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Smoothness of hairs for some entire functions

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1 Preliminaries

Let f be an entire function and f^n denote the n-th iterate of f. Recall that the *Fatou set* $F(f)$ and the *Julia set* $J(f)$ of f are defined as follows:

$$\begin{aligned} F(f) &:= \{z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}, \\ J(f) &:= \mathbb{C} \setminus F(f). \end{aligned}$$

By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if f is a polynomial, while it is non-compact if f is transcendental. This is due to the fact that ∞ is an essential singularity of f.

The purpose of this paper is to construct so-called *hairs*, which is subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_\lambda(z) = \lambda e^z$ $(\lambda \in \mathbb{C} \setminus \{0\})$ in 1984 ([**DK**]). Here we briefly explain their results. Define

$$B_l := \{z \mid (2l-1)\pi < \operatorname{Im} z + \theta < (2l+1)\pi\}, \quad \theta = \arg \lambda \in [-\pi, \pi),\ l \in \mathbb{Z}$$

then we can define itinerary $S(z) := \boldsymbol{s} = (s_0, s_1, \cdots, s_n, \cdots) \in \mathbb{Z}^{\mathbb{N}}$ for a point $z \in \mathbb{C}$ by $E_\lambda^n(z) \in B_{s_n}$.

Theorem 1.1 (Devaney-Krych, 1984). *If $\boldsymbol{s} \in \mathbb{Z}^{\mathbb{N}}$ satisfies the following "growth condition":*

$$^\exists x_0 \in \mathbb{R},\ ^\forall n,\ (2|s_n|+1)\pi + |\theta| \leq g^n(x_0), \quad g(t) := |\lambda| e^t,$$

then there exists $h_{\boldsymbol{s}}(t) \subset J(E_\lambda)$ which satisfies the following:

(i) $E_\lambda(h_{\boldsymbol{s}}(t)) = h_{\sigma(\boldsymbol{s})}(g(t))$, where σ is the shift map on $\mathbb{Z}^{\mathbb{N}}$,

(ii) $E_\lambda^n(h_{\boldsymbol{s}}(t)) \to \infty$ $(n \to \infty)$ for every t.

The curve $h_{\boldsymbol{s}}(t)$ is called *a hair.* Viana showed that this hair $h_{\boldsymbol{s}}(t)$ is a C^∞ curve ([**V**]).

In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. For simplicity, we state the result for the easiest case, that is, for a "fixed" itinerary $s = (s_0, s_0, s_0, \cdots)$. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) := P(z)e^{Q(z)}$ as an application of our general results. Finally in §6 we briefly explain how to construct hairs for general itineraries.

2 C^0 a priori estimates — existence of a hair $h(t)$ —

Our setting is as follows:

A: Let $U,\ V \subset \mathbb{C}$ be unbounded domains, $f : U \to V$ a holomorphic diffeomorphism and $g : [\tau_*, \infty) \to \mathbb{R}$ the reference mapping, i.e., an increasing C^∞ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^n(t) \to \infty$ $(n \to \infty)$.)

B: (Initial curves) : There exist continuous curves $h_0, h_1 : [\tau_*, \infty) \to \mathbb{C}$ and a continuous increasing function $R : [\tau_*, \infty) \to \mathbb{R}_+$ and a constant $0 < {}^\exists\kappa < 1$ which satisfy the following:

- $|h_1(t) - h_0(t)| \leq (1-\kappa)R(t)$ for $t \in [\tau_*, \infty)$; (1)
- If $|w - h_0(g(t))| \leq R(g(t))$ for some $t \in [\tau_*, \infty)$, then $w \in V$ and
$$|f'(z)|\frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \text{ where } z = (f|_U)^{-1}(w) \tag{2}$$

(This is equivalent to that $f : B_f(t) \to \overline{\mathbb{D}(h_0(g(t)), R(g(t)))}$ is a homeomorphism with $|f'(z)|\dfrac{R(t)}{R(g(t))} \geq \dfrac{1}{\kappa},\ z \in B_f(t)$, where $B_f(t) := \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\}$.)

