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DEVELOPMENTS IN NONCOMMUTATIVE KOROVKIN-TYPE THEOREMS

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1. INTRODUCTION

The classical approximation theorem due to P. P. Korovkin [11] in 1953, unified many existing approximation processes such as Bernstein polynomial approximation of continuous functions. Korovkin's discovery inspired many researchers that lead to Korovkin-type theorems and Korovkin sets in various settings such as more general function spaces, Banach algebras, Banach lattices and operator algebras. Another major advancement was the discovery of geometric theory of Korovkin sets by Y. A. Šaškin [21] and D. E. Wulbert in 1968 [7]. A detailed survey of most of these developments can be found in the article of Berens and Lorentz in 1975 [7], monograph of Altomare and Campiti [1] most recent survey by Altomare [2] which contains several new results also.

This article aims at providing a rather short survey of the developments in the so called noncommutative Korovkin-type approximation theory and Korovkin sets (quantization of Korovkin theorems, W. B. Arveson [6]) in the settings of C^* and W^*-algebras. Due to technical reasons only important theorems are quoted that too without proofs. However an attempt is made to provide illustrative examples, a few new results and research problems. First we quote three major theorems due to Korovkin following the article of Berens and Lorentz [7]. We designate these as Korovkin's type I, type II and type III theorems. The survey will be about 'quantization' of these three theorems!

Type I Korovkin's theorem. Let $\{\Phi_n : n = 1, 2, 3, \ldots\}$ be a sequence of positive linear maps on $C[a, b]$ and for each of the functions $g_k(x) = x^k$, $x \in [a, b]$, $k = 0, 1, 2$, let

$$\lim_{n\to\infty} \Phi_n(g_k) = g_k \text{ uniformly on } [a, b],\ k = 0, 1, 2.$$

Then

$$\lim_{n\to\infty} \Phi_n(f) = f \text{ uniformly on } [a, b] \text{ for all } f \text{ in } C[a, b].$$

Definition 1.1. A set S in $C[a, b]$ is called a <u>test set</u> or Korovkin set for positive linear operators on $C[a, b]$ if for every sequence $\{\Phi_n\}$ of positive linear operators on $C[a,b]$, $\lim_{n\to\infty}\Phi_n(s)=s$ uniformly on $[a,b]$ for every s in S implies that $\lim_{n\to\infty}\Phi_n(f{=}f$ uniformly of $[a,b]$ for all $f\in C[a,b]$.

Type I theorem says that $\{1,x,x^2\}$ is a test set.

Type II Korovkin's theorem. There is not test set for $C[a,b]$ consisting only of two functions. Thus the cardinality of a test set is atleast 3.

Type III Korovkin's theorem. A triple $\{f_0,f_1,f_2\}$ is a test set of $C[a,b]$ exactly when it is a Čebyšev system on $a,b]$.

This article is divided into four sections. The next three sections are devoted to noncommutative Type I, Type II and Type III theorems. The last section contains some aspects of weak Korovkin type theorems and its geometric formulation.

2. Type I Theorems

This section deals with type I Korovkin theorems in the settings of C^*-algebras. It is assumed that the C^*-algebras considered here are over complex numbers and always contain identity unless otherwise specified.

Definition 2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be complex C^*-algebras with identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ respectively and let $T:\mathscr{A}\to\mathscr{B}$ be a positive linear contraction (a linear map T that preserves positivity and such that $T(1_{\mathscr{A}})\le 1_{\mathscr{B}}$). For a subset H of $\mathscr{A}$ the Korovkin closure $K_+(H,T)$ is defined as

$$\Big\{a\in\mathscr{A}\,\Big|\,\lim_\alpha\Phi_\alpha(a)=T(a)\text{ for every net }\{\Phi_\alpha\}_{\alpha\in I}\text{ of positive linear contractions from }\mathscr{A}\text{ to }\mathscr{B}\text{ such that }\lim_\alpha\Phi_\alpha(h)=T(h)\text{ for all }h\text{ in }H\ \Big\}$$

Here convergence considered is the norm convergence unless otherwise stated explicitly. It seems that the first noncommutative Korovkin type theorem was due to W. B. Arveson in 1970 for $*$-homomorphisms where $\mathscr{A}$ is $C(X)$, the C^*-algebra of all complex continuous functions on a compact, Hausdorff space X. We recall this, being the first of its kind. Recall that for a subset H of $C(X)$, the Choquet boundary $\partial_H^+(X)$ is defined as the set of all points x in X such that the evaluation functionals $\epsilon_x|H$ has the unique positive linear extension ϵ_X to $C(X)$. Also the support K_T of $T:C(X)\to\mathscr{B}$ is defined as set of all x in X such that $f(x)=0$ whenever $T(f)=0$.

2.1. **Theorem.** Let H be a subset of $C(X)$ containing 1 and let $T:C(X)\to\mathscr{B}$ be a $*$ homomorphism such that $K_T\subseteq\partial_H^+(X)$. Then $K_+(H,T)=C(X)$.

