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Asynchrony, Markov, Interference

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Abstract

Spread spectrum (SS) codes generated by a Markov chain reduce the variance of interference in code division multiple access (CDMA) system. The amount of interference depends on chip waveform, since in an asynchronous CDMA system, transmitted signals are delayed for continuous-valued random time, and discrete-time SS code signals are interpolated by a chip waveform. In this report, the variance of interference of bandlimited as well as timelimited signature waveform is analyzed. We give several remarks on design of such a signature waveform for CDMA systems.

1 Introduction

The purpose of communication is to transmit some amount of information through noisy channel. Communication was mathematically formulated by Shannon; the amount of information is measured by *entropy* of a random variable of transmit signal and the channel is characterized by a conditional probability of output signals given input signals. Thus the nature of communication is described by a stochastic process [1].

Spread spectrum (SS) communication is a system where the frequency bandwidth of a data signal is expanded by a pseudo-random sequence, which is independent of data signal. Such pseudorandom sequences are referred to as SS codes. SS codes are shared by the transmitter and the receiver and kept unknown to other persons. SS communications have many desirable properties: 1) anti-jamming, 2) security provided by a SS code and low power frequency spectrum, 3) capability to multi-path fading, and 4) capability to multiple-access.

This report enhances the result of Mazzini et. al [2, 3], which says that SS codes generated from a Markov chain are better than i.i.d. codes in asynchronous CDMA systems. In [2, 3, 4, 5], a rectangular chip waveform was assumed but it is not bandlimited, while communication channels are usually bandlimited. We analyze cross-interferences of bandlimited as well as time-limited signature waveforms and give several remarks on design of such a signature waveform in CDMA systems. It is shown that if we assume chip waveforms are bandlimited but there is no restriction on time-width of a waveform, a set of Welch bound equality (WBE) sequences filtered by an ideal low pass filter, which cannot be realizable, is an optimal set of signature waveforms.

2 Piecewise Linear Map generating Markov sequences

Let $g(\omega)$ be a function from an interval $I = [d, e]$ onto itself and consider a recursion $\omega_n = g(\omega_{n-1})$ for $n = 1, 2, 3, \ldots$, where ω_0 is an initial value or a seed of the real-valued sequence $\{\omega_n\}$. Let $\Theta(\omega)$ be a threshold function with $\Theta(\omega) = +1$ for $\omega \geq \theta$ and $\Theta(\omega) = -1$ for $\omega < \theta$ for a threshold $\theta \in I$. Kohda and Tsuneda proposed to employ such a sequence as a spread spectrum (SS) codes [6, 1], where Chebyshev map with degree $k \geq 2$ $T_k(\omega) = \cos(k \cos^{-1}(\omega))$ was selected as a one dimensional map. It was proved [7] that for a suitable choice of a threshold function, a binary sequence $\{\Theta(T_k^n(\omega_0))\}_n$ generated from Chebyshev map with degree k is independent and identically distributed (i.i.d.) random variables for almost all initial value ω_0.

An interesting fact was pointed out by Mazzini et al. [2] that SS codes generated from piecewise linear Markov map can reduce bit error rate (BER) of asynchronous CDMA systems. This result astonished researchers who believed i.i.d. sequences were optimum for SS codes. Kohda and Fujisaki [4] proved that the multiple-access interference (MAI) in asynchronous CDMA system was minimized by Markov codes. Definition of MAI will be explained in the next section.