Definition 2.1. Let $\rho : [\tau, \infty) \to \mathbb{R}_+$ be a continuous increasing function and define a norm $||\psi||_{\rho,\tau}$ for $\psi : [\tau, \infty) \to \mathbb{C}$ by
$$||\psi||_{\rho,\tau} := \sup_{t \geq \tau} |\psi(t)|\rho(t).$$
We call ρ *a weight function.* Then it is easy to see that the space $X_{\rho,\tau} := \{\psi \mid ||\psi||_{\rho,\tau} < \infty\}$ becomes a Banach space.

Note that if we put $\rho_*(t) := 1/R(t)$ then the condition **B** (1) can be read as
$$||h_1 - h_0||_{\rho_*,\tau_*} \leq 1 - \kappa.$$

Under the above setting, we can show the existence of a hair $h(t)$:

Lemma 2.2. *Under the assumptions* **A** *and* **B**, *there exist continuous functions* $h_n : [\tau_*, \infty) \to \mathbb{C}$ $(n = 2, 3, \dots)$ *such that for* $n = 0, 1, 2, \dots,$
$$||h_n - h_0||_{\rho_*,\tau_*} \leq 1 - \kappa^n; \tag{3}$$
$$f \circ h_{n+1}(t) = h_n \circ g(t) \quad \textit{for} \quad t \geq \tau_*; \tag{4}$$
$$||h_{n+1} - h_n||_{\rho_*,\tau_*} \leq (1-\kappa)\kappa^n. \tag{5}$$

Therefore there exists a continuous function $h(t) = \lim_{n\to\infty} h_n(t)$ *satisfying*

$$f \circ h(t) = h \circ g(t) \quad \text{for} \quad t \geq \tau_* \quad \text{and} \quad |h(t) - h_0(t)| \leq R(t). \tag{6}$$

□

Of course, $f^n(h(t)) \to \infty$ $(n \to \infty)$ holds for $^\forall t \geq \tau_*$, since we have $f^n(h(t)) = h(g^n(t))$ and $g^n(t) \to \infty$ $(n \to \infty)$.

3 C^1 estimates

From (4), we have

$$\log h'_{n+1} = \log h'_n \circ g + \log g' - \log f' \circ h_{n+1}. \tag{7}$$

So define

$$\psi_n(t) := \log h'_n(t), \tag{8}$$

Then we have

$$\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n). \tag{9}$$

If $\psi_n - \psi_{n-1} \to 0$ as $t \to \infty$, by composing g, $(\psi_n - \psi_{n-1}) \circ g$ may go to 0 faster. This can be formulated in terms of $||\cdot||_{\rho_0,\tau_*}$ with an appropriate weight function $\rho_0 : [\tau_*, \infty) \to \mathbb{R}^+$ (which is assumed to be increasing). In fact, for a function $\psi : [\tau_*, \infty) \to \mathbb{C}$ (for our case, $\psi = \psi_n - \psi_{n-1}$), we have

$$\begin{aligned} ||\psi \circ g||_{\rho_0,\tau} &= \sup_{t\geq\tau} |\psi(g(t))|\rho_0(t) = \sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))|\rho_0(g(t)) \\ &\leq \left(\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))}\right) \cdot \left(\sup_{t'\geq g(\tau)} |\psi(t')|\rho_0(t')\right) = \left(\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))}\right) ||\psi||_{\rho_0,g(\tau)}. \end{aligned} \tag{10}$$

So if $\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$, then $||\cdot||_{\rho_0,\tau}$-norm is contracted by composing g. This implies the possibility to prove the geometric convergence of (9).

For further estimates (C^k, $k = 1, 2, \dots$), we need to prepare the following.

Definition 3.1. (1) Let ρ_k, $\sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$, $(k = 0, 1, 2, \dots)$ be *weight functions* with $\sigma_k(t) \leq \rho_k(t)$. These are to measure the norm $||\cdot||_{\rho_k,\tau}$ of $\psi_{n+1}^{(k)} - \psi_n^{(k)}$ and the norm $||\cdot||_{\sigma_k,\tau}$ of $\psi_n^{(k)}$.