The proof of the above theorem uses the lattice theoretic properties of the selfadjoint part of $C(X)$. It is to be recalled that the selfadjoint part of a C^*-algebra $\mathscr{A}$ is a lattice in the natural order if and only of $\mathscr{A}$ is commutative. In 2009, W. B. Arveson published a paper [6] in which he worked out a relation between 'noncommutative Choquet boundary' and 'hyper-rigid subspaces' (Korovkin sets) for general C^* algebras. In what follows a brief sketch of the developments from 1970 to 2010 is provided. Important theorems that appeared in the articles published during the above period by various authors, are regarding the following two questions.

(1) When does the Korovkin closure has an algebraic structure

(2) When is the Korovkin closure the full C^* algebra. Mainly four types of maps are considered in this settings. A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ is called

(a) Positive if $\Phi(x^*x)$ is positive for all $x \in A$

(b) Schwarz map if $\Phi(x^*x) \geq \phi(x)^*\Phi(X)$ for all $x \in \mathscr{A}$

(c) Completely positive if $\Phi^{(n)} : \mathscr{A} \otimes M_n(\mathbb{C}) \to \mathscr{B} \otimes M_n(\mathbb{C})$ is a positive for all positive integers n, where $M_n(\mathbb{C})$ is the set of all $n \times n$ matrices over $\mathbb{C}$ and $\Phi^{(n)}$ is the map on $\mathscr{A} \otimes \mathscr{B}$ defined by

$$\Phi^{(n)}(a_{ij}) = (\Phi(a_{ij})), \text{ where } (a_{ij}) \in \mathscr{A} \otimes M_n(\mathbb{C})$$

(d) Completely contractive if $\Phi^{(n)}$ is contractive for each positive integer n.

The main tools being used in the development of the commutative theory for positive linear maps are Kadison-Schwarz type inequalities and Choquet boundary theory. It is to be mentioned that every completely positive map of norm ≤ 1 is a Schwarz map and the very definition of Schwarz map implies the Schwarz type inequality. For general positive linear map $\Phi : \mathscr{A} \to \mathscr{B}$ with norm ≤ 1 Kadison proved that

$$\Phi(x^2) \geq \Phi(x)^2 \forall x \in \mathscr{A}, x^* = x.$$

This fundamental inequality is known as Kadison-Schwarz inequality and has been improved by many mathematicians like M. D. Choi [9] and T. Furuta [10]. These improvements will have some effect on the study of Korovkin sets. However this possibility is yet to be investigated.

Using Kadison-Schwarz inequality, W. M. Priestley [17] proved the following theorem in 1976.

2.2. **Theorem.** Let $\mathscr{A}$ be a C^*-algebra and let $\{\Phi_\alpha\}_{\alpha \in I}$ be a net of positive linear maps on $\mathscr{A}$ such that

$$\Phi_\alpha(1_{\mathscr{A}}) \leq 1_{\mathscr{A}} \quad \forall \alpha \in I.$$

Then the set

$$J := \{x \in \mathscr{A} | \lim_\alpha \Phi_\alpha(a) = a \ \forall a \in \{x, x^* \circ x, x^2\}\}$$

is a J^*-algebra in $\mathscr{A}$.

Recall that a J^*-algebra in $\mathscr{A}$ is a norm closed, $*$ closed subset of $\mathscr{A}$ which is also closed under the Jordan product $\circ$, namely

$$a \circ b = ab + ba, \ a, b \in \mathscr{A}$$

For Schwarz maps A. G. Robertson [18] proved the following theorem.

2.3. **Theorem.** Let $\mathscr{A}$ be a C^*-algebra with identity $1_{\mathscr{A}}$ and let $\{\Phi_\alpha\}_{\alpha\in I}$ be a net of Schwarz maps on $\mathscr{A}$ such that

$$\Phi_\alpha(1_{\mathscr{A}}) \leq 1_{\mathscr{A}}, \quad \forall \alpha \in I.$$

Then the subset

$$K = \{x \in \mathscr{A} \mid \lim_\alpha \Phi_\alpha(a) = a \text{ for } a \in \{x, x^*x, xx^*\}\}$$

is a C^*-algebra in $\mathscr{A}$.

Subsequently B. V. Limaye and M. N. N. Namboodiri [12] improved the results of Priestley and Robertson to obtain the following theorem in 1982:

2.4. **Theorem.** Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ respectively. Let $\{\Phi_\alpha\}_{\alpha\in I}$ be a net of positive linear maps from $\mathscr{A}$ to $\mathscr{B}$ such that

$$\Phi_\alpha(1_{\mathscr{A}}) \leq 1_{\mathscr{B}} \quad \forall \alpha \in I.$$

Let $T : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Then the subset

$$J := \{x \in \mathscr{A} \mid \lim_\alpha \Phi_\alpha(a) = T(a), \text{ for } a \in \{x, x^* \circ x\}\}$$

is a J^*-algebra in $\mathscr{A}$.

