to\emptyset}(t) \uparrow \{x \in \mathbb{R}^2_{++} \mid x_i > I_i\}$ $(\lambda \uparrow \infty)$ and $V_{i\to\emptyset}(x_i, t) \uparrow V_i(x_i, t), S^i_{i\to\emptyset}(t) \downarrow S^i_i(t)$ $(\lambda \downarrow 0)$.

Now, consider a case in which the investment opportunity for project i occurs at an instantaneous rate λdt, where a positive constant λ denotes the intensity of the Poisson arrival. I assume that the occurrence is independent of $X(t)$. The occurrence of an opportunity in which to invest will be caused by deregulation, the exit of rival firms, technological innovation, etc. For $X_i(t) = x_i$, the option value prior to its arrival is equal to the following expectation:

$$V_{\emptyset\to i}(x_i, t) := \mathbb{E}_t^{x_i}[\int_0^\infty e^{-ry} V_i(X_i(t+y), t+y)\lambda e^{-\lambda y} dy], \qquad (7)$$

where the subscript $\emptyset \to i$ means that project i may be available in future. A firm waits for the arrival of the option, and after the arrival it adopts the optimal policy for problem (2). At the time of arrival, the option value jumps upward from $V_{\emptyset\to i}(X_i(t), t)$ to $V_i(X_i(t), t)$. Note that $V_{\emptyset\to i}(x_i, t) \uparrow V_i(x_i, t)$ $(\lambda \uparrow \infty)$ and $V_{\emptyset\to i}(x_i, t) \downarrow 0$ $(\lambda \downarrow 0)$.

2.2 Max-option

As a benchmark, this subsection considers an option to invest in a single project between projects 1 and 2. The model applies not only to a case in which two projects are exclusive (e.g., alternative land use in [10]) but also to a case where a firm faces a budget constraint. This type of option is identified as an American max-option. European max-options have been studied in [22, 13], while American max-options have been studied in [10, 5, 23, 8]. Although a typical max-option has a multidimensional state variable, [7] studies a max-option based on a one-dimensional state variable in order to investigate investment timing with an alternative scale choice.

For $X(t) = x$, the option value is equal to the value function of the optimal stopping problem as follows:

$$V_{1,2}(x,t) := \sup_{\tau\in\mathcal{T}_t} \mathbb{E}_t^x[1_{\{\tau\leq T\}}\mathrm{e}^{-r(\tau-t)} \underbrace{\max_{i=1,2}\{X_i(\tau) - I_i\}}_{\text{project choice}}], \tag{8}$$

where the subscript $1,2$ represents the option to invest between projects 1 and 2.

The optimal stopping time for problem (2) becomes $\tau_{1,2}(t) := \inf\{s \geq t \mid X(s) \in S_{1,2}(s) = S^1_{1,2}(s) \cup S^2_{1,2}(s)\}$, where the stopping region $S^i_{1,2}(s)$ is defined by

$$S^i_{1,2}(s) := \{x \in \mathbb{R}^2_{++} \mid V_{1,2}(x,s) = x_i - I_i\} \tag{9}$$

for $i = 1$ and 2. The max-option literature (e.g., [10, 5]) shows the following properties:
(Convexity of the value function) $V_{1,2}(x,t)$ is convex with respect to x.
(Convexity of each stopping region) $S^i_{1,2}(t)$ is a convex set.
(Monotonicity of each stopping region) $x \in S^i_{1,2}(t) \Rightarrow x' \in S^i_{1,2}(t)$ $(\forall x'_i \geq x_i, \forall x'_j \leq x_j)$, where $i \neq j$.
(Behavior on the indifference line) $x_1 - I_1 = x_2 - I_2 \Rightarrow x \notin S_{1,2}(t)$.

The monotonicity of $S^i_{1,2}(t)$ implies that an increase (decrease) in the value of project i ($j \neq i$) facilitates investment in project i. The behavior on the indifference line means that a firm waits and sees which project is better when the values of two projects equal.

3 Main Results

This section links the standard option in Section 2.1 and the max-option in Section 2.2 via a random variable distributed exponentially. Section 3.1 investigates a max-option that may change to the standard option, while Section 3.2 investigates an option that may change to the max-option. I show several properties of the option values and exercise regions. The model applies to the decision-making process about land development with an alternative land use choice under regulatory risks.

3.1 Max-option that will change to the standard option

As in Section 2.2, consider an option to invest between projects 1 and 2. Assume that the investment opportunity for project 2 is killed at an instantaneous rate $\lambda \mathrm{d}t$ which is independent of $X(t)$.[6] For $X(t) = x$, the option value is equal to the value function of the optimal stopping problem as follows:[7]

$$\begin{aligned} V_{1,2\to1}(x,t) \;\; := \;\; & \sup_{\tau\in\mathcal{T}_t} \mathbb{E}_t^x[\int_0^\infty 1_{\{\tau\leq T\}}\{1_{\{\tau<t+y\}}\mathrm{e}^{-r(\tau-t)}\max_{i=1,2}\{X_i(\tau) - I_i\} \\ & +1_{\{\tau\geq t+y\}}\mathrm{e}^{-ry}V_1(X_1(t+y), t+y)\}\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y], \end{aligned} \tag{1}$$

[6][18] models rival preemption for the max-option endogenously. Indeed, [18] derives the equilibrium of a preemption game in which two firms compete for two alternatives.