(2) For given weight functions ρ_k, σ_k, define

$$\begin{aligned} \alpha_k(t) &:= \frac{\rho_k(t)|g'(t)|^k}{\rho_k(g(t))}, \\ \bar{\alpha}_k(\tau) &:= \sup_{t\geq\tau} \alpha_k(t), \\ D_k(t) &:= \sup_{z\in B_f(t)} \left|(\log f')^{(k)}(z)\right|, \quad k = 0, 1, 2, \dots, \quad t, \ \tau \geq \tau_*, \\ B_f(t) &:= \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\} \end{aligned}$$

Now in order to prove that $h(t)$ is C^1, we assume that there exist weight functions $\rho_0, \sigma_0 : [\tau_*.\infty) \to \mathbb{R}_+$ satisfying the following conditions $\mathbf{C}_0$, $\mathbf{D}_0$ and $\mathbf{F}_0$:

$\mathbf{C}_0$: h_0, h_1 are C^1 with $h_0'(t), h_1'(t) \neq 0$ and $\psi_0(t) = \log h_0'(t)$, $\psi_1(t) = \log h_1'(t)$ satisfy

$$||\psi_1 - \psi_0||_{\rho_0,\tau_*} < \infty \quad \text{and} \quad ||\psi_0||_{\sigma_0,\tau_*} < \infty.$$

$\mathbf{D}_0$: $\lim_{\tau\to\infty} \bar{\alpha}_0(\tau) = \limsup_{t\to\infty} \dfrac{\rho_0(t)}{\rho_0(g(t))} < 1.$

$\mathbf{F}_0$: $K_0 := \sup_{t\geq\tau_*} D_1(t)R(t)\rho_0(t) < \infty.$

Lemma 3.2. *Suppose* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_0$, $\mathbf{D}_0$ *and* $\mathbf{F}_0$ *are satisfied. Then* h_n *are* C^1 $(n = 2, 3, \dots)$ *and there exists* $\kappa_0 < 1$ *and* C_0 *such that* $\psi_n(t) = \log h_n'(t)$ *satisfy*

$$||\psi_{n+1} - \psi_n||_{\rho_0,\tau_*} \leq C_0\kappa_0^n \quad (n = 0, 1, 2, \dots). \tag{11}$$

Therefore the limit $h(t)$ *is also* C^1 *and* $\psi(t) = \log h'(t)$ *satisfies*

$$||\psi - \psi_0||_{\rho_0,\tau_*} \leq \frac{C_0}{1-\kappa_0} \quad \text{and} \quad ||\psi||_{\sigma_0,\tau_*} \leq \frac{C_0}{1-\kappa_0} + ||\psi_0||_{\sigma_0,\tau_*} < \infty.$$

□

4 Higher order derivatives — estimate for $\psi_n^{(k)} (k = 1, 2, \dots)$ —

Differentiating (7) and using $h_{n+1}' = e^{\psi_{n+1}}$, we have

$$\psi_{n+1}' = (\psi_n' \circ g) \cdot g' + (\log g')' - \left((\log f')' \circ h_{n+1}\right) e^{\psi_{n+1}}, \tag{12}$$

$$\begin{aligned}\psi_{n+1}'' &= (\psi_n'' \circ g) \cdot (g')^2 + (\psi_n' \circ g) \cdot g'' + (\log g')'' \\ &\quad - \left((\log f')'' \circ h_{n+1}\right) e^{2\psi_{n+1}} - \left((\log f')' \circ h_{n+1}\right) e^{\psi_{n+1}} \psi_{n+1}'.\end{aligned} \tag{13}$$

More generally, the following holds:

