数理解析研究所講究録
第 1737 巻 2011 年 31-34

On estimations for parametrized operator means
作用素平均族の評価式について

大阪教育大学・教養学科・情報科学　藤井　淳一 (Jun Ichi Fujii)
Departments of Arts and Sciences (Information Science)
Osaka Kyoiku University

For a nonnegative operator monotone function f on $[0,\infty)$, Kubo and Ando [2] introduced an *operator mean* for positive operators m_f:

$$A\,m_f B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

where the semi-continuity $\lim_{\varepsilon\downarrow 0}(A+\varepsilon I) m_f (B+\varepsilon I) \downarrow A\,m_f B$ assures that we may assume operators are invertible. Recently Kittaneh-Manasrah [1] gave a refined Young inequality, which is immediately extended to an inequality among operator means in the sense of by Furuichi-Lin [3] : Let t be a weight $t\in[0,1]$, then

$$A\nabla_t B - A\#_t B \geq \min\{2t, 2(1-t)\}\,(A\nabla B - A\# B)$$

where $A\nabla_t B=(1-t)A+tB$; the arithmetic mean and $A\#_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}$; the geometric one (for convenience' sake, we omit t if $t=\frac{1}{2}$).

In this talk, we generalize it for parametrized operator means: For $-1 \leqq r \leqq 1$, it is known that the functions $f_{r,t}(x) = (1-t+tx^r)^{\frac{1}{r}}$ are operator monotone, so they define operator means

$$A\,\mathrm{m}_{r,t} B = A^{\frac{1}{2}}\left((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}}.$$

If $r=0$, $A\#_t B$ is the limit $\lim_{r\to 0} A\,\mathrm{m}_{r,t} B$. Then we have

Theorem. *For* $0 \leqq s \leqq r \leqq 1$,

$$A\,\mathrm{m}_{r,t} B - A\,\mathrm{m}_{s,t} B \geq \min\{(2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}}\}\,(A\,\mathrm{m}_r B - A\,\mathrm{m}_s B)\,.$$

We have only to show the numerical inequality:

$$f_{r,t}(x) - f_{s,t}(x) \geqq \min\{(2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}}\}\left(f_{r,\frac{1}{2}}(x) - f_{s,\frac{1}{2}}(x)\right). \tag{0}$$

It is easy to show the above inequality if $r=1/n$, but I have not yet a simple proof for the general case. I have shown it by the following properties:

Lemma. The following properties hold:

(0) f is operator monotone.

(1) $F_{t,x}(r) = (1-t+tx^r)^{\frac{1}{r}}$ is monotone-increasing for r.

(2) $\ell_x(r) = \frac{x^r-1}{r}$ is monotone-increasing for r.

(3) If $0 < x \leqq 1$, then $H(r) = g(r)^{\frac{1}{r}-1} = (1-t+tx^r)^{\frac{1}{r}-1}$ is monotone-increasing for r.

(4) Let $J_{r,x}(t) = (1-t+tx^r)^{\frac{1}{r}}$. Then, $rtJ'_{r,x}(t) = J_{r,x}(t) - J_{r,x}(t)^{1-r}$. Moreover, if $0 < x \leqq 1$, then $J'_{r,x}(t) \geqq J'_{s,x}(t)$.

Proof. (0) Considering the analytic continuation of$f$ to the upper half plane $\operatorname{Im} z > 0$, we have

$$0 < \operatorname{Arg}(1-t+tz^r) < \operatorname{Arg} z^r,$$

so that

$$0 < \operatorname{Arg} f_{r,t}(z) = \operatorname{Arg}(1-t+tz^r)^{1/r} < \operatorname{Arg} z.$$

It follows that $\operatorname{Im} f_{r,t}(z) > 0$, which shows f is operator monotone.