[7]This problem is the same as the max-option problem (8) replaced $X_2(t)$ with the killed process.

where y denotes the term until the death. The subscript $1,2 \to 1$ represents the max-option that may change to the standard option to invest in project 1.

Using the similar calculation to (5), I can rewrite problem (1) as

$$V_{1,2\to 1}(x,t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x[1_{\{\tau \le T\}} e^{-(r+\lambda)(\tau - t)} \max_{i=1,2}\{X_i(\tau) - I_i - V_{\emptyset \to 1}(X_1(\tau), \tau)\}] + V_{\emptyset \to 1}(x_1, t), \quad (2)$$

where $V_{\emptyset\to 1}(x_1,t)$ is defined by (7). The optimal stopping time for problem (2) becomes $\tau_{1,2\to 1}(t) := \inf\{s \ge t \mid X(s) \in S_{1,2\to 1}(s) = S^1_{1,2\to 1}(s) \cup S^2_{1,2\to 1}(s)\}$, where the stopping region $S^i_{1,2\to 1}(s)$ is defined by

$$S^i_{1,2\to 1}(s) := \{x \in \mathbb{R}^2_{++} \mid V_{1,2\to 1}(x,s) = x_i - I_i\} \quad (3)$$

for $i = 1$ and 2.

Proposition 1
(Convexity of the value function) $V_{1,2\to 1}(x,t)$ is convex with respect to x.
(Convexity of each stopping region) $S^i_{1,2\to 1}(t)$ is a convex set.
(Monotonicity of each stopping region) $x \in S^i_{1,2\to 1}(t) \Rightarrow x' \in S^i_{1,2\to 1}(t)$ $(\forall x'_i \ge x_i, \forall x'_j \le x_j)$, where $i \ne j$.
(Behavior on the indifference line) $x_1 - I_1 = x_2 - I_2 \Rightarrow x \notin S_{1,2\to 1}(t)$.
(Comparison) $\max\{V_1(x_1,t), V_{2\to\emptyset}(x_2,t)\} \le V_{1,2\to 1}(x,t) \le V_{1,2}(x,t)$, $S^1_{1,2}(t) \subset S^1_{1,2\to 1}(t) \subset S^1_1(t)$, and $S^2_{1,2}(t) \subset S^2_{1,2\to 1}(t) \subset S^2_{2\to\emptyset}(t)$.

Proposition 1 extends previous findings by [10, 5, 8] to a case in which the investment opportunity may be killed. The properties for the max-option in Section 2.2 remain true for the generalized case. The monotonicity of $S^i_{1,2\to 1}(t)$ implies that an increase (decrease) in the value of project i ($j \ne i$) encourages investment in project i. The behavior on the indifference line means that a firm will wait and see which project is better when the values of both projects are equal. Clearly, $V_{1,2\to 1}(x,t)$ ($S^i_{1,2\to 1}(t)$) monotonically decreases (increases) with the intensity λ. This means that an increased possibility of the disappearance reduces the value of waiting and encourages investment. Note that $V_{1,2\to 1}(x,t) \downarrow \max\{x_2 - I_2, V_1(x_1,t)\}, S^1_{1,2\to 1}(t) \uparrow S^1_1(t) \setminus \{x \in \mathbb{R}^2_{++} \mid x_2 - I_2 \ge V_1(x_1,t)\}, S^2_{1,2\to 1}(t) \uparrow \{x \in \mathbb{R}^2_{++} \mid x_2 - I_2 > V_1(x_1,t)\}$ $(\lambda \uparrow \infty)$ and $V_{1,2\to 1}(x,t) \uparrow V_{1,2}(x,t), S^i_{1,2\to 1}(t) \downarrow S^i_{1,2}(t)$ $(\lambda \downarrow 0)$.

A higher volatility σ_i increases $V_{1,2}(x,t)$ and $V_1(x_1,t)$, and hence, it increases (decreases) $V_{1,2\to 1}(X(t),t)$ ($S_{1,2\to 1}(t)$). On the other hand, an increase in the correlation coefficient ρ tends to decrease (increase) $V_{1,2\to 1}(X(t),t)$ ($S_{1,2\to 1}(t)$). This is because $V_{1,2}(x,t)$ tends to increase with ρ (see [10, 8]). Note that the option value jumps downward from $V_{1,2\to 1}(X(t),t)$ to $V_1(X_1(t),t)$ at the time of the Poisson death (see Figure 1). The jump size is endogenously determined as $V_{1,2\to 1}(X(t),t) - V_1(X_1(t),t)$.[8] The jump size decreases with λ and ρ. Typically, $V_1(X_1(t),t)$ is more volatile than $V_{1,2\to 1}(X(t),t)$ in which $X_1(t)$ and $X_2(t)$ diversify the risk. Accordingly,

[8]In contrast, in a model based on the discontinuous stochastic process the jump size must be exogenously presumed.

the volatility of the option value jumps upward on the same timing. This may account for empirical observations (e.g., [2, 6]) that the volatility of a stock price increases when the stock price decreases.