Lemma 4.1. *For* $k = 1, 2, \dots,$ *we have*

$$\begin{aligned}\psi_{n+1}^{(k)} &= \left(\psi_n^{(k)} \circ g\right) (g')^k + \sum_{\substack{1\leq\ell<k \\ j_1\geq\cdots\geq j_\ell\geq 1 \\ j_1+\cdots+j_\ell=k}} \mathit{const}\, \left(\psi_n^{(\ell)} \circ g\right) g^{(j_1)} \dots g^{(j_\ell)} + (\log g')^{(k)} \\ &\quad - \sum_{\substack{1\leq\ell\leq k,\, 0\leq\nu \\ j_1\geq\cdots\geq j_\nu\geq 1 \\ \ell+j_1+\cdots+j_\nu=k}} \mathit{const}\, \left((\log f')^{(\ell)} \circ h_{n+1}\right) e^{\ell\psi_{n+1}} \psi_{n+1}^{(j_1)} \dots \psi_{n+1}^{(j_\nu)},\end{aligned} \tag{14}$$

where the coefficients "const" are some constants depending the indices $\ell, j_1, j_2, \dots$. □

Note that in the right hand side of (14), only the first term contains k-th derivative of ψ_n and all other terms involve lower order derivatives of ψ_n (or none). Therefore if lower order derivatives are "under control," it is expected that we can proceed as in the previous section.

For the exponential map $f(z) = \lambda e^z$ and $g(t) = |\lambda|e^t$, we have $(\log f')' \equiv 1$ and $(\log f')^{(\ell)} \equiv 0$ ($\ell > 1$). So the formula (14) simplifies substantially. Moreover $g^{(j_1)} \dots g^{(j_\ell)}$ is a constant multiple of $g(t)^\ell$ which also simplifies the expression.

Suppose weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$ are given. We require the following conditions:

$\mathbf{C}_k$: h_0, h_1 are C^{k+1} and $\psi_0 = \log h_0'$ and $\psi_1 = \log h_1'$ satisfy

$$||\psi_1^{(k)} - \psi_0^{(k)}||_{\rho_k,\tau_*} < \infty \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k,\tau_*} < \infty.$$

$\mathbf{D}_k$: $\lim_{\tau\to\infty} \bar{\alpha}_k(\tau) < 1$.

$\mathbf{E}_k$: For $1 \le \ell < k$ and $j_1, \dots, j_\ell \ge 1$ with $j_1 + \dots + j_\ell = k$,

$$\sup_{t\ge\tau_*} \frac{\rho_k(t)|g^{(j_1)}(t)\cdots g^{(j_\ell)}(t)|}{\rho_\ell(g(t))} < \infty.$$

$\mathbf{F}_k$: For $1 \le \ell \le k$, $\nu \ge 0$, $j_1, \dots, j_\nu \ge 1$ with $\ell + j_1 + \dots + j_\nu = k$,

$$\sup_{t\ge\tau_*} D_{\ell+1}(t)R(t)\frac{\rho_k(t)}{\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)} < \infty;$$

$$\sup_{t\ge\tau_*} D_\ell(t)\frac{\rho_k(t)}{\rho_0(t)\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)} < \infty;$$

$$\text{if } \nu \ge 1, \text{ then for } 1 \le i \le \nu, \quad \sup_{t\ge\tau_*} D_\ell(t)\frac{\rho_k(t)}{\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)}\frac{\sigma_{j_i}(t)}{\rho_{j_i}(t)} < \infty.$$

Here if $\nu = 0$, set $\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t) = 1$. Note that the last condition should be satisfied only when $\nu \ge 1$.