(1) Let $g(r) = 1-t+tx^r$, $y = F_{t,x}(r) = g(r)^{1/r}$. Then $\log y = \log g(r)/r$. By the convexity of $\eta(x) = x\log x$, we have

$$\begin{aligned} F'_{t,x}(r) = y' &= \frac{yg'(r)}{rg(r)} - \frac{y\log g(r)}{r^2} = \frac{y^{1-r}rg'(r)}{r^2} - \frac{y\log g(r)}{r^2} \\ &= \frac{y^{1-r}}{r^2}(tx^r\log x^r - g(r)\log g(r)) = \frac{y^{1-r}}{r^2}(t\eta(x^r) - \eta(g(r))) \\ &\geqq \frac{y^{1-r}}{r^2}(t\eta(x^r) - (1-t)\eta(1) - t\eta(x^r)) = 0, \end{aligned}$$

which shows $F_{t,x}(r)$ is monotone increasing.

(2) By the Klein inequality $\log y \geqq 1 - 1/y$,

$$\ell'_x(r) = \frac{rx^r\log x - (x^r-1)}{r^2} = \frac{x^r\log x^r - (x^r-1)}{r^2} \geqq \frac{x^r - 1 - (x^r-1)}{r^2} = 0.$$

(3) Suppose $0 < x \leqq 1$ and $1-t+tx^r \leqq 1$. By $1-r < 1-s$ and (1), we have

$$H(r) = (1-t+tx^r)^{(1-r)/r} \geqq (1-t+tx^s)^{(1-r)/s} \geqq (1-t+tx^s)^{(1-s)/s} = H(s).$$

(4) The former inequality follows from

$$\begin{aligned} rtJ'_{r,x}(t) &= t(1-t+tx^r)^{\frac{1-r}{r}}(x^r-1) = (1-t+tx^r)^{\frac{1-r}{r}}(1-t+tx^r-1) \\ &= (1-t+tx^r)^{\frac{1}{r}} - (1-t+tx^r)^{\frac{1-r}{r}} = J_{r,x}(t) - J_{r,x}(t)^{1-r}. \end{aligned}$$

Since

$$J'_{r,x}(t) = (1-t+tx^r)^{1/r-1}\frac{x^r-1}{r},$$

It follows from (2) and (3) that $J'_{r,x}(t) \geqq J'_{s,x}(t)$. □

Proof of theorem. Suppose $0 < x \leqq 1$. For $t \leqq 1/2$, put $K(t) = \frac{J_{r,x}(t)-J_{s,x}(t)}{(2t)^{1/r}}$. It follows from Lemma that

$$\begin{aligned} K'(t) &= \frac{J'_{r,x}(t)-J'_{s,x}(t)}{(2t)^{1/r}} - \frac{2}{r}\frac{J_{r,x}(t)-J_{s,x}(t)}{(2t)^{1/r+1}} \\ &= \frac{2}{r(2t)^{1/r+1}}\left(tr(J'_{r,x}(t)-J'_{s,x}(t)) - (J_{r,x}(t)-J_{s,x}(t))\right) \\ &= \frac{2}{r(2t)^{1/r+1}}\left(J_{s,x}(t)^{1-s} - J_{r,x}(t)^{1-r}\right) = \frac{2}{r(2t)^{1/r+1}}\left(H(s)-H(r)\right) \leqq 0, \end{aligned}$$

which shows K is monotone decreasing and attains the minimum $J_{r,x}(1/2) - J_{s,x}(1/2)$ at $t = 1/2$.

Next suppose $t > 1/2$. Putting $L(t) = \frac{J_{r,x}(t)-J_{s,x}(t)}{(2(1-t))^{1/r}}$, we have by Lemma that

$$\begin{aligned} L'(t) &= \frac{2}{r(2(1-t))^{1/r+1}}\left((1-t)r(J'_{r,x}(t)-J'_{s,x}(t)) + (J_{r,x}(t)-J_{s,x}(t))\right) \\ &= \frac{2}{r(2(1-t))^{1/r+1}}\left(r(J'_{r,x}(t)-J'_{s,x}(t)) + (J_{r,x}(t)^{1-r} - J_{s,x}(t)^{1-s})\right) \\ &= \frac{2}{r(2(1-t))^{1/r+1}}\left(r(J'_{r,x}(t)-J'_{s,x}(t)) + (H(r)-H(s))\right) \geqq 0. \end{aligned}$$