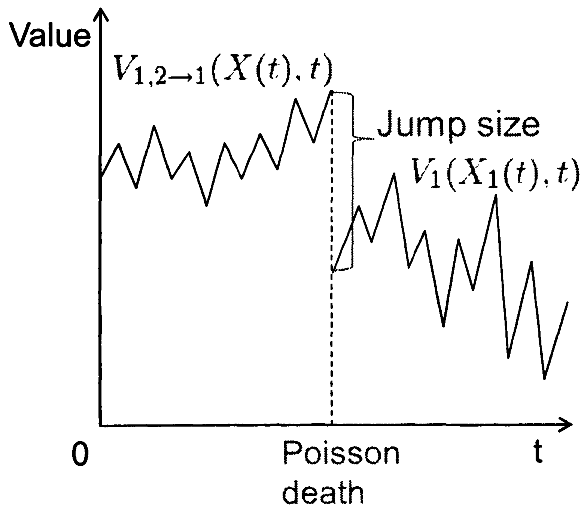

Figure 1: The downward jump caused by the disappearance of the alternative.

3.2 Option that will change to the max-option

This subsection considers an option that may change to the max-option. This option contrasts with the option studied in Section 3.1. Assume that the investment opportunity for project 2 is created at an instantaneous rate λdt which is independent of $X(t)$. For $X(t)=x$, the option value prior to the Poisson arrival is equal to the value function of the optimal stopping problem as follows:

$$\begin{aligned} V_{1\to 1,2}(x,t) \quad := \quad & \sup_{\tau\in\mathcal{T}_t} \mathrm{E}_t^x[\int_0^\infty 1_{\{\tau\le T\}}\{1_{\{\tau<t+y\}}\mathrm{e}^{-r(\tau-t)}(X_1(\tau)-I_1) \\ & +1_{\{\tau\ge t+y\}}\mathrm{e}^{-ry}V_{1,2}(X(t+y),t+y)\}\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y] \end{aligned} \tag{4}$$

where y denotes the term until the arrival. The subscript $1\to 1,2$ denotes the option that may change to the max-option. Note that $V_{1\to 1,2}(x,t)$, unlike $V_1(x_1,t)$, depends not only on x_1 but also on x_2 because of the potential arrival of project 2.

Using the similar calculation to (5), I can rewrite problem (4) as

$$V_{1\to 1,2}(x,t) = \sup_{\tau\in\mathcal{T}_t} \mathrm{E}_t^x[1_{\{\tau\le T\}}\mathrm{e}^{-(r+\lambda)(\tau-t)}(X_1(\tau)-I_1-V_{\emptyset\to 1,2}(X(\tau),\tau))]+V_{\emptyset\to 1,2}(x,t), \tag{5}$$

where $V_{\emptyset\to 1,2}(x,t)$ is defined by

$$V_{\emptyset\to 1,2}(x,t) = \mathrm{E}_t^x[\int_0^\infty \mathrm{e}^{-ry}V_{1,2}(X(t+y),t+y)\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y]. \tag{6}$$

The optimal stopping time for problem (5) becomes $\tau_{1\to 1,2}(t) := \inf\{s \geq t \mid X(s) \in S^1_{1\to 1,2}(s)\}$, where the stopping region $S^1_{1\to 1,2}(s)$ is defined by

$$S^1_{1\to 1,2}(s) := \{x \in \mathbb{R}^2_{++} \mid V_{1\to 1,2}(x,s) = x_1 - I_1\}. \tag{7}$$

Note that, the exercise region $S^1_{1\to 1,2}(s)$ exists only for project 1 although it also depends on the value of project 2.

Proposition 2

(Convexity of the value function) $V_{1\to 1,2}(x,t)$ is convex with respect to x.
(Convexity of the stopping region) $S^1_{1\to 1,2}(t)$ is a convex set.
(Monotonicity of the stopping region) $x \in S^1_{1\to 1,2}(t) \Rightarrow x' \in S^1_{1\to 1,2}(t)$ $(\forall x'_1 \geq x_1, \forall x'_2 \leq x_2)$.
(Comparison) $V_1(x_1,t) \leq V_{1\to 1,2}(x,t) \leq V_{1,2}(x,t)$ and $S^1_{1,2}(t) \subset S^1_{1\to 1,2}(t) \subset S^1_1(t)$.

To my knowledge, there has been no studies that investigates the option that will change to the max-option. Proposition 2 presents the first result for this type of option. Propositions 1 and 2 bridge the gap between the standard option and the max-option from both sides. The exercise region $S^1_{1\to 1,2}(t)$ has the same convexity and monotonicity as $S^1_{1,2}(t)$ for the max-option. The monotonicity of $S^1_{1\to 1,2}(t)$ implies that an increase (decrease) in the value of project 1 (2) accelerates investment in project 1. In the presence of the potentially available project 2, a high value of project 2 increases the option value and delay investment in project 1. It follows from expression (5) that $S^1_{1\to 1,2}(t)$ is contained in $\{x \in \mathbb{R}_{++} \mid x_1 \geq I_1 + V_{\emptyset\to 1,2}(x,t)\}$. In particular, by

$$\begin{aligned} V_{\emptyset\to 1,2}(x,t) &\geq \mathbb{E}^x_t\Big[\int_0^\infty \mathrm{e}^{-ry}(X_2(t+y) - I_2)\lambda \mathrm{e}^{-\lambda y}\mathrm{d}y\Big] \\ &= \frac{\lambda x_2}{r+\lambda-\mu} - \frac{\lambda I_2}{r+\lambda}, \end{aligned}$$