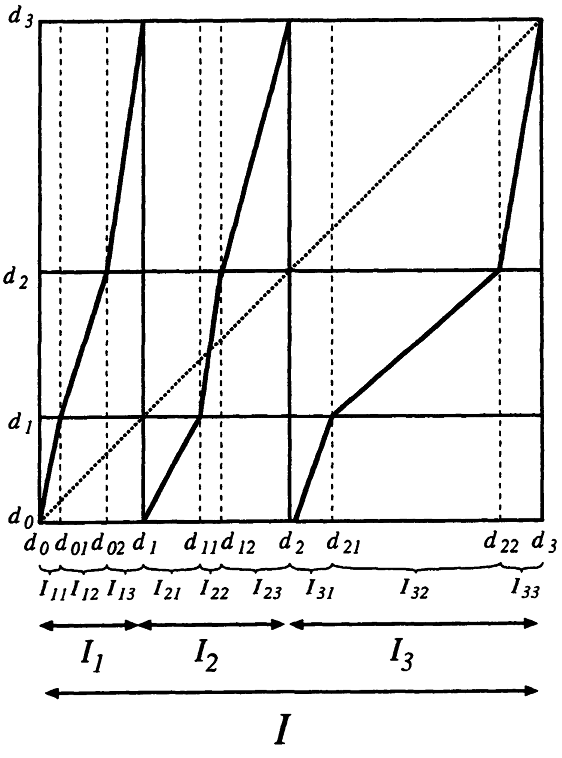

Figure 1: A piecewise linear map with nine linear intervals. A three state Markov chain can be embedded into this map.

Fig. 1 shows a piecewise linear map in which a Markov chain is embedded [5]. The interval $I = [d, e]$ is separated into M sub-intervals, denoted by $I_1, I_2, \ldots, I_M$, where

$I_k = [d_{k-1}, d_k]$ with $d_0 = d$ and $d_M = e$ ($M = 3$ in Fig.1). Each subinterval I_k is separated into M sub-subintervals $I_{k,1}, I_{k,2}, \cdots, I_{k,M}$, where $I_{k,j} = [d_{k,j-1}, d_{k,j}]$ with $d_{k,0} = d_k$ and $d_{k,M} = d_{k+1}$. Define a piecewise linear function as

$$g(\omega) = \frac{|I_k|}{|I_{j,k}|}(\omega - d_{j,k}) + d_k, \quad \text{if } \omega \in I_{j,k}. \tag{1}$$

Initial value ω_0 is uniformly chosen from the whole interval I. We consider ω_n is in the j-th state at a time instant n if $\omega_n \in I_j$ for $n = 0, 1, 2, \ldots$ and $j = 1, \ldots, M$. The conditional probability that ω is in I_k at time $n+1$ given that ω is in I_j at time n is equal to $|I_{jk}|/|I_j|$. This implies a Markov chain with a transition matrix

$$P = \begin{pmatrix} \frac{|I_{1,1}|}{|I_1|} & \cdots & \frac{|I_{1,M}|}{|I_1|} \\ \vdots & & \vdots \\ \frac{|I_{M,1}|}{|I_M|} & \cdots & \frac{|I_{M,M}|}{|I_M|} \end{pmatrix} \tag{2}$$

is embedded into $g(\omega)$. Conversely, for a given P, we can construct a one-dimensional piecewise linear map using the formula Eq. (1) if all elements of P is nonzero. The method of constructing $g(\omega)$ from P is referred to as Kalman's procedure [5].

A one dimensional PL map with $M = 2$ together with threshold function $\Theta(\omega) = -1$ for $d \leq \omega < d_1$ and $\Theta(\omega) = 1$ for $d_1 \leq \omega < e$ generates $\{+1, -1\}$-valued sequences.

3 Interference Reduction

It is desirable for a CDMA system if cross-interference between every pair of users is small. Interestingly, Markov codes have smaller interference than the i.i.d. codes. In this section, we review Kohda and Fujisaki's proof [4] of the above fact.