Under these assumptions, we can show the following:

Lemma 4.2. *Let $k \ge 1$. Suppose* **A**, **B**, $\mathbf{C}_j$ *($0 \le j \le k$),* $\mathbf{D}_j$ *($0 \le j \le k$),* $\mathbf{E}_j$ *($1 \le j \le k$) and* $\mathbf{F}_j$ *($0 \le j \le k$) are satisfied. Then h_n are C^{k+1} ($n = 2, 3, \dots$) and there exist constants $0 < \kappa_k < 1$ and C_k such that*

$$||\psi_{n+1}^{(k)} - \psi_n^{(k)}||_{\rho_k,\tau_*} \le C_k\kappa_k^n \quad (n = 0, 1, 2, \dots). \tag{15}$$

Therefore the limit $h(t)$ is also C^{k+1} and $\psi = \log h'$ satisfies

$$||\psi^{(k)} - \psi_0^{(k)}||_{\rho_k,\tau_*} \le \frac{C_k}{1-\kappa_k} \quad \textit{and} \quad ||\psi_n^{(k)}||_{\sigma_k,\tau_*},\ ||\psi^{(k)}||_{\sigma_k,\tau_*} \le C_k'.$$

□

5 Examples

As an application of our results, we consider the following function:

$$f(z) = P(z)e^{Q(z)}, \quad P(z) = b_m z^m + \dots + b_0, \quad Q(z) = a_d z^d + \dots + a_1 z + a_0$$
$$m = \deg P \ge 0,\ d = \deg Q \ge 1,\ (a_d \ne 0,\ b_m \ne 0).$$

By a linear change of coordinate and multiplying P by e^{a_0}, we may assume that $a_d = 1$ and $a_0 = 0$. Let $g(t) = t^m e^{t^d}$ be the "reference function" to compare.

Lemma 5.1. *For any $\varepsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \varepsilon$, $P(t(1+w))e^{Q(t(1+w))} = t^m e^{t^d}$ and $|tw| \leq C$, where C is a constant.* □

By using this function $w(t)$, we define $h_0(t)$ and start constructing $h_n(t)$.

Proposition 5.2. *There exist $\tau_* > 0$ and C^∞-function $h_0 : [\tau*, \infty) \to \mathbb{C}$ such that $h_0'(t) \neq 0$ and*

$$f \circ h_0(t) = g(t) \ (= t^m e^{t^d}) \tag{16}$$

$$h_0(t) := t(1 + w(t)) = t + O(1) \quad (as\ t \to \infty) \tag{17}$$

$$(\log h_0'(t))^{(k)} = O\left(\frac{1}{t^{k+2}}\right) \quad (k = 0, 1, 2, \dots). \tag{18}$$

Moreover h_0, $h_1 := f^{-1}(h_0 \circ g)$ satisfies **A** *and* **B** *with $R(t) = \dfrac{const}{t^{d-1}g(t)}$.* □

Proposition 5.3. *Let $\sigma_k(t) = t^{k+2}$ $(k = 0, 1, 2, \dots)$. Suppose that $\rho_k(t)$ $(k = 0, 1, 2, \dots)$ satisfy*

$$\sigma_k(t) \leq \rho_k(t) \tag{19}$$

$$\limsup_{t \to \infty} \frac{\rho_k(t) t^{k(d-1)} (g(t))^k}{\rho_k(g(t))} < 1 \tag{20}$$

$$\rho_k(t) \leq const \, \frac{\rho_\ell(g(t))}{t^{k(d-1)}(g(t))^\ell} \quad (1 \leq \ell < k) \tag{21}$$

$$\rho_k(t) \leq const \cdot t^k g(t) \tag{22}$$

$$\rho_k(t) \leq const \, \frac{\rho_0(t)}{t^{d-k}} \quad (k \geq 1) \tag{23}$$

$$\rho_k(t) \leq const \, \frac{\rho_j(t)}{t^{d+j-1}} \quad (1 \leq j < k). \tag{24}$$

Then $\mathbf{C}_j$ *$(0 \leq j \leq k)$,* $\mathbf{D}_j$ *$(0 \leq j \leq k)$,* $\mathbf{E}_j$ *$(1 \leq j \leq k)$ and* $\mathbf{F}_j$ *$(0 \leq j \leq k)$ are satisfied.* □

Corollary 5.4. *For a suitable choice of const and $\mu_k > 0$, $\rho_k(t) = const \dfrac{e^{\varepsilon t}}{t^{\mu_k}}$ satisfies the hypothesis.* □

6 General cases

In this section we briefly explain how to construct hairs for general itineraries. We consider the following general setting:

Setting: Let U_l, $V_l \subset \mathbb{C}$ be unbounded domains and $f_l : U_l \to V_l$ be holomorphic diffeomorphisms $(l = 0, 1, 2, \cdots)$. Let $g : [\tau_*, \infty) \to \mathbb{R}$ be a reference mapping, i.e., an increasing C^∞ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^l(t) \to \infty$ $(l \to \infty)$). Set $\tau_l := g^l(\tau_*)$.