I can show that $x_1 \geq I_1 + \lambda x_2/(r+\lambda-\mu) - \lambda I_2/(r+\lambda)$ for $x \in S^1_{1\to 1,2}(t)$. This indicates that for a high x_2, $S^1_{1\to 1,2}(t)$ is much smaller than $S^1_1(t) = \{x \in \mathbb{R}^2_{++} \mid x_1 \geq x^*_1(t)\}$. Clearly, $V_{1\to 1,2}(x,t)$ $(S_{1\to 1,2}(t))$ monotonically increases (decreases) with the intensity λ. This means that an increased possibility of the arrival enhances the value of waiting and discourages investment. I have $V_{1\to 1,2}(x,t) \uparrow V_{1,2}(x,t), S^1_{1\to 1,2}(t) \downarrow S^1_{1,2}(t)$ $(\lambda \uparrow \infty)$ and $V_{1\to 1,2}(x,t) \downarrow V_1(x_1,t), S^1_{1\to 1,2}(t) \uparrow S^1_1(t)$ $(\lambda \downarrow 0)$.

As in Section 3.1, $V_{1\to 1,2}(X(t),t)$ $(S_{1\to 1,2}(t))$ increases (decreases) with the volatility σ_i and decreases (increases) with the correlation coefficient ρ. Note that the option value jumps upward from from $V_{1\to 1,2}(X(t),t)$ to $V_{1,2}(X(t),t)$ at the time of the Poisson arrival (see Figure 2). The jump size, which is endogenously determined as $V_{1,2}(X(t),t) - V_{1\to 1,2}(X(t),t)$, decreases with λ.

3.3 Extensions and limitations

This section explains several extensions of the results. First, consider the max option that has n investment opportunities. Assume that projects $n, n-1, \ldots, m+1$ will be killed sequentially

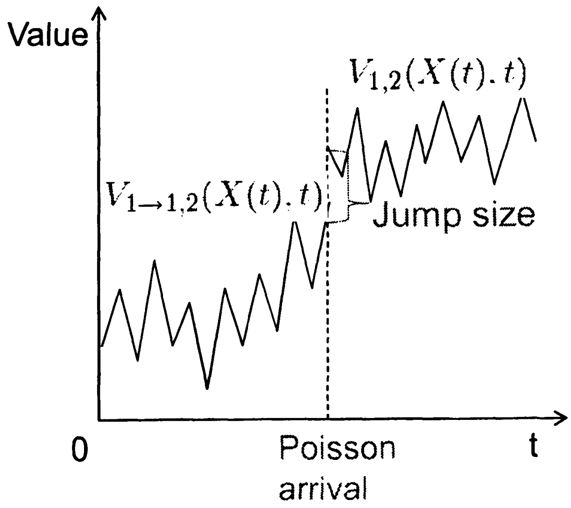

Figure 2: The upward jump caused by the occurrence of the alternative.

with the intensity λ. For $X(t) = x$, the option value prior to the Poisson death is equal to the value function of the optimal stopping problem as follows:

$$\begin{aligned}
& V_{1,\dots,n\to1,\dots,n-1\to\cdots\to1,\dots,m}(x,t) \\
:= \; & \sup_{\tau\in\mathcal{T}_t} \mathrm{E}_t^x[\int_0^\infty 1_{\{\tau\le T\}}\{1_{\{\tau<t+y\}}\mathrm{e}^{-r(\tau-t)} \max_{i=1,\dots,n}\{X_i(\tau)-I_i\} \\
& +1_{\{\tau\ge t+y\}}\mathrm{e}^{-ry}V_{1,\dots,n-1\to1,\dots,n-2\to\cdots\to1,\dots,m}(X(t+y),t+y)\}\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y] \quad (n>m),
\end{aligned}$$

which is defined backward from

$$\begin{aligned}
& V_{1,\dots,m\to1,\dots,m}(x,t) \\
:= \; & V_{1,\dots,m}(x,t) \\
:= \; & \sup_{\tau\in\mathcal{T}_t} \mathrm{E}_t^x[1_{\{\tau\le T\}}\mathrm{e}^{-r(\tau-t)} \max_{i=1,\dots,m}\{X_i(\tau)-I_i\}].
\end{aligned}$$

This is a generalized version of problem (1). Note that $V_{1,\dots,m}(x,t)$ is convex with respect to x. Then, using backward induction, I can show the same properties as Proposition 1.