Consider an asynchronous CDMA with K users. Let $u_k(t)$ and τ_k, respectively, be a signature signal and a time delay of k-th user ($k = 0, 1, \ldots, K-1$). Assume that each user's signature waveform has a unit energy, i.e., $\int_{-\infty}^{\infty} |u_k(t)|^2 dt = 1$ for $k = 0, \ldots K-1$ and that user indices are sorted in a ascending order of time delays, i.e., $\tau_0 < \tau_1 < \cdots < \tau_{K-1}$. The MAI for k-th user and p-th data period is defined by

$$I_{k,p} = \sum_{j \neq k} \{d_p^{(j)} C_{j,k}(\tau_{j,k}) + d_{p+1}^{(j)} C_{j,k}(\tau_{j,k} - T)\}, \tag{3}$$

where $\tau_{j,k} = \tau_j - \tau_k$ and τ_j is a time delay of j-th user, T is a data duration, $d_p^{(j)}$ is a data for j-th user and p-th period, and $C_{j,k}(\tau)$ is the cross-correlation of j-th and k-th users signal with relative time delay τ, given by

$$C_{j,k}(\tau) = \int_{-\infty}^{\infty} u_j(t) u_k(t + \tau) dt. \tag{4}$$

Suppose $u_k(t)$ is selected from a set of waveforms

$$u_k(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s_{n,k} \Pi_{T_c}(t - nT_c), \tag{5}$$

where $\boldsymbol{s}_k = (s_{0,k}, \ldots, s_{N-1,k})^T$ is a spread spectrum (SS) code for the j th user, N is the spreading factor, T_c is a chip duration, and $\Pi_{T_c}(t)$ is a rectangular chip waveform

$$\Pi_{T_c}(t) = \begin{cases} \frac{1}{\sqrt{T_c}} & (0 \le t < T_c), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let S_0 be an i.i.d. random variable for $s_{n,k}$ with $\text{Prob}(S = \pm 1) = \frac{1}{2}$. The variance of MAI is one of the performance criteria in CDMA or SSMA communications, defined by

$$\mathbf{E}_{S_0}[I_{k,p}^2] - (\mathbf{E}_{S_0}[I_{k,p}])^2 \tag{7}$$

where $\mathbf{E}_{S_0}$ denotes the expectation with respect to S_0. Note that for i.i.d. codes,

$$\mathbf{E}_{S_0}[I_{k,p}] = 0 \tag{8}$$

since

$$\mathbf{E}_{S_0}[s_{n,j} s_{m,k}] = \delta_{j,k} \delta_{n,m}. \tag{9}$$

We have

Lemma 1 *For an integer* $0 \le \ell \le N-1$ *and a fraction* $0 \le \epsilon < 1$,

$$C_{j,k}((\ell + \epsilon)T_c) = (1 - \epsilon)C_{j,k}(\ell T_c) + \epsilon C_{j,k}((\ell + 1)T_c). \tag{10}$$

Lemma 2 *The variance of cross-correlation function with an integer time delay* $0 \le \ell \le N-1$ *is*

$$\mathbf{E}_{S_0}[C_{j,k}^2(\ell T_c)] = 1 - \frac{\ell}{N}, \quad j \ne k \tag{11}$$

$$\mathbf{E}_{S_0}[C_{j,k}(\ell T_c) C_{j,k}((\ell+1)T_c)] = 0, \quad j \ne k \tag{12}$$

These lemmas give

Lemma 3 *Suppose* $d_p^{(j)}$ *and* $d_{p+1}^{(j)}$ *are i.i.d. random variables with* $P(d_p^{(j)} = \pm 1) = \frac{1}{2}$, *then*

$$\mathbf{E}_D[\mathbf{E}_{S_0}[I_{k,p}^2]] = \sum_{j=0, j \ne k}^{K-1} ((1 - \epsilon_{j,k})^2 + \epsilon_{j,k}^2), \tag{13}$$

where $\epsilon_{j,k}$ *is a fraction part of* $\tau_{j,k} = \tau_j - \tau_k$.

Finally, assuming $\epsilon_{j,k}$ be uniformly distributed in $[0,1)$ gives $\mathbf{E}_\epsilon[\mathbf{E}_D[\mathbf{E}_{S_0}[I_{k,p}^2]]] = \frac{2}{3}(K-1)$. This is the variance of MAI for i.i.d. spreading codes averaged over balanced i.i.d. data sequence and uniformly distributed time delay.