Thus L is monotone decreasing and attains the maximum at $t = 1/2$. Therefore

$$J_{r,x}(t) - J_{s,x}(t) \geqq (2\min\{1-t,t\})^{\frac{1}{r}}\left(J_{r,x}(1/2) - J_{s,x}(1/2)\right),$$

that is,

$$(1-t+tx^r)^{1/r} - (1-t+tx^s)^{1/s} \geqq (2\min\{1-t,t\})^{\frac{1}{r}}\left(\left(\frac{1+x^r}{2}\right)^{1/r} - \left(\frac{1+x^s}{2}\right)^{1/s}\right)$$

holds for $0 < x \leqq 1$. By the homogeneity of x, it also holds for $x > 1$. □

If the following conjecture holds, we immediately have a simple proof for (0):

Conjecture. For $0 < s \leqq r \leqq 1$,

$$\begin{cases} \left(\frac{(1-t+tx^r)^{\frac{1}{r}} - (t(1+x^r))^{\frac{1}{r}}}{1-(2t)^{\frac{1}{r}}}\right)^{\frac{1}{r}} \geqq \left(\frac{(1-t+tx^s)^{\frac{1}{r}} - (t(1+x^s))^{\frac{1}{r}}}{1-(2t)^{\frac{1}{r}}}\right)^{\frac{1}{s}} & (t < \frac{1}{2}) \\ \left(\frac{(1-t+tx^r)^{\frac{1}{r}} - ((1-t)(1+x^r))^{\frac{1}{r}}}{1-(2(1-t))^{\frac{1}{r}}}\right)^{\frac{1}{r}} \geqq \left(\frac{(1-t+tx^s)^{\frac{1}{r}} - ((1-t)(1+x^s))^{\frac{1}{r}}}{1-(2(1-t))^{\frac{1}{r}}}\right)^{\frac{1}{s}} & (t > \frac{1}{2}). \end{cases}$$

In fact, it is also shown easily for $r = \frac{1}{n} > s$: For the case $t < 1/2$, the Jensen inequality for the power function for $ns < 1$ implies

$$\begin{aligned}
\text{LHS} &= \left(\frac{(1-t+tx^{\frac{1}{n}})^n - \left(2t\left(\frac{1+x^{\frac{1}{n}}}{2}\right)\right)^n}{1-(2t)^n} \right)^n \\
&= \left(\frac{\sum_{k=0}^{n-1}(2t)^k(1-t+tx^{\frac{1}{n}})^{n-k-1}\left(\frac{1+x^{\frac{1}{n}}}{2}\right)^k}{\sum_{k=0}^{n-1}(2t)^k} \right)^{ns\cdot\frac{1}{s}} \\
&\geqq \left(\frac{\sum_{k=0}^{n-1}(2t)^k((1-t+tx^{\frac{1}{n}})^{ns})^{n-k-1}\left((\frac{1+x^{\frac{1}{n}}}{2})^{ns}\right)^k}{\sum_{k=0}^{n-1}(2t)^k} \right)^{\frac{1}{s}} \\
&= \left(\frac{\sum_{k=0}^{n-1}(2t)^k(1-t+tx^{s})^{n-k-1}\left(\frac{1+x^{s}}{2}\right)^k}{\sum_{k=0}^{n-1}(2t)^k} \right)^{\frac{1}{s}} \\
&= \left(\frac{(1-t+tx^s)^n - (t(1+x^s))^n}{1-(2t)^n} \right)^{\frac{1}{s}} = \text{RHS}.
\end{aligned}$$

参考文献

[1] F.Kittaneh and Y.Manasrah: Improved Young and Heinz inequalities for matrices, J. Math. Anal. Appl. **361**(2010) 262–269.

[2] F.Fubo and T.Ando: Means of positive linear operators, Math. Ann. **246** (1980), 205–224.

[3] S.Furuichi and M.Lin: On refined Young inequalities, Preprint, `http://arxiv.org/PS_cache/arxiv/pdf/1001/1001.0195v1.pdf`