Similarly, a generalized version of problem (4) is expressed as

$$\begin{aligned}
& V_{1,\dots,m\to1,\dots,m+1\to\cdots\to1,\dots,n}(x,t) \\
:= \; & \sup_{\tau\in\mathcal{T}_t} \mathrm{E}_t^x[\int_0^\infty 1_{\{\tau\le T\}}\{1_{\{\tau<t+y\}}\mathrm{e}^{-r(\tau-t)} \max_{i=1,\dots,m}\{X_i(\tau)-I_i\} \\
& +1_{\{\tau\ge t+y\}}\mathrm{e}^{-ry}V_{1,\dots,m+1\to\cdots\to1,\dots,n}(X(t+y),t+y)\}\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y].
\end{aligned}$$

I can show the same properties as Proposition 2 by the backward induction. Of course, it does not matter if the intensities vary over projects. Furthermore, I can show the same properties even if the order of the disappearances or occurrences is not presumable.

For the problem in Section 3.2, a new value $X_2(t)$ may be unobservable until the arrival. For example, value of an alternative that proceeds from some technical innovation may not be evaluated correctly prior to the innovation. This partial information problem is essentially different from problem (4) in that the option value and the investment strategy depend only on $X_1(t)$.

4 Numerical Examples

This section provides numerical examples of the options studied in the previous section. I use base parameter values as follows[9]:

$$r = 8\%,\ \mu_1 = \mu_2 = 0\%,\ \sigma_1 = \sigma_2 = 20\%, \rho = 0\%. \tag{1}$$

For expositional purposes, I set the investment cost $I_1 = I_2 = 100$ and the option values are computed at the money, i.e., $x = (100, 100)$. The maturity of the option is set at $T - t = 3$ years. For the intensity λ, the probability that an opportunity in which to invest disappears or occurs within 3 years is expressed as

$$\int_0^3 \lambda \mathrm{e}^{-\lambda t} \mathrm{d}t = 1 - e^{-3\lambda}.$$

I set this probability at $25\%, 50\%$, and 75%. These correspond to $\lambda = 0.096, 0.231$, and 0.462. In the numerical procedure, I make a discretization with 200 time steps per 1 year, and use a bivariate version of the lattice binomial method (e.g., [4]). Technically, I compute the lattice model for maturity $T = 4$ years, and draw the investment regions for $t = 1$ year.

First, I set the intensity $\lambda = 0.231$. Figure 3 illustrates the exercise regions $S_{1,2\to 1}(t) = S^1_{1,2\to 1}(t) \cup S^2_{1,2\to 1}(t)$ and $S^1_{1\to 1,2}(t)$ for the base parameter values (1). For comparison, the figure also presents the exercise regions $S_{1,2}(t) = S^1_{1,2}(t) \cup S^2_{1,2}(t)$ and $S^1_1(t)$. I can check the convexity and monotonicity of $S^i_{1,2\to 1}(t)$ and $S^1_{1\to 1,2}(t)$, as well as the relationship that $S^1_{1,2}(t) \subset S^1_{1,2\to 1}(t) \subset S^1_1(t), S^2_{1,2}(t) \subset S^2_{1,2\to 1}(t)$, and $S^1_{1,2}(t) \subset S^1_{1\to 1,2}(t) \subset S^1_1(t)$.

In Figure 3, let us take a look at the max-option that will change to the standard option. When $X(t)$ hits the boundary of $S^1_{1,2\to 1}(t)$ (the lower-right curve), a firm invests in project 1. On the contrary, when $X(t)$ hits the boundary of $S^2_{1,2\to 1}(t)$ (the upper-left curve), the firm invests in project 2. The firm delays the decision on project choice for $X(t)$ between the two curves, although project 2 may be killed in the waiting time. The figure indicates that the difference between $S^2_{1,2\to 1}(t)$ and $S^2_{1,2}(t)$ is greater than the difference between $S^1_{1,2\to 1}(t)$ and $S^1_{1,2}(t)$. The prospective disappearance of project 2 encourages investment in project 2 prior to the disappearance, while it does not have a significant influence on the investment timing in project 1. The boundary of $S^2_{1,2\to 1}(t)$ approaches the threshold $x^*_2(t)$ replaced $r = 0.08$ with $r + \lambda = 0.08 + 0.231 = 0.311$ when $x_1 \to 0$. Then, for $x_1 \approx 0$, $S^2_{1,2\to 1}(t)$ is much larger than $S^2_{1,2}(t)$. On the other hand, both boundaries of $S^1_{1,2}(t)$ and $S^1_{1,2\to 1}(t)$ converges to the threshold $x^*_1(t)$ when $x_2 \to 0$. Then, for $x_2 \approx 0$, $S^1_{1,2\to 1}(t)$ is almost the same as $S^1_{1,2}(t)$.

In Figure 3, let us now turn to the option that may change to the max-option. When $X(t)$ hits the boundary of $S^1_{1\to 1,2}(t)$ (the lower-right curve), a firm invests in project 1. Otherwise,

[9]These parameter values are similar to [10, 8]. I carried out a lot of computations with varying parameter values and distilled robust results into this section.

the firm delays investment, and in the waiting time project 2 may become available. I see that the gap between $S^1_{1\to 1,2}(t)$ and $S^1_1(t)$ increases with x_2. This is because a higher x_2 increases an incentive for the firm to wait for the occurrence of project 2. On the other hand, for $x_2 \approx 0$ the boundary of $S^1_{1\to 1,2}(t)$ converges to the threshold $x^*_1(t)$, and hence there is no gap between them.