Now let us discuss the variance reduction by negatively correlated codes. Consider that $\{s_{n,k}\}_{n=0}^{N-1}$ is generated from a Markov chain with a transition probability matrix

$$P = \begin{pmatrix} \frac{1+\lambda}{2} & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & \frac{1+\lambda}{2} \end{pmatrix}. \tag{14}$$

Then Eq. (9) is replaced by

$$\mathbf{E}_{S_\lambda}[s_{n,j}s_{m,k}] = \delta_{j,k}\lambda^{|n-m|}, \tag{15}$$

where $\mathbf{E}_{S_\lambda}$ stands for the expectation with respect to correlated codes, and $-1 < \lambda < 1$ is an eigenvalue of a transition matrix of a Markov chain except for one [1]. Note that Markov codes with $\lambda = 0$ implies i.i.d. codes. We give

Lemma 4 *For a large N, we have*

$$\mathbf{E}_{S_\lambda}[C_{j,k}^2(\ell T_c)] \approx \left(1 - \frac{\ell}{N}\right)\frac{1+\lambda^2}{1-\lambda^2}, \tag{16}$$

$$\mathbf{E}_{S_\lambda}[C_{j,k}(\ell T_c)C_{j,k}((\ell+1)T_c)] \approx \left(1 - \frac{\ell}{N}\right)\frac{2\lambda}{1-\lambda^2}. \tag{17}$$

This lemma gives

Theorem 1 (Kohda-Fujisaki[4])

$$\begin{aligned}\mathbf{E}_D[\mathbf{E}_{S_\lambda}[I_{k,p}^2]] = \sum_{j=1, j\neq k}^{K} \Big\{ &(1 - 2\epsilon_{j,k} + 2\epsilon_{j,k}^2)\frac{1+\lambda^2}{1-\lambda^2} \\ &+ 2\epsilon_{j,k}(1-\epsilon_{j,k})\frac{2\lambda}{1-\lambda^2}\Big\} \end{aligned}\tag{18}$$

Suppose that $\epsilon_{j,k}$ is uniformly distributed in $[0,1)$. Then the expectation of Eq.(18) is $\mathbf{E}_\epsilon[\mathbf{E}_D[\mathbf{E}_{S_\lambda}[I_{k,p}^2]]] = (K-1)\frac{2}{3}\frac{1+\lambda^2+\lambda^2}{1-\lambda^2}$, which is minimized by $\lambda^* = -2+\sqrt{3}$ [4].

The superiority of negatively correlated codes was first reported by Mazzini et. al [2, 3]. Kohda and Fujisaki [4] gave a proof why Markovian SS codes can reduce the variance of MAI[1].

What is the condition for Markov codes to be better than i.i.d. one? If $\epsilon_{j,k} = 0$, then Eq. (18) is $\frac{1+\lambda^2}{1-\lambda^2}$, which is minimized by $\lambda = 0$. This implies asynchrony is a necessary. Moreover, the shape of chip waveform is important. Optimal λ depends on the choice of chip waveforms, as we will show at the last part of this section.

[1]The paper [4] gave an important remark on the order of the expectation operation; it was shown that if $P(d_p^{(j)} = -1) = \mu_j$ and $P(d_p^{(j)} = 1) = 1-\mu_j$, then $\mathbf{E}_\epsilon[\mathbf{E}_{S_\lambda}[\mathbf{E}_D[I_{k,p}^2]]] \geq 4\mu_j(1-\mu_j)\mathbf{E}_\epsilon[\mathbf{E}_{S_\lambda}[I_{k,p}^2]]$.