If all Φ_α, $\alpha \in I$ are Schwarz maps and T a $*$-homomorphism, then J is a C^*-algebra in $\mathscr{A}$.

In the above cases the test set was symmetric with respect to $*$ operation. For the case when this symmetry is not assumed, B. V. Limaye, M. N. N. Namboodiri in 1984 [14] and A. G. Robertson in 1986 [19] proved the following theorems.

It is known that Korovkin type approximation theory leads to deeper understanding of the structure under consideration. The following interesting theorem in [14] reveals this.

2.5. **Theorem.** Let $\mathscr{A} = \mathscr{B}$ be a noncommutative C^* algebra with identity $1_{\mathscr{A}}$ and let

$$D = \{x \in \mathscr{A} \mid \lim_\alpha \Phi_\alpha(a) = a \text{ for all } a \in \{x, x^*x\}\}$$

where $\{\Phi_\alpha\}_{\alpha\in I}$ is a net of Schwarz maps on $\mathscr{A}$ with norm ≤ 1. Then D is a subalgebra of $\mathscr{A}$. Also D is $*$ closed if and only if $\mathscr{A} = M_2(\mathbb{C})$, the set of all 2×2 matrices over $\mathbb{C}$.

The above theorem shows that $M_2(\mathbb{C})$ behaves like a commutative C^*-algebra and this is the only noncommutative one! In fact, Limaye and Namboodiri proved that, among all finite dimensional noncommutative C^* algebra $M_2(\mathbb{C})$ is the only one for which D in Theorem 2.5 is $*$ closed. Robertson proved that finite dimensionality assumption can be dropped.

We bypass several important developments of Korovkin approximation theory for commutative as well as noncommutative Banach algebras. Excellent exposition can be found by Micheal Panneberg, [1, Appendix A], Ferdinand Beckhoff [1, Appendix B] and F. Altomare [2]. However M. Uchiyama's paper [23] give estimates the norm related to Schwarz map. He also obtains several extensions of Korovkin type theorems by using operator monotone functions and T. Ando's inequality. In this paper he also unified several earlier results by introducing o^*-subalgebras and the associated generalized Schwarz maps with respect to the product o. More over the proofs that he gives are simpler than the earlier ones.

For the sake of completion we quote a couple of theorems of Uchiyama [23] for the C^* algebra $C(X)$.

2.6. **Theorem.** [23, Theorem 3.1]

Let $S \subset C(X)$ and $C^*(S)$ be the C^*-algebra generated by S. Let f be a operator monotone function defined on $[0, \infty]$ such that $f(0) \leq 0$ and $f(\infty) = \infty$. Set $g = f^{-1}$ then we have

$$C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u|^2) : u \in S\})$$

if $f(0) = 0$ or $1 \in S$. Where the set on the right side denotes the Korovkin closure.

2.7. **Theorem.** [23, Theorem 2.12] Let $\{\Phi_n\}$ be a sequence of Schwarz maps from $\mathscr{A}$ to $\mathscr{B}$ where $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras with identities, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Let f be an operator monotone function on $[0, \infty]$ with $f(0) = 0$, $f(\infty) = \infty$. Set $g = f^{-1}$. Then the set

$$C = \{a \in \mathscr{A} | \Phi_n(x) \to \Phi(x) \text{ for } x = a, g(a^*a) \text{ and } g(aa^*)\}$$

is a C^*-subalgebra.

Another important development was the use of Krein-Millman theorem for compact convex sets and the associated unique extension property. For noncommutative C^*-algebras, the following theorem was proved by Takahasi in 1979 [22].

2.8. **Theorem.** Let ϕ be an extreme state of a C^* algebra $\mathscr{A}$ and let $x \in \mathscr{A}_+$ peaks for ϕ, that is the supports of x and ϕ in the enveloping von Neumann algebra add up to 1. Let $\{\phi_\alpha\}_{\alpha \in I}$ be a net of positive linear functionals of $\mathscr{A}$ and assume that $\lim_\alpha \phi_\alpha(1 - \mathscr{A}) = 1$ and $\lim_\alpha \phi_\alpha(x) = 0$. Then we have

$$\lim_\alpha \phi_\alpha(a) = 0 \text{ for all } a \text{ om } \mathscr{A}.$$

Though the above theorem is elegant, for an arbitrary C^*-algebras there is no way of finding extreme states (or pure states), where as for commutative C^* algebras extreme states are point evaluations. In the case of C^*-algebra $B(H)$, H a Hilbert space, ϕ a vector state better results are known to exist. See for example Limaye-Namboodiri (1979) [1, Appendix B.] , Altomare 1987 [1] and Dieckmann 1992 [1].