Now, consider the comparative statics. First, I explore the effects of the intensity λ. Figures 4 and 5 draw the exercise regions $S_{1,2\to 1}(t) = S^1_{1,2\to 1}(t) \cup S^2_{1,2\to 1}(t)$ and $S^1_{1\to 1,2}(t)$ with varying levels of λ. As mentioned in Section 3.2, $S^i_{1,2\to 1}(t)$ increases with λ, while $S^1_{1\to 1,2}(t)$ decreases with λ. For the max-option that may change to the standard option for project 1, a higher λ increases an incentive for a firm to invest in project 2 prior to the disappearance. On the other hand, for the option that may change to the max-option, a higher λ increases the value of waiting for project 2. Comparing Figures 4 and 5, I find that the impact of λ on the exercise policy is stronger for the option that will change to the max-option. This suggests that a firm should take into careful consideration an alternative which may be available in future rather than an alternative which may disappear in future.

Next, I explore the effects of the correlation coefficient ρ. Figures 6 and 7 illustrate the exercise regions $S_{1,2\to 1}(t) = S^1_{1,2\to 1}(t) \cup S^2_{1,2\to 1}(t)$ and $S^1_{1\to 1,2}(t)$ with varying levels of ρ. In both figures, the intensity λ is fixed at $\lambda = 0.231$. I see that $S^i_{1,2\to 1}(t)$ and $S^1_{1\to 1,2}(t)$ monotonically enlarge with ρ. It is known (e.g., [10, 8]) that $S^i_{1,2}(t)$ tends to increase with ρ because a higher ρ decreases the value of the option to delay the decision concerning project choice. Figures 6 and 7 demonstrate that the same result holds even if the disappearance or occurrence of an alternative is taken into consideration. This means the robustness of the previous findings.

The effects of ρ also appear in Tables 1 and 2. The tables present the option values $V_{1,2\to 1}(x,t)$ and $V_{1\to 1,2}(x,t)$ for varying levels of λ and ρ. Note that $V_{1,2\to 1}(x,t)$ and $V_{1\to 1,2}(x,t)$ are computed at the money, i.e., $x = (100, 100)$. The option values monotonically decrease with ρ. The impact of ρ on $V_{1,2\to 1}(x,t)$ weakens with λ, while the impact of ρ on $V_{1\to 1,2}(x,t)$ strengthens with λ. This can be explained as follows. When $\lambda \to \infty$, $V_{1,2\to 1}(x,t)$ approaches $V_1(x,t)$ which is independent of ρ. On the other hand, $V_{1\to 1,2}(x,t)$ approaches $V_1(x,t)$ when $\lambda \to 0$. Then, a higher (lower) λ weakens the impact of ρ on $V_{1,2\to 1}(x,t)(V_{1\to 1,2}(x,t))$. The impact of λ is significant to $V_{1\to 1,2}(x)$ especially for a low ρ. Indeed, for $\rho = -50\%$, the Poisson arrival with the intensity $\lambda = 0.096, 0.231$, and 0.462 enhances the option value by 21%, 44%, and 66%, respectively. For $\rho = 0\%$ the Poisson arrival with the intensity $\lambda = 0.096, 0.231$, and 0.462 increases the option value by 18%, 36%, and 54%, respectively.

Let us examine Tables 1 and 2 from a different aspect. As mentioned in Section 3.1 and 3.2 (see Figures 1 and 2), the gap between $V_{1,2\to 1}(x,t)$ and $V_1(x,t) = 11.7$ is equal to the jump size at the time of the Poisson death, while the gap between $V_{1\to 1,2}(x,t)$ and $V_{1,2}(x,t)$ is equal to the jump size at the time of the Poisson arrival. Comparing Tables 1 and 2, I find that the downward jump size tends to be larger than the upward jump size. This finding may be related to empirical findings (e.g., [21]) that the stock price response to bad news is larger than the stock price response to good news.

Table 1: The option values $V_{1,2\to 1}(x,t)$.

$\lambda \setminus \rho$	−75%	−50%	−25%	0%	25%	50%	75%
0	22.8	22.1	21.28	20.32	19.2	17.82	15.98
0.096	21.07	20.46	19.75	18.93	17.97	16.8	15.26
0.231	19.36	18.83	18.23	17.55	16.75	15.79	14.53
0.462	17.5	17.07	16.59	16.04	15.42	14.68	13.73

Table 2: The option values $V_{1\to 1,2}(x,t)$.

$\lambda \setminus \rho$	−75%	−50%	−25%	0%	25%	50%	75%
0	11.7	11.7	11.7	11.7	11.7	11.7	11.7
0.096	14.42	14.24	14.02	13.77	13.47	13.12	12.65
0.231	17.16	16.81	16.39	15.9	15.33	14.62	13.7
0.462	19.93	19.41	18.79	18.07	17.22	16.18	14.8

5 Conclusion

The paper has investigated the nature of two types of option; the max-option that will change to the standard option by the Poisson death, and the option that will change to the max-option by the Poisson arrival. These uncertain changes in real options can be caused by changes in regulatory, political, competitive, and technological environment. The model, unlike the previous studies, directly captures these catastrophic risks in capital budgeting. In the model, I have revealed how the possibility of the disappearance or occurrence influences the optimal exercise policy and the option value. The results are summarized as follows.