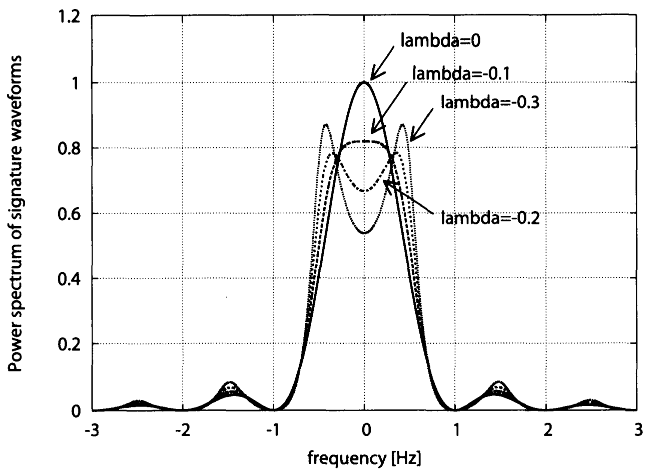

Figure 2: The expected power spectrum of $u_k(t)$ for i.i.d. codes ($\lambda = 0$) and Markov codes ($\lambda = -0.1, -0.2, -0.3$).

The expectation of the power spectrum of $u_k(t)$ is irrespective of k, given by [8]

$$\mathbf{E}_{S_\lambda}[|U_k(f)|^2] = \mathrm{sinc}(fT_c)^2 \frac{1-\lambda^2}{1+\lambda^2 - 2\lambda\cos(2\pi f T_c)}, \tag{19}$$

where $U_k(f) = \int_{-\infty}^{\infty} u_k(t) e^{-2\pi \mathbf{i} f t} dt$ is the Fourier transform of $u_k(t)$, where $\mathbf{i}$ is an imaginary unit $\sqrt{-1}$. Eq.(19) is illustrated in Fig. 2 for $\lambda = 0, -0.1, -0.2$ and -0.3. It is shown Markov codes with $\lambda = -0.2$ and -0.3 give two peaks in their power spectrum at approximately $f = \pm 0.4$, as well as increases the sidelobes. It can be understood that Markov codes with negative correlation flatten the power spectrum of a rectangular pulse.

It was assumed in [4] that the fraction part of the time delay ϵ_{ij} is distributed uniformly in $[0, 1)$. On the other hand, Pursley [9] assumed the time delay τ_{ij} is uniformly distributed in $[0, T)$. Under this assumption, we may evaluate the squared cross-correlation averaged over $\tau \in [0, T)$, which is calculated in frequency domain as

$$\begin{aligned} &\frac{1}{T}\int_0^T \left\{ |C_{j,k}(\tau)|^2 + |C_{j,k}(\tau - T)|^2 \right\} d\tau \\ &= \frac{1}{T}\int_{-\infty}^{\infty} |U_j(f)|^2 |U_k(f)|^2 df. \end{aligned} \tag{20}$$

The expectation of $|U_j(f)|^2$ and $|U_k(f)|^2$ are both given by (19). In Fig. 3, $\mathbf{E}_{S_\lambda}[|U_j(f)|^2|U_k(f)|$ is illustrated. The expectation of (20) can be expressed as the area under the curve of Fig. 3.

Remark: The assumption that $\tau_{jk} = (\ell_{jk} + \epsilon_{jk})T_c$ is uniformly distributed in $[0, T]$ allows us to use Parseval's identity of Eq.(20). However, this is stronger assumption