2.9. **Theorem (Altomare** [1]**).** Let $T \in B(H)$, be a non zero compact operator, λ a simple eigenvalue of T such that $\|T\| = |\lambda|$, x a corresponding unit eigenvector and let $\phi = \langle \cdot x, x \rangle$ be the vector state corresponding to x.

Put $S = I_H + \frac{1}{|\lambda|^2}T^*T - \frac{1}{\lambda}T - \frac{1}{\bar{\lambda}}T^*$. Then

$$K_+(\{I_H, S\}, \phi) = B(H).$$

Here I_H denotes the identity operator on H.

The following theorem is due to Dieckmann

2.10. **Theorem (Dieckmann).** Let $T \in B(H)$ be a strictly positive compact operator on H and let $\mathscr{A}(H) = C^*(I_H, K(H))$, where $K(H)$ is the set of all compact operators on H. Let ϕ be the complex homomorphism on $\mathscr{A}(H)$ defined by $\phi(\lambda I_H + K) = \lambda$, $K \in K(H)$. Then $K_+(\{I_H, T\}, \phi) = \mathscr{A}(H)$.

Finally we go through Arveson's contributions to noncommutative Korovkin type theorems and Korovkin sets in 2009 [6] via his own theory of noncommutative Choquet boundary theory of operator systems. In the fundamental papers during 1969–70 and 2008, he introduced and proved many concepts and the theorems related to non commutative Choquet boundary and Silov boundary ideals corresponding to operator systems. This is quite analogous to classical theory of Choquet and Silov boundaries for function systems. Analogous to the work of Saskin, Arveson studied the relation between noncommutative Korovkin sets and noncommutative Choquet boundary in 2009 [6]. He proved many interesting theorems in this settings, though some of these results were already known to exist. We start with the notion of hyperrigid set of generators of C^* algebras [6].

2.11. **Definition.** A finite of countably infinite set G of generators of C^* algebra $\mathscr{A}$ is said to be hyperrigid if for every faithful representation $\pi(\mathscr{A}) \subseteq B(H)$ of $\mathscr{A}$ on a Hilbert space H and every sequence of unit preserving completely positive maps (UCP) $\Phi_n : B(H) \to B(H)$, $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} \|\Phi_n(\pi(g) - \pi(g))\| = 0 \forall g \in G \Rightarrow \lim_{n\to\infty} \|\|\Phi_n(\pi(a) - \pi(a))\| = 0,$$

$\forall a \in \mathscr{A}$. He then proves the following basic theorem.

2.12. **Theorem.** For every separable operator system S that generates a C^*-algebra $\mathscr{A}$, the following are equivalent.

(i) S is hyperrigid

(ii) For every non degenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space H and every sequence $\Phi_n : \mathscr{A} \to B(H)$

of UCP maps;

$$\lim_{n\to\infty} \|\Phi_n(s) - \pi(s)\| = 0 \quad \forall s \in S \Rightarrow \lim_{n\to\infty} \|\Phi_n(a) - \pi(a)\| = 0$$

for all $a \in \mathscr{A}$.

(iii) For every non degenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space, π/S has the unique extension property. That is, π/S has a unique completely positive linear extension to $\mathscr{A}$.

(iv) For every unital C^* algebra $\mathscr{B}$, every unital homomorphism of C^*-algebras $\theta : \mathscr{A} \to \mathscr{B}$ and every UCP map $\Phi' : \mathscr{B} \to \mathscr{B}$

$$\Phi(x) = x \quad \forall x \in \theta(S) \Rightarrow \Phi(x) = x \forall x \in \theta(\mathscr{A}).$$

One of the main results (Theorem 3.3 [6]) that Arveson obtains as a consequence of the above theorem, is known to exist. In fact much better theorem can be found in [6]. However he proves a very strong theorem [6, Theorem 5.1] which is as follows.

2.13. **Theorem.** Let S be a separable operator system whose generated C^*-algebra $\mathscr{A}$ has countable spectrum such that every irreducible representation of $\mathscr{A}$ is a boundary representation for S. Then S is hyperrigid.

2.14. **Arveson's conjecture.** If every irreducible representation of $\mathscr{A}$ is a boundary representation for a separable operator system S, then S is hyperrigid.

Now recall that the following theorem was proved by Y. A. Saskin for positive linear contractions and D. E. Wulbert for linear contractions [7].

2.15. **Theorem.** Let G be a subset of $C(X)$ that separates points of X and contains the constant function 1_X. Then G is a Korovkin set for linear contractions or positive linear contractions if and only if $\partial_{\mathrm{Ch}} G_0 = X$, $G_0 = \operatorname{span} G$.

The noncommutative Choquet boundary was defined by Arveson [6] in the following way.