A higher intensity of the disappearance (occurrence) decreases (increases) the option value and encourages (discourages) investment. The properties, such as the monotonicity and convexity shown by the max-option literature, remain true when the random change is taken into consideration. The impact of the uncertain change is relatively greater for the option that may change to the max-option. Notably, the impact is significant for a weaker or negative correlation between the project values. The results offer rational explanations for the behavior of an owner of farmland which has not been cultivated in many years. The results have the potential to account for the asymmetric market reaction to good and bad news, although the model is not intended to investigate the stock price reactions.

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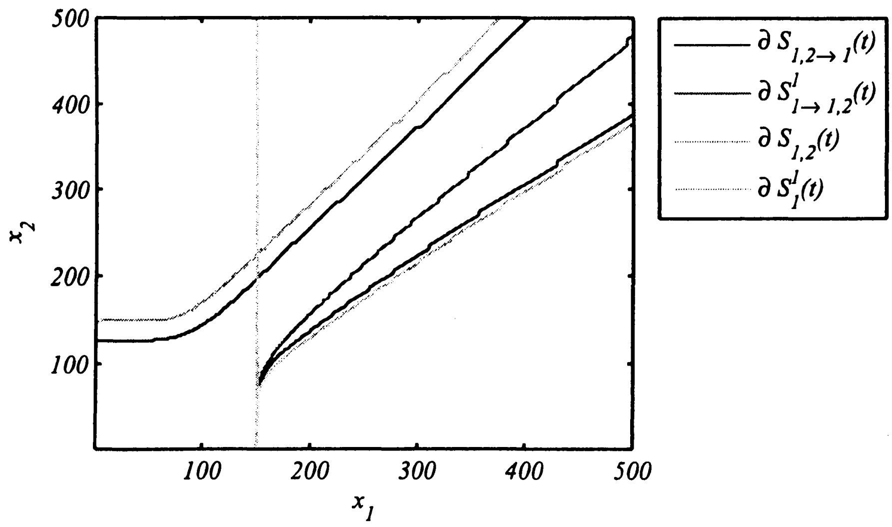

Figure 3: The exercise regions for the four options. This figure plots the boundaries of the exercise regions $S_{1,2\to1}(t) = S^1_{1,2\to1}(t) \cup S^2_{1,2\to1}(t)$, $S^1_{1\to1,2}(t)$, $S_{1,2}(t) = S^1_{1,2}(t) \cup S^2_{1,2}(t)$ and $S^1_1(t)$. The parameter values are set at the base case (1) with the intensity $\lambda = 0.231$.

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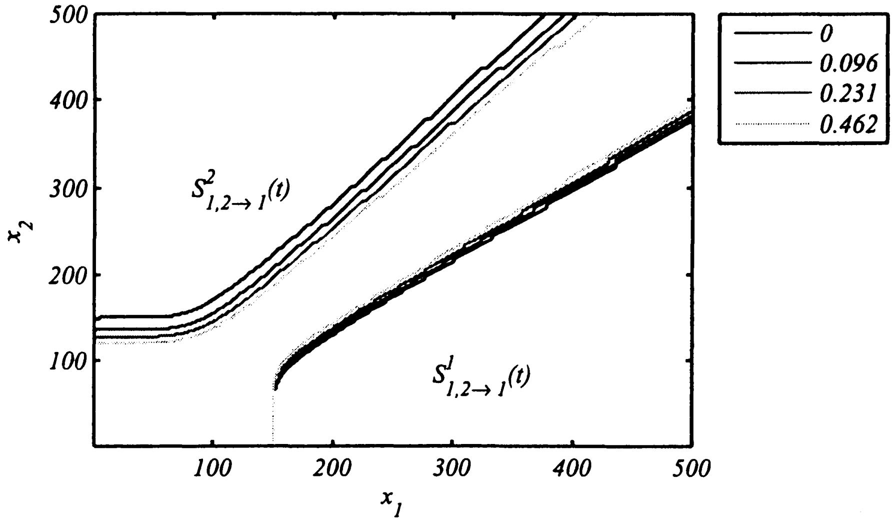

Figure 4: The exercise regions for the max-option that will change to the standard option. This figure plots the boundaries of the exercise regions $S_{1,2\to1}(t) = S^1_{1,2\to1}(t) \cup S^2_{1,2\to1}(t)$ for the intensity $\lambda = 0, 0.096, 0.231$, and 0.462. For $\lambda = 0$, the region is equal to the exercise region for the max-option, $S_{1,2}(t)$. The parameter values are set at the base case (1).

