数理解析研究所講究録
第1741巻 2011年 34-38

A generalization of Shelah's omitting types theorem

竹内 耕太 (Kota Takeuchi)
筑波大学数理物質科学研究科
(Graduate School of Pure and Applied Sciences,
University of Tsukuba)

Abstract

This note gives a generalization of Shelah's omitting types theorem.

Lを可算言語、TをL-理論とする。Shelahのタイプ排除定理は、連続濃度未満の完全タイプの集合Rについて、Rの全てのタイプを排除するモデルが存在することを保証する。一方、Newelskiの研究により、濃度ω_1のタイプの集合Rで、Rの元をすべて排除するようなモデルが存在しないような理論の存在が、ZFCと矛盾しないことも知られている[1]。よって、Shelahのタイプ排除定理における、タイプが完全であるという仮定をどの程度弱められるかという問いは、興味深い問題と言える。本研究では、次の形に拡張されたShelahのタイプ排除定理の証明を行う。

Theorem *Let T be a theory formulated in a countable language L and L_0 a sublanguage of L. Let R be a set of nonisolated complete L_0-types such that $|R| < 2^\omega$. Let S be a countable set of nonisolated L-types. Then there is a model $M \models T$ omitting all the members of $R \cup S$.*

Proof:

Throughout, L is a countable language and T is a countable first-order theory formulated in L. (T may be incomplete.) We always work under T. L-formulas are denoted by φ, $\psi, \theta, \chi, \ldots$. We fix a sublanguage $L_0 \subset L$. L_0-formulas are denoted by $\xi, \ldots$. Types are (possibly inocomplete) L-types over the empty set. We say a type $p(\bar{x})$ is a complete L_0-type if p consists of only L_0-formulas, and if for every $\xi(\bar{x}) \in L_0$, ξ or $\neg\xi$ is in p.

Definition 1 Let $L_0 \subset L$ and $\varphi_i(\bar{x}) \in L$ satisfiable.

1. We say that two L-formulas $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ are L_0-separable in $\bar{x}' \subset \bar{x}$ if there are L_0-formulas $\xi_0(\bar{x}')$ and $\xi_1(\bar{x}')$ such that $T \models \varphi_k(\bar{x}) \to \xi_k(\bar{x}')$ $(k = 0, 1)$, and ξ_0 and ξ_1 are incompatible in T.

2. We say $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ are essentially L_0-separable in $\bar{x}'$ if there are satisfiable L-formulas $\varphi'_k(\bar{x})$ $(k = 0, 1)$ with $T \models \varphi'_k(\bar{x}) \to \varphi_k(\bar{x})$ $(k = 0, 1)$ such that φ'_0 and φ'_1 are L_0-separable in $\bar{x}'$.

3. Let $\Phi = \varphi_0(\bar{x}), ..., \varphi_n(\bar{x})$ be a sequnece of satisfiable L-formulas. We say that Φ is maximally L_0-separated if for each $i \neq j$ and each subsequence $\bar{x}' \subset \bar{x}$, whenever $\varphi'_i(\bar{x})$ and $\varphi'_j(\bar{x})$ are essentially L_0-separable in $\bar{x}'$ then they are L_0-separable in $\bar{x}'$.

   A maximally L_0-separated sequence $\Phi' = \varphi'_0(\bar{x}), ..., \varphi'_n(\bar{x})$ will be called a maximal L_0-separation of Φ if $T \models \varphi'_i(\bar{x}) \to \varphi_i(\bar{x})$ $(i = 0, ..., n)$.

Lemma 2 *Let $\Phi = \varphi_0(\bar{x}), ..., \varphi_n(\bar{x})$ be satisfiable L-formulas. Then there are satisfiable L-formulas $\varphi'_i(\bar{x})$ $(i \leq n)$ such that $\Phi' = \varphi'_0(\bar{x}), ..., \varphi'_n(\bar{x})$ is a maximal L_0-separation of Φ.*

Proof: Let $\bar{y} \subset \bar{x}$ and suppose that $\varphi_i(\bar{y})$ and $\varphi_j(\bar{y})$ are essentially L_0-separable in $\bar{y}$. Choose L-formulas $\varphi'_i(\bar{x})$ and L-formulas $\varphi'_j(\bar{x})$ witnessing the essential L_0-seprability. Then we replace $\varphi_i(\bar{x})$ and $\varphi_j(\bar{x})$ by $\varphi'_i(\bar{x})$ and $\varphi'_j(\bar{x})$, respectively. We repeat this process (finitely many times) and finally we get a desired maximal L_0-separation.