2.16. **Definition.** Let S be an operator system in a C^*-algebra $\mathscr{A}$, i.e., a self adjoint linear subspace of $\mathscr{A}$ such that $1_{\mathscr{A}} \in S$ and $\mathscr{A} = C^*(S)$ the C^*-algebra generated by S and $1_{\mathscr{A}}$. A boundary representation for S is an irreducible representation π of $\mathscr{A}$ such that π/S has a unique completely positive linear extension to $\mathscr{A}$. The set ∂_S of all unitary equivalence classes of all boundary representations for S id defined as the noncommutative Choquet boundary of the operator system S.

Since irreducible representation of the function space $C(X)$ can be identified with points in X itself, Arveson's notion of Choquet boundary for operator systems is an exact noncommutative analogue of the classical one for function systems.

In what follows we examine the possibility of extending Arveson's theorem quoted here for linear contractions. Since extension theorem for completely positive maps is not available, we need to define hyperrigidity separately. So we introduce strong hyperrigidity so as to suit completely contractive maps.

2.17. **Definition.** A finite or countably infinite set G of generators of a C^*-algebra $\mathscr{A}$ is said to be strongly hyperrigid if for every faithful representation π of $\mathscr{A}$ in $B(H)$ and for every sequence Φ_n completely contractive maps from $\pi(\mathscr{A})$ to $B(H)$

$$\begin{aligned} &\lim_{n\to\infty} \|\Phi_n(\pi(g)) - \pi(g)\| = 0 \quad \forall g \in G \\ &\Rightarrow \lim_{n\to\infty} \|\Phi_n(\pi(a)) - \pi(a)\| = 0 \quad \forall a \in \mathscr{A}. \end{aligned}$$

2.18. **Remarks.** It can be seen that the strong hyperrigidity coincide with hyperrigidity for UCP as a consequence of Arveson's extension theorem for CP maps. However hyperrigidity of G need not imply strong hyperrigidity. To overcome this difficulty it would be reasonable to assume that G is closed under $*$ operation if necessary.

In what follows we aim at identifying 'obstructions' to strong hyperrigidity. We also assume that the operator system is $*$ closed and contains identity element.

2.19. **Characterisation theorem.** For separable operator system S that generates a C^*-algebra $\mathscr{A}$, the following are equivalent:

(i) S is strongly hyperrigid.

(ii) For every non degenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space H and every sequence $\Phi_n : \mathscr{A} \to B(H)$ completely contractive maps,

$$\begin{aligned}&\lim_{n\to\infty} \|\Phi_n(s) - \pi(s)\| = 0 \quad \forall s \in S\\ &\Rightarrow \lim_{n\to\infty} \|\Phi_n(a) - \pi(a)\| = 0 \quad \forall a \in \mathscr{A}.\end{aligned}$$

(iii) For every non degenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space H, π/S has the unique extension property.

(iv) For every unital C^*-algebra $\mathscr{B}$, every unital homomorphism of C^*-algebras $\theta : \mathscr{A} \to \mathscr{B}$ and every (for every UCP map $\Phi : \theta(\mathscr{A}) \to \mathscr{B}$) map $\Phi\theta \to \mathscr{B}$

$$\Phi(x) = x \quad \forall x \in \theta(S) \Rightarrow \Phi(x) = x \quad \forall x \in \theta(\mathscr{A}).$$

Proof. The proof is more or less the same as that of Arveson. However the details are provided. We show that

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii)

Let $\pi : \mathscr{A} \to B(H)$ be a non degenerate representation of $\mathscr{A}$ on a separable Hilbert space H and let $\Phi_n : \mathscr{A} \to B(H)$ be a sequence of (completely contractive) linear maps such that

$$\lim_{n\to\infty} \|\Phi_n(s) - \pi(s)\| = 0 \quad \forall s \in S.$$

Let $\sigma : \mathscr{A} \to B(H)$ be a faithful representation of $\mathscr{A}$ on another separable Hilbert space K.

Then $\sigma \oplus \pi : \mathscr{A} \to B(K \oplus H)$ is a faithful representation of $\mathscr{A}$ on $K \oplus H$. Define maps $\mu_n : (\sigma \oplus \pi)(\mathscr{A}) \to B(K \oplus H)$ by

$$\mu_n(\sigma(a) \oplus \pi(a)) = \sigma(a) \oplus \Phi_n(a), \quad a \in \mathscr{A}.$$

Then μ_n is completely contractive.

Also $\mu_n(\sigma(s) \oplus \pi(s) \to \sigma(s) \oplus \pi(s))$, for all $s \in S$.

Also $\mu_n(\sigma(a) \oplus \pi(a) \to \sigma(a) \oplus \pi(a))$, for all $a \in \mathscr{A}$.

