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Low theories and the number of independent partitions

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1 Introduction

In this paper, we simply say that T is a theory if it is a complete first order theory formulated in a countable language. There are a number of important notions which classify theories. Simplicity, introduced by Shelah in [4], is one of such notions. A simple theory is characterized as a theory in which the length of a dividing sequence of types is bounded ($< \infty$). The notion of lowness was defined by Buechler in [1]. A low theory is characterized by the following property: For each formula $\varphi(x, y)$ there is a number $n_{\varphi} \in \omega$ such that whenever $\{\varphi(x, a_i) : i < m\}$ satisfies (1) $\{\varphi(x, a_i) : i < m\}$ is consistent, and (2) $\varphi(x, a_i)$ divides over $A_i = \{a_j : j < i\}$ $(i < m)$, then $m \leq n_{\varphi}$. It is easy to see that a low theory is a simple theory. However, a simple theory need not to be low.

In [2], Casanovas constructed a simple nonlow theory. His theory T_1 is the theory of the structure $M = (M, P, P_1, P_2, ..., Q, R)$, where

1. M is the disjoint union of P and Q;

2. P_n's are dijoint copies of ω;

3. P is the disjoint union of $\bigcup_{i \in \omega} P_i$ and ω;

4. Q is the set of all sequences $(A_1, A_2, ..., A_{\omega})$, where A_n is an n-elment subset of P_n, and $A_{\omega} \in G$, where G is a fixed class of subsets of ω such that (i) whenever $X_1, ..., X_k, Y_1, ..., Y_l \in G$ are distinct then $\bigcap X_i \cap \bigcap Y_i^c \neq \emptyset$, and (ii) for any distinct elements $m_1, ..., m_k, n_1, ..., n_k \in \omega$ there is $X \in G$ with $m_1, ..., m_k \in X$ and $n_1, ..., n_k \in X^c$.

5. $R \subset P \times Q$;

6. $R(a, (A_1, A_2, ..., A_\omega))$ if (i) $a \in P_n$ and $a \in A_n$ $(\exists n \in \omega)$ or (ii) $a \in P \setminus \bigcup_{n\in\omega} P_n$ and $a \in A_\omega$.

T_1 is not supersimple and furthermore $R(x, y)$ defines infinitely many mutually independent partitions in the following sense: If we enumerate P_n as $P_n = \{a_{nm} : m \in \omega\}$, then

- for each $\eta \in \omega^\omega$, $\{R(a_{n\eta(n)}, y) : n \in \omega \setminus \{0\}\}$ is consistent, and
- for each $n \in \omega \setminus \{0\}$, $\{R(a_{nm}, y) : m \in \omega\}$ is $(n+1)$-inconsistent.

By modifying this example, Casanovas and Kim [3], showed the existence of a supersimple nonlow theory T_2. This T_2 does not have infinitely many mutually independent partitions. However, there is a formula $\varphi(x, y)$ such that for each $k \in \omega$ we can find parameter sets $A_i = \{a_{ij} : j \in \omega\}$ $(i < k)$ defining k independent partitions.

For explaining the above situation more precisely, we will define a rank $D_{\text{inp}}(*, \varphi(\bar{x}, \bar{y}))$, which bounds the number of independent partitions. Namely, we let $D_{\text{inp}}(\Sigma(\bar{x}), \varphi(\bar{x}, \bar{y}))$ be the first cardinal κ such that there are no κ-many independent partitions $\Psi_i = \{\varphi(\bar{x}, \bar{a}_{ij}) : j \in \omega\}$ $(i < \kappa)$ of Σ. Then, for T_1, $D_{\text{inp}}(x = x, R(y, x))$ is ω_1. For T_2, we can show that $D_{\text{inp}}(\bar{x} = \bar{x}, \varphi(\bar{x}, \bar{y})) \leq \omega$ is for any φ, and that $D_{\text{inp}}(x = x, \varphi(x, y)) = \omega$ for some φ. So it is natural to ask whether there is a simple nonlow theory T such that $D_{\text{inp}}(\bar{x} = \bar{x}, \varphi(\bar{x}, \bar{y})) < \omega$ for any φ. We prove in this paper that there is no such theory.

2 On Simplicity and Lowness

We fix T and work in a large saturated model of T. From now on x, y, will denote finite tuples of variables. First we recall definitions of basic ranks.

Definition 1 Let $\Sigma(x)$ be a set of formulas and $\varphi(x, y)$ a formula. Let $k \in \omega$.