Definition 3 Let $\psi(x_1, ..., x_n)$ be an L-formula and $s(\bar{y})$ an L-type. We say $\psi(x_1, ..., x_n)$ totally omits $s(\bar{y})$ if whenever $M \models T$ and $a_1, ..., a_n \in M$ satisfes $\psi(\bar{a})$ then no tuple from $\{a_1, ..., a_n\}$ realizes $s(\bar{y})$. Let Σ be a finite set of formulas. We simply say that Σ totally omits s if $\bigwedge \Sigma$ totally omits s.

Remark 4

- Let $s(\bar{x})$ be a nonisolated type. Then for every satisfiable L-formula $\varphi(\bar{x})$ there is a satisfiable L-formula $\varphi'(\bar{x})$ with $T \models \varphi'(\bar{x}) \to \varphi(\bar{x})$ such that φ' and s are inconsistent.

- It is easy to check that for every satisfiable L-formula $\varphi(\bar{x})$ and nonisolated type $s(\bar{y})$, there is a satisfiable L-formula $\psi(\bar{x})$ with $T \models \psi \to \phi$ such that ψ totally omits s.

Next lemma is easy but important for our proof of the theorem.

Lemma 5 *Let $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ be satisfiable L-formulas such that they are not essentially L_0-separable in $\bar{x}' \subset \bar{x}$. Then φ_0 and φ_1 isolate the same complete L_0-type $p(\bar{x}')$.*

Proof: Suppose otherwise. Then it is easy to find an L_0-formula $\chi(\bar{x}')$ such that both $\varphi_0 \wedge \chi$ and $\varphi_1 \wedge \neg\chi$ are satisfiable. Two L-formulas $\varphi_0 \wedge \chi$ and $\varphi_1 \wedge \neg\chi$ are L_0-separable in $\bar{x}'$. Since $T \models \varphi_0 \wedge \chi \to \varphi_0$ and $T \models \varphi_1 \wedge \neg\chi \to \varphi_1$, this means that φ_0 and φ_1 are essentiall L_0-separable. A contradiction.

Suppose $Z = \{z_i | i < \omega\}$ is a fixed countable set of new variables. We denote a sequence $z_0, z_1, \cdots, z_{i-1}$ by $\bar{z}_i$. Enumerate S as $S = \{s_i(\bar{x}_i) : i \in \omega\}$. We may assume that for each $s_n(\bar{x}_n)$, $|\bar{x}_n| \leq n$. Let $\{\theta_i(\bar{z}_i, z_i)\}$ be an enumeration of L-formulas having the form $\exists x \varphi(\bar{z}_i, x) \to \varphi(\bar{z}_i, z_i)$.

By induction, we construct a binary tree $\{\Sigma_\eta(\bar{z}_{len(\eta)}) | \eta \in 2^{<\omega}\}$ of finite sets of L-formulas with the following properties: For every $n \in \omega$ and every $\eta \in 2^n$,

1. If $m < n$ then $\Sigma_{\eta|m} \subset \Sigma_{\eta|n}$;
2. $\{\bigwedge \Sigma(\bar{z}_n)\}_{\sigma \in 2^n}$ is maximally separated;
3. Σ_η is consistent;
4. Σ_η contains θ_n;
5. Σ_η totally omits each of s_i $(i \leq n)$.

Let $\Sigma_{\langle\rangle} = \emptyset$ and suppose $\Sigma_\sigma(\bar{z}_n)$ is defined for every $\sigma \in 2^n$. Take two copies of $\Sigma_\sigma(\bar{z}_n)$ and set

$$\Sigma_\sigma^{0,k}(\bar{z}_n) = \Sigma_\sigma(\bar{z}_n) \ (k = 0, 1).$$

Then, by Lemma 2, there is a set $\{\psi_{\sigma,k}(\bar{z}_n)\}_{\sigma \in 2^n, k=0,1}$ which is a maximal L_0-separation of $\{\bigwedge \Sigma_\sigma^{0,k}(\bar{z}_n)\}_{\sigma \in 2^n, k=0,1}$. Set

$$\Sigma_\sigma^{1,k}(\bar{z}_n) = \Sigma_\sigma^{0,k}(\bar{z}_n) \cup \{\psi_{\sigma,k}(\bar{z}_n)\}.$$