Now,

$$\lim_{n\to\infty} \sup \|\Phi_n(a) - \pi(a)\| \leq \lim_{n\to\infty} \text{sub}\|\sigma(a) \oplus \Phi(a) - \sigma(a) \oplus \pi(a)\|$$
$$= \text{sub}\|\mu_n(\sigma(a) \oplus \pi(a)) - \sigma(a) \oplus \pi(a)\|$$

Therefore

$$\lim_{n\to\infty} \|\Phi_n(a) - \pi(a)\| = 0 \quad a \in \mathscr{A}.$$

Now (iii) $\Rightarrow$ (iv): Let $\theta : \mathscr{A} \to \mathscr{B}$ be an identity preserving homomorphism of C^*-algebras and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a UCP that satisfies $\Phi(\theta(s)) = \theta(s)$, $s \in S$. We have to show that

$$\Phi(\theta(a)) = \theta(a) \quad \forall a \in \mathscr{A}.$$

Let B_0 be the separable C^*-algebra in $\mathscr{B}$ generated by

$$\theta(\mathscr{A}) \cup \Phi(\theta(\mathscr{A})) \cup \Phi^2(\theta(\mathscr{A})) \cup \cdots$$

It is clear that $\Phi(B_0) \subseteq B_0$.

By considering a faithful representation of B_0 on a separable Hilbert space H, we may assume that $B_0 \subseteq B(H)$. Let $\bar{\Phi} : B(H) \to B(H)$ is a UCP map either $\overline{\Phi}/B_0 = \Phi$. Here $\bar{\Phi}(\theta(s)) = \theta(s)$, $\forall s \in S$. Since $\theta : \mathscr{A} \to B(H)$ is a representation on H, we must have

$$\Phi(\theta(a)) = \bar{\Phi}(\theta(a)) = \theta(a) \quad \forall a \in A.$$

Hence the proof.

(iv) $\Rightarrow$ (i) Let $\pi : \mathscr{A} \to B(H)$ be a faithful representation of $\mathscr{A}$ on $B(H)$ for some Hilbert space H. Put $\mathscr{B} = B(H)$. Consider the C^*-algebras of all bounded seqences $l^\infty(\mathscr{A})$ and $l^\infty(\mathscr{B})$ in $\mathscr{A}$ and $\mathscr{B}$ respectively. Let $\Phi_n : \pi(A) \to B(H)$ be the sequence of all completely contractive maps such that

$$\lim_{n\to\infty} \|\Phi_n(\pi(s) - \pi(s))\| = 0 \quad \forall s \in S.$$

To show that

$$\lim_{n\to\infty} \|\Phi_n(\pi(a) - \pi(a)\| > 0 \quad \forall a \in \mathscr{A}.$$

Let

$$\tilde{\Phi} : l^\infty(\mathscr{A}) \to l^\infty(\mathscr{B}) \text{ be defined as}$$
$$\tilde{\Phi}(a_1, a_2 \ldots a_n, \ldots) = (\Phi_1(a_1), \Phi_2(a_2), \ldots)$$
$$(a_1, a_2, \ldots, a_n, \ldots) \in l^\alpha(\mathscr{A})$$

First we show that $\tilde{\Phi}$ is completely contractive. It is quite easy to see that $l^\infty(\mathscr{A}) \otimes M_n(C)$ can be identified isometrically with $l^\alpha(\mathscr{A} \otimes M_n(C))$. The same way identify $l^\infty(\mathscr{B} \otimes M_n(C))$ with $l^\infty(\mathscr{B}) \otimes M_n(C)$.

Thus $\tilde{\Phi}^{(n)}$ can be regarded as a map from $l^\infty(\mathscr{A} \otimes M_n(C))$ to $l^\infty(\mathscr{B} \otimes M_n(C))$, for each positive integer n. $\tilde{\Phi}^{(n)}$ is the map induced by $\tilde{\Phi}$ for each n.

It is easy to see that $\tilde{\Phi}^{(n)}$ is contractive since $\Phi^{(n)}$ is contractive and $\Phi^{(n)}(\pi(I)) \to \pi(I)$ as $n \to \infty$.

Let $C_0(\mathscr{A})$ (respectively $C_0(\mathscr{B})$) denotes the ideal of all sequences in $\mathscr{A}$ (respectively $\mathscr{B}$) that converges to zero in norm. Consider the map

$$\tilde{\Phi}_0 : \frac{l^\infty(\mathscr{A})}{C_0(\mathscr{A})} \to \frac{l^\infty(\mathscr{B})}{C_0(\mathscr{B})}$$

defined by

$$\tilde{\Phi}_0(x + C_0(\mathscr{A})) = \tilde{\Phi}(x) + C_0(\mathscr{B}), \quad x \in l^\infty(\mathscr{A})$$

Then $\tilde{\Phi}_0$ is completely contractive. Consider the embedding $\theta : \mathscr{A} \to l^\infty(\mathscr{A})$ defined by

$$\begin{aligned} \theta(a) &= (a, a, \dots) + C_0(\mathscr{A}) \\ \text{Therefore}\tilde{\Phi}_0(\theta(s)) &= (\Phi_1(s), \Phi_2(s), \dots) + C_0(\mathscr{A}) \\ &= (s, s, \dots, s, \dots) + C_0(\mathscr{A}) \\ &= \theta(s) \quad \forall s \in S \end{aligned}$$