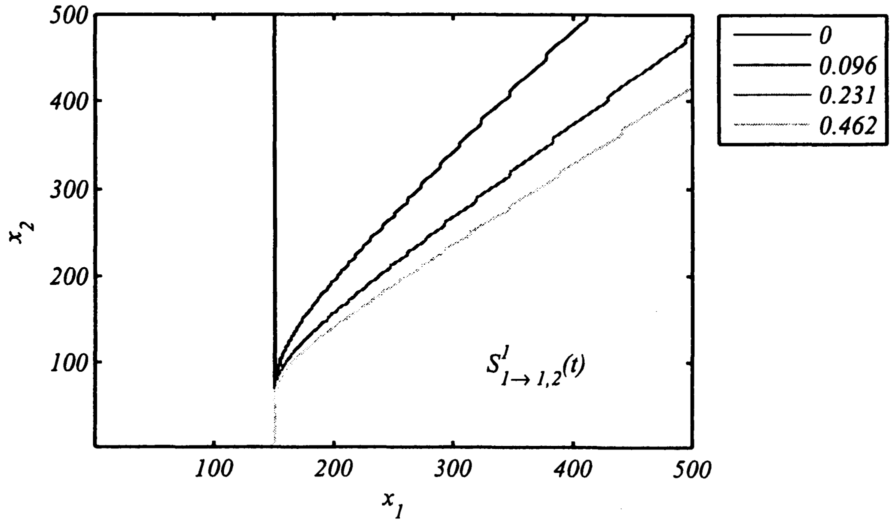

Figure 5: The exercise regions for the option that will change to the max-option. This figure plots the boundaries of the exercise regions $S^1_{1\to1,2}(t)$ for the intensity $\lambda = 0, 0.096, 0.231$, and 0.462. For $\lambda = 0$, the region is equal to the exercise region for the standard option, $S^1_1(t)$. The parameter values are set at the base case (1).

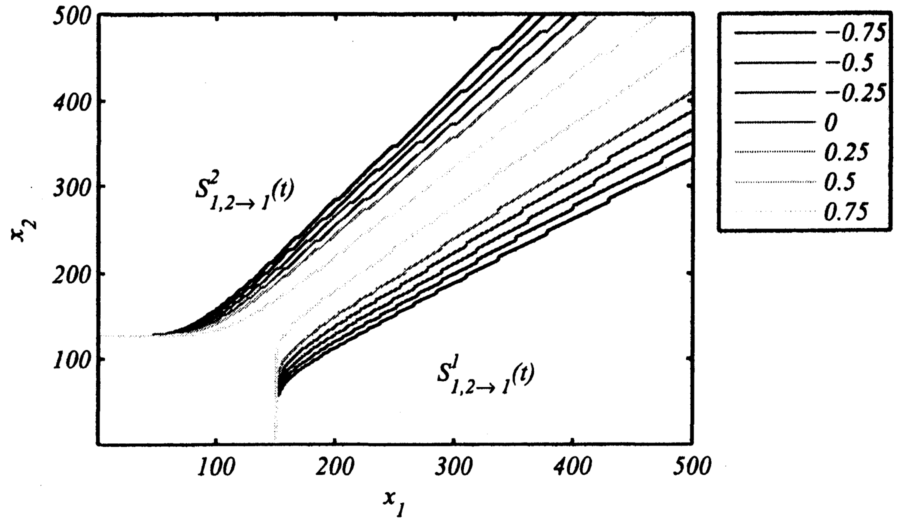

Figure 6: The exercise regions for the max-option that will change to the standard option. This figure plots the boundaries of the exercise regions $S_{1,2\to 1}(t) = S^1_{1,2\to 1}(t) \cup S^2_{1,2\to 1}(t)$ for the correlation coefficient $\rho = -75\%, -50\%, -25\%, 0\%, 25\%, 50\%$, and 75%. The intensity is set at $\lambda = 0.231$. The other parameter values are set at the base case (1).

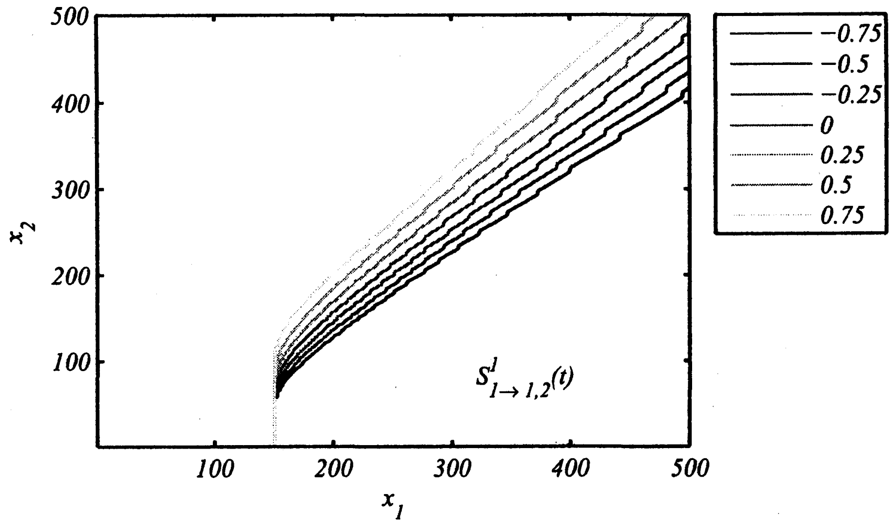

Figure 7: The exercise regions for the option that will change to the max-option. This figure plots the boundaries of the exercise regions $S^1_{1\to 1,2}(t)$ for the correlation coefficient $\rho = -75\%, -50\%, -25\%, 0\%, 25\%, 50\%$, and 75%. The intensity is set at $\lambda = 0.231$. The other parameter values are set at the base case (1).