Next, for each $\sigma \in 2^n$, take a satisfiable L-formula $\chi_{\sigma,k}(\bar{z}_n) \models \Sigma_\sigma^{1,k}(\bar{z}_n)$ such that $\chi_{\sigma,k}$ totally omits $s_i(\bar{x}_i)$ for every $i \leq n$. (Such formula exists by Remark 4.) Set

$$\Sigma_\sigma^{2,k}(\bar{z}_n) = \Sigma_\sigma^{1,k}(\bar{z}_n) \cup \{\chi_{\sigma,k}(\bar{z}_n)\}.$$

Finally set $\Sigma_{\sigma\hat{}k} = \Sigma_{\sigma}^{2,k}(\bar{z}_n) \cup \{\theta_n(\bar{z}_n, z_n)\}$. It is easy to check that $\{\Sigma_\eta(\bar{z}_{n+1})\}_{\eta\in 2^{n+1}}$ satisfies the required conditions 1-5 (with n replaced by $n+1$). So we have succeeded to construct all Σ_η's. Now, for a path $\eta \in 2^\omega$, we define $\Sigma_\eta(Z)$ by $\Sigma_\eta = \bigcup_{n\in\omega} \Sigma_{\eta|n}$. Recall that θ_n has the form $\exists x\varphi(\bar{z}_n, x) \to \varphi(\bar{z}_n, z_n)$. So, by the condition 4, every M_η realizing $\Sigma_\eta(Z)$ is a model of T. By the condition 5, M_η omits all types in S.

Claim A *For each $p \in R$, $\{\eta \in 2^\omega \mid M_\eta \models \exists\bar{x}p(\bar{x})\}$ is countable.*

We fix $p(\bar{x}) \in R$ and $\bar{z} \subset Z$ with $|\bar{x}| = |\bar{z}|$. Suppose $\Sigma_\eta(Z)\cup p(\bar{z})$ is consistent. Take any $\eta' \neq \eta$. If $\Sigma_{\eta'}(Z) \cup p(\bar{z})$ is also consistent, then $\Sigma_{\eta|n}$ and $\Sigma_{\eta'|n}$ are not essentially L_0-separable in $\bar{z}$, where n is chosen so that $\bar{z} \subset \bar{z}_n$. Hence p must be isolated by a L-formula, by Lemma 5. But R is a set of nonisolated types, a contradiction. So, for each $p \in R$ and $\bar{z} \subset Z$, $\{\eta \in 2^\omega | \Sigma_\eta(Z) \cup p(z)$ consistant$\}$ has at most one element. This proves the claim, since there are only countably many possible choices of $\bar{z} \subset Z$. (End of Proof of Claim)

Finally, by the claim above and the assumption that $|R| < 2^\omega$, we can find a path $\eta \in 2^\omega$ such that M_η omits R.

Corollary 6 Suppose $\alpha < 2^\omega$. Let T_0 be a complete L-theory and $p, q_i \in S(T)$ $(i < \alpha)$. If for every $i < \alpha$ there is a model M such that M omits q_i and M realizes p, then there is a model N such that N omits all q_i's but N realizes p.

Definition 7 Let M be an L-structure. We say that M is finitely generated if there is a tuple $\bar{a} \in M$ such that $M = \mathrm{acl}_M(\bar{a})$.

In [4], Tsuboi generalized Steinhorn's omitting types theorem. He showed the next result in his paper,

Theorem 8 Let M be an L-structure. Suppose M is not finitely generated. Let $p(\bar{x})$ be a type that is not realized in M. Then $p(\bar{x})$ is not isolated in the theory $\mathrm{Th}_{L(M)}(M) \cup \{y \neg a | a \in M\}$. So, there is a proper elementary extension N of M, which omits p.

We also have a generalization of Tsuboi's result, by our generalization of Shelah's omitting types theorem.

Corollary 9 Let M be an L-structure and $\kappa < 2^\omega$. Suppose M is not finitely generated. Let $p_\eta(\bar{x}_\eta) \in S(T)$ be a complete type that is not realized in M, for each $\eta < \kappa$. Then, there is a proper elementary extension N of M, which omits p_η for all $\eta < \kappa$.

References

[1] Newelski, L., "Omitting types and the real line," *The Journal of Symbolic Logic,* vol.52(1987), pp.1020-1026.

[2] Shelah, S., *Classification Theory and the Number of Nonisomorphic Models,* North-Holland, 1991. second edition.

[3] Tsuboi, A., "Models omitting Given Complete Types," *Notre Dame Journal of Formal Logic,* vol.49(2008), number 4, pp. 393-399.

[4] Tsuboi, A.,"A note on Steinhorn's omitting types theorem," *Colloquium of Mathematicum* vol.115 (2009), pp.189-194