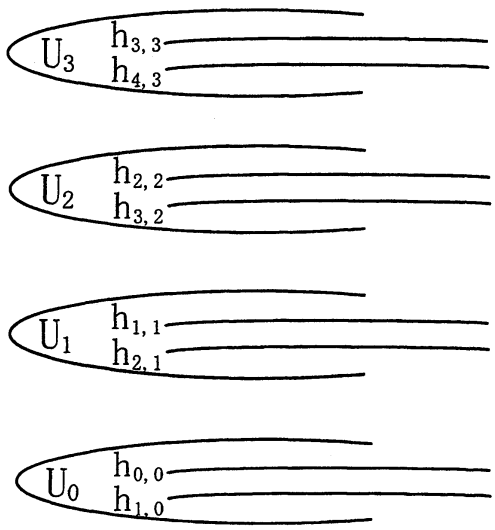
Our goal is to construct $h_l : [\tau_l, \infty) \to \mathbb{C}$, $(l = 0, 1, 2, \cdots)$ such that

$$f_l \circ h_l(t) = h_{l+1} \circ g(t).$$

Strategy: Construct initial curves $h_{l,l}$ $(l = 0, 1, 2, \cdots)$. Then define $h_{n,l} : [\tau_l, \infty) \to \mathbb{C}$, $(0 \le l < n)$ by lifting successively so that

$$f_l \circ h_{n,l}(t) = h_{n,l+1} \circ g(t).$$

See the figure and the diagram below:



$$\begin{array}{ccccccccc}
 & h_{0,0} & & & & & & & \\
 & h_{1,0} & h_{1,1} & & & & & & \\
 & h_{2,0} & h_{2,1} & h_{2,2} & & & & & \\
 & h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
 & h_{l,0} & h_{l,1} & h_{l,2} & h_{l,3} & \cdots & h_{l,l} & & \\
 & h_{l+1,0} & h_{l+1,1} & h_{l+1,2} & h_{l,3} & \cdots & h_{l+1,l} & h_{l+1,l+1} & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots \\
(l \to \infty) & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \ddots \\
 & h_0 & h_1 & h_2 & h_3 & \cdots & h_l & h_{l+1} & \cdots
\end{array}$$

Under the similar assumptions as in the previous sections, we can show the existence and smoothness of hairs $h_l(t)$ $(l = 0, 1, 2, \cdots)$. We omit the details. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $\boldsymbol{s} \in \{0, 1, \cdots, d-1\} \times \mathbb{Z}^{\mathbb{N}}$, where $d = \deg Q$. For the details, see [**Ki**]. So by taking f_l to be the restriction of f to a suitable domain according to $\boldsymbol{s}$, we can apply our results for general setting and obtain the smooth hair $h_{\boldsymbol{s}}(t)$ corresponding to $\boldsymbol{s}$.

References

[**DK**] R. Devaney, M. Krych, *Dynamics of Exp(z)*, Ergod. Th. & Dynam. Sys. **4** No.1 (1984), 35–52.

[**Ki**] M. Kisaka, *Dynamics of structurally finite transcendental entire functions.*, Studies on complex dynamics and related topics (Kyoto, 2000), Sūrikaisekikenkyū-sho Kōkyūroku, No.1220 (2001), 17–25.

[**V**] M. Viana da Silva, *The differentiability of the hairs of exp(Z)*, Procs. Amer. Math. Soc. **103** No. 4 (1988), 1179-1184.