Thus

$$\tilde{\Phi}_0 : \frac{l^\infty(\mathscr{A})}{C_0(\mathscr{A})} \to \frac{l^\infty(\mathscr{B})}{C_0(\mathscr{B})}$$

such that

$$\tilde{\Phi}_0(\theta(s)) = \theta(s) \quad s \in S. \qquad\qquad \Rightarrow \tilde{\Phi}_0$$

is a UCP since identity $1_{\mathscr{A}} \in S$. This is because, if $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras with identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ and if $\Phi : \mathscr{A} \to \mathscr{B}$ is a contractive linear map such that

$$\|\Phi(1_{\mathscr{A}})\| = \|\Phi\|,$$

then Φ is positivity preserving. Then $\tilde{\Phi}_0(\theta(a)) = \theta(a)$ for all $a \in \mathscr{A}$. That is,

$$\begin{aligned} &(\Phi_1(a), \Phi_2(a), \dots) + C_0(\mathscr{A}) \\ &\qquad = (a, a, \dots) + C_0(\mathscr{A}) \quad a \in \mathscr{A} \\ &\qquad \Rightarrow \|\Phi_n(a) - a\| \to 0 \text{ as } n \to \infty \end{aligned}$$

□

2.20. **Remarks.** The above theorem is a noncommutative analogue of Wulberts theorem for hyperrigidity in function spaces such as $C(X)$. This is because every contractive linear map on $C(X)$ is completely contractive [3, 4]. So most of Arveson's theorem for hyperrigidity for C^* algebras is valid for strong hyperrigidity also. We state some of these without proof.

2.21. **Corollary.** Let S be a strongly hyperrigid separable operator system, with generated C^*-algebra $\mathscr{A}_1$ let K be an ideal in $\mathscr{A}$ and let $a \in \mathscr{A} \mapsto \dot{a} \in \mathscr{A}/K$ be the quotient map. Then $\dot{S}$ is a strongly hyperrigid operator system in $\mathscr{A}/K$.

2.22. **Theorem.** Let $x \in B(H)$ be a self adjoint operator with atleast 3 points in its spectrum and let $\mathscr{A}$ be the C^*-algebra generated by x and 1. Then

(i) $G = \{1, x, x^2\}$ is a strongly hyperrigid operator system for $\mathscr{A}$, while
(ii) $G_0 = \{1, x\}$ is not a strongly hyperrigid generator for $\mathscr{A}$.

2.23. **Theorem.** Let $\{u_1, u_2, \dots u_n\}$ be a set of isometrices that generate a C^*-algebra $\mathscr{A}$ and let

$$G = \{u_1, u_2, \dots u_n, u_1^*, u_1 + u_2^* u_2 + \cdots + u_n^* u_n\}.$$

Then G is a strongly hyperrigid generator for $\mathscr{A}$.

2.24. **Corollary.** The set $G = \{u_1, u_2, \dots, u_n\}$, where $\sum_{k=1}^{n} u_k u_k^* = I$, of generators of the Cuntz algebra θ_n is strongly hyperrigid.

We conclude this section by remarking that many more implication of 'strong hyperrigidity theorem' are to be investigated. However such results will appear elsewhere.

3. Type II Korovkin theorems

Recall that this section deals with the size of a test set H in a C^*-algebra $\mathscr{A}$ generated by H. In $C[a, b]$, there is no test set containing only two elements.

Observe that (i) of 2.22 is already known, where as (ii) does not seem to exist in this generality. Does this result have a noncommutative analogue?

3.1. **Question.** Let $x \in B(H)$. Let $\mathscr{A}$ be the C^*-algebra generated by I ad X. Then it is known that $\{I, x, x^*x + xx^*\}$ is hyperrigid in $\mathscr{A}$. If spectrum $\sigma(x)$ has atleast 3 distinct points, then is it true that $\{I, x\}$ is not a hyperrigid generator of $\mathscr{A}$?

The following simple modification of 2.22 is possible.

3.2. **Proposition.** Let $x \in B(H)$ be normal. Then $G = \{1, x, x^*x\}$ is a hyperrigid set of generators for $\mathscr{A} = C^*(x)$. If $\sigma(x)$ contains three distinct points λ_1, λ_2 and λ_3, on some straight line, then $\{1, x\}$ will not be a hyperrigid generator for $\mathscr{A}$.

We provide the proof for the sake of completion.

Proof. Statement (i) is already known. Now we prove (ii) using Arveson's argument. Let $S = \text{span}\{1, x\}$ and let $\sigma(x)$ denote the spectrum of x. For $f \in C(\sigma(x))$, let $\phi_k(f(x)) = f(\lambda_k)$, $k = 1, 2, 3$.