1. $D(\Sigma(x), \varphi(x, y), k) \geq 0$ if $\Sigma(x)$ is consistent. $D(\Sigma(x), \varphi(x, y), k) \geq n+1$ if there is an indiscernible sequence $\{b_i : i \in \omega\}$ over $\text{dom}(\Sigma)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi(x, y), k) \geq n$ for all $i \in \omega$, and $\{\varphi(x, b_i) : i \in \omega\}$ is k-inconsistent.

2. $D(\Sigma(x), \varphi(x,y)) \geq 0$ if $\Sigma(x)$ is consistent. For a limit ordinal δ, $D(\Sigma(x), \varphi(x,y)) \geq \delta$ if $D(\Sigma(x), \varphi(x,y)) \geq \alpha$ for all $\alpha < \delta$. $D(\Sigma(x), \varphi(x,y)) \geq \alpha+1$ if there is an indiscernible sequence $\{b_i : i \in \omega\}$ over $\mathrm{dom}(\Sigma)$ such that $D(\Sigma(x) \cup \{\varphi(x,b_i)\}, \varphi(x,y)) \geq \alpha$ $(i \in \omega)$, and $\{\varphi(x,b_i) : i \in \omega\}$ is inconsistent.

Fact 2 *1. $D(\Sigma(x), \varphi(x,y), k) \geq n$ if there is a tree $A = \{a_\nu : \nu \in \omega^{\leq n}\}$ such that (1) $\Sigma(x) \cup \{\varphi(x, a_{\eta|i}) : 1 \leq i \leq n\}$ is consistent ($\forall \eta \in \omega^n$), and (2) $\{\varphi(x, a_{\nu^\frown i}) : i \in \omega\}$ is k-inconsistent ($\forall \nu \in \omega^{<n}$).*

2. $D(\Sigma(x), \varphi(x,y)) \geq n$ if there is a tree $A = \{a_\nu : \nu \in \omega^{\leq n}\}$ and numbers $k_0, ..., k_{n-1}$ such that (1) $\Sigma(x) \cup \{\varphi(x, a_{\eta|i}) : 1 \leq i \leq n\}$ is consistent ($\forall \eta \in \omega^n$), and (2) $\{\varphi(x, a_{\nu^\frown i}) : i \in \omega\}$ is $k_{\mathrm{lh}(\nu)}$-inconsistent ($\forall \nu \in \omega^{<n}$).

From the fact above, we see the following:

1. T is simple if and only if $D(\Sigma(x), \varphi(x,y), k) \in \omega$ for any φ and k.
2. T is simple if and only if $D(\Sigma(x), \varphi(x,y)) < \infty$ for any φ.
3. T is low if and only if $D(\Sigma(x), \varphi(x,y)) \in \omega$ for any φ.

Now we define a rank assining a cardinal to each set of formulas.

Definition 3 $D_{\mathrm{inp}}(\Sigma(x), \varphi(x,y))$ is the minimum cardinal κ for which there is no matrix $A = \{a_{ij} : (i,j) \in \kappa \times \omega\}$ such that (1) $\Sigma(x) \cup \{\varphi(x, a_{i\eta(i)}) : i < \kappa\}$ is consistent ($\forall \eta \in \omega^\kappa$), and (2) for all $i < \kappa$, $\{\varphi(x, a_{ij}) : j \in \omega\}$ is k_i-inconsistent, for some $k_i \in \omega$.

Remark 4 Let $(M, P, P_1, ..., Q, R)$ be the structure explained in the introduction. For each n, let $\{a_{nm} : m \in \omega\}$ be an enumeration of P_n. Then we see the following

- for each $\eta \in \omega^\omega$, $\{R(a_{n\eta(n)}, y) : n \in \omega \setminus \{0\}\}$ is consistent, and
- for each $n \in \omega \setminus \{0\}$, $\{R(a_{nm}, y) : m \in \omega\}$ is $(n+1)$-inconsistent.