Then ϕ_k is a multiplicative positive linear functional of norm 1 which is an irreducible representation of $\mathscr{A}$ on $\mathbb{C}$. But

$$\begin{aligned} \phi_k(\lambda 1_\alpha + \mu_x) &= \lambda + \mu \phi_k(x) \\ &= \lambda + \mu \lambda_k, \;\; k = 1, 2, 3. \end{aligned}$$

But λ_2 is a convex combination (assume without loss of generality) of λ_1 and λ_3. Therefore $\lambda_2 = t\lambda_1 + (1 - t)\lambda_3$, $0 < t < 1$.

Therefore $\phi_2(\lambda 1_{\mathscr{A}} + \mu x) = t\phi_1(\lambda 1_{\mathscr{A}} + \mu x) + (1 - t)\phi_3\lambda 1_{\mathscr{A}} + \mu x)$. Thus the positive linear functional $\phi = t\phi_1 + (1 - \phi)\phi_3$ and ϕ_2 are two different completely positive extensions of ϕ_2/S. Therefore the irreducible representation ϕ_2 fails to have unique extension property therefore one x is not hyperrigid. □

We conclude this section by stating a problem of Arveson [6].

3.3. **Question.** Let $I = [a, b]$, $f : I \to R$ and $A \in B(H)$ be selfadjoint. Is $[1, A, f(A)]$ hyperrigid in $C^*(A)$? Arveson observes that in case A has discrete spectrum in $[a, b]$ and if f is either strictly convex or strictly concave, the answer is affirmative.

4. Type III Korovkin theorems

Recall the classical Korovkin theorem says that $\{f_1, f_2, f_3\}$ is hyperrigid in $C[a, b]$ exactly when span $\{f_1, f_2, f_3\}$ is a Čebyšev system. It would be interesting to examine its non commutative counterpart using Čebyšev systems in C^*-algebra. First we recall the notion of Čebyšev system in Banach spaces.

4.1. **Definition.** Let M be a subspace of a Banach space. N is called a Čebyšev system if each vector N admits a unique closest point in M.

A. Haar in 1918 [20] obtained the following characterization of finite dimensional Čebyšev subspaces of $C(X)$, X compact and Hausdorff. For C^* algebras the study was carried out by A. G. Robertson, David Yost and G. K. Pederson [16]

4.2. **Proposition.** [7] Let M be an n-dimensional subspace of $C(X)$. Then M is a Čebyšev system if and only if no non zero function in M has more that $n - 1$ zeros.

4.3. **Theorem.** [7] Let X denote an interval $[a, b]$ or the unit circle T. Then each Čebyšev system $S = \{g_0, g_1, \ldots, g_m\}$ $m \geq 2$ is a Korovkin set (hyperrigid)

It is to be remarked that finite Korovkin sets have been studied for function spaces, commutative Banach algebras and for some special types of C^* algebras [1]. We pose the following problem whose answer is not yet known.

4.4. **Question.** Let M be an n dimensional subspace of C^* algebras $\mathscr{A}$, where $n \geq 3$. Is it true that M is hyperrigid if it is a Čebyšev subspace?

We conclude this section by mentioning few things regarding weak Korovkin type theorems.

When approximation in the weak sense by completely positive linear maps on $B(H)$ is considered, Korovkin type results have been obtained in [13]. For example, recall the definition of weak Korovkin set introduced in [13]. It is as follows:

4.5. **Definition.** A subset S of $B(H)$ is called a weak Korovkin set if for each net Φ_α of completely positive maps satisfying $\Phi_\alpha(I) \leq I$, the relation $\Phi_\alpha(s) \to s$ weakly, $s \in S$ implies $\Phi_\alpha(T) \to T$ weakly $T \in B(H)$.

One of the main theorems proved in [13] is as follows.

4.6. **Theorem.** Let S be an irreducible set $B(H)$ such that S contains the identity operator I and $C^*(S)$ contains a non zero compact operator. Then S is a weak Korovkin set in $B(H)$ if and only if $\mathrm{id}|_S$ has a unique completely positive linear extension to $C^*(S)$ namely $\mathrm{id}|_{C^*(S)}$.

4.7. **Remarks.** The condition '$\mathrm{id}|_S$ has a unique completely positive linear extension to $C^*(S)$' means that the identity representation of $C^*(S)$ is boundary representation for S in the sense of Arveson.

The following boundary theorem of Arveson enables to identify a number of weak Korovkin sets.

4.8. **Boundary theorem of Arveson.** Let S be a irreducible set in $B(H)$ such that S contains the identity operator and $C^*(S)$ contains a non zero compact operator. Then the identity representation of $C^*(S)$ is a boundary representation for S, if and only if the quotient map $q : B(H) \to B(H)/K(H)$ is not completely isometric on span$(S + S^*)$ where $K(H)$ denote the set of all compact operators on H.

One of the implications provides the following example.

4.9. **Example.** Let S be an irreducible operator which is almost normal but not normal, then the set $S = \{I, s, s^*s + ss^*\}$ is a weak Korovkin set.

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