This imples that $D_{\text{inp}}(x = x, R(x,y)) \geq \omega_1$. Now we work in an elementary extension of M. Suppose, for a contradiction, that there is an $\omega_1 \times \omega$ matrix $A = \{a_{ij}\}_{i\in\omega_1, j\in\omega}$ witnessing $D_{\text{inp}}(x = x, R(x,y)) \geq \omega_2$. Then, by compactness, we can assume that for each i, $I_i = \{a_{ij} : j \in \omega\}$ is an indiscernible sequence. If $I_i \cap \bigcup_{n\in\omega} P_n = \emptyset$, then $\{R(x,b) : b \in I_i\}$ is a consistent set. So, for each $i < \omega_1$, we can choose $n_i \in \omega$ such that $I_i \subset P_{n_i}$. Now we can choose $n \in \omega$ and an infinite set subset $J \subset \omega_1$ such that $n_i = n$ for all $i \in J$. But, then $\{R(a_{i\eta(i)}, y) : i \in J\}$ is n-inconsistent, contradicting the choice of A.

Proposition 5 *Suppose that T is simple. Suppose also that $D_{\text{inp}}(x = x, \varphi(x,y))$ is finite. Then $D(x = x, \varphi(x,y)) < \omega$.*

Proof: Choose $k \in \omega$ with $D_{\text{inp}}(x = x, \varphi(x,y)) = k$. By way of contradiction, we assume that $D(x = x, \varphi(x,y)) \geq \omega$. Fix $m \in \omega$. By $D(x = x, \varphi(x,y)) \geq \omega$, there is a set $A = \{a_\nu : \nu \in \omega^{<m(k+1)}\}$ witnessing $D(x = x, \varphi(x,y)) \geq m(k+1)$. Then we have (1) $\{\varphi(x, a_{\eta|i}) : i < m(k+1)\}$ is consistent for any $\eta \in \omega^{<m(k+1)}$, and (2) $\{\varphi(x, a_{\nu\frown i}) : i \in \omega\}$ is $k_{\text{lh}(\nu)}$-inconsistent for any ν with $\text{lh}(\nu) + 1 < m(k+1)$. We can assume that A is an indiscernible tree. For $l < m$ and $\nu = \nu_0\frown n \in \omega^{l+1}$, we define

$$a^*_\nu = a_{\nu_0{}^*\frown n\frown 0}, a_{\nu_0{}^*\frown n\frown 0^2} \ldots, a_{\nu_0{}^*\frown n\frown 0^k},$$

where

$$\nu^*_0 = \nu_0(0), 0^k, \nu(1), 0^k, \ldots, \nu(l-1), 0^k.$$

We let $\varphi^*(x, y_1, \ldots, y_k)$ denote the formula $\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k)$. Notice that the definition of φ^* does not depend on m.

Claim A $\{\varphi^*(x, a^*_{\nu_0\frown m}) : m \in \omega\}$ *is k-contradictory.*

Suppose this is not the case. Then there is a k-element subset $F = \{i_1, \ldots, i_k\}$ of ω such that

$$\{\varphi^*(x, a^*_{\nu_0\frown i_1}), \ldots, \varphi^*(x, a^*_{\nu_0\frown i_k})\}$$

is consistent. In particular, by the definition of φ^*, we see that the following set is consistent.

$$\{\varphi(x, a_{\nu_0^*\frown i_1\frown 0}), \ldots, \varphi(x, a_{\nu_0^*\frown i_k\frown 0^k})\}$$

Then, by the indiscernibility of A, the following Γ_ν is also consistent, for each sequence ν of length k:

$$\Gamma_\nu = \{\varphi(x, a_{\nu_0^*\frown i_1\frown \nu(1)}), \varphi(x, a_{\nu_0^*\frown i_2\frown 0\frown \nu(2)}), \ldots, \varphi(x, a_{\nu_0^*\frown i_k\frown 0^{k-1}\frown \nu(k)})\}.$$

On the other hand, by our choice of the tree A, for each $l = 1, ..., k$, the set

$$\{\varphi(x, a^*{}_{\nu_0 ^\frown i_l ^\frown 0^{l-1} ^\frown i}) : i \in \omega\}$$

is inconsistent ($k_{\mathrm{lh}(\nu_0)+(1+l)}$-inconsistent). This yields $D_{\mathrm{inp}}(x = x, \varphi(x, z)) \geq k + 1$, a contradiction. (End of Proof of Claim)

By claim A, the set $\{\varphi^*(x, a^*{}_\nu) : \nu \in \omega^m\}$ witnesses $D(x = x, \varphi^*, k) \geq m$. Since m is arbitrary, we conclude $D(x = x, \varphi^*, k) = \infty$, contradicting the simplicity of T.

Corollary 6 *Suppose that T is simple. Suppose also that $D_{\mathrm{inp}}(x = x, \varphi(x, y))$ is finite for all φ. Then T is low.*

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