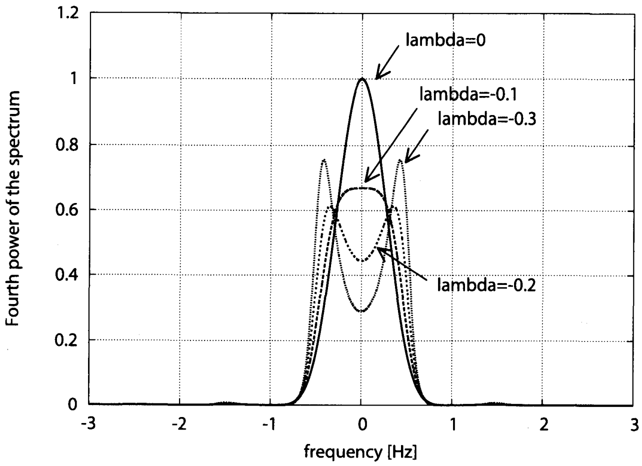

Figure 3: The variances of cross-interference of signature waveforms using i.i.d. codes ($\lambda = 0$) and Markov codes ($\lambda = -0.1, -0.2$ and $-0,3$) are, respectively, equal to the areas under the corresponding curves.

than the assumption that ϵ_{jk} is distributed in $[0, 1]$. Note that Eq.(18) shows that the variance of MAI is independent of the integer part of τ_{jk}/T_c.

Remark: Replacing i.i.d. random variables with Markov codes with eigenvalue $\lambda = -2 + \sqrt{3}$ reduces the variance of MAI from $\frac{2}{3}(K-1)$ to $\frac{1}{\sqrt{3}}(K-1)$. This happens if a rectangular waveform is used, which however is not bandlimited. A wireless channel is separated into many channels and is shared by several communication and broadcast services. Thus, signature waveforms must be bandlimited. Moreover, sampling theorem tells us that if a signal is not bandlimited, we cannot reconstruct the original signal from its samples. Hence a continuous time signal $u_j(t)$ defined by Eq. (5) contains more information than its samples $s_{n,j}$. This is one of the reasons why we can reduce the variance of MAI when time delays take continuous values randomly.

The variance of MAI for a CDMA system with a chip waveform other than the rectangular one is also reduced by Markov codes. Denote a general chip waveform by $v(t)$. Eq. (19) is replaced by

$$\mathbf{E}_{S_\lambda}[|U_j(f)|^2] = |V(f)|^2 \frac{1-\lambda^2}{1+\lambda^2 - 2\lambda\cos(2\pi f T_c)}, \tag{21}$$

where $V(f)$ is the Fourier transform of $v(t)$. Thus,

$$\begin{aligned} &\mathbf{E}_{S_\lambda}\left[\frac{1}{T}\int_{-\infty}^{\infty}\left\{|C_{j,k}(\tau)|^2 + |C_{j,k}(\tau - T)|^2\right\} d\tau\right] \\ &= \frac{1}{T}\int_{-\infty}^{\infty} |V(f)|^4 \left(\frac{1-\lambda^2}{1+\lambda^2 - 2\lambda\cos(2\pi f T_c)}\right)^2 df. \end{aligned} \tag{22}$$

Periodicity of $\frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos(2\pi f T_c)}$ implies that we can replace $|V(f)|^4$ in Eq.(22) by $\sum_p |V(f-\frac{p}{T_c})|^4$ and the infinite integral by a finite integral with an interval $1/T_c$, i.e,

$$\frac{1}{T}\sum_{p=-\infty}^{\infty}\int_{-1/2T_c}^{1/2T_c}\left|V\left(f-\frac{p}{T_c}\right)\right|^4\left(\frac{1+\lambda^2}{1-\lambda^2-2\lambda\cos(2\pi f T_c)}\right)^2 df.$$

We conclude that the variance of MAI is reduced by nonzero λ, unless $\sum_p |V(f-\frac{p}{T_c})|^4$ is not a constant function of f.

4 Welch bound

In the previous section, we calculate the expectation of the variance of interferences for i.i.d. random spreading. The variance is reduced if we replace i.i.d. codes by Markov codes. In this section, lower bound of the variance of interference is discussed. For symbol-synchronous systems, such sequences that minimizes the square value of interferences are called Welch bound equality (WBE) sequences. We will discuss the asynchronous version of WBE sequences. Interestingly, such asynchronous WBE sequences have negative autocorrelations.

The total sum of the square value of (3) has been discussed as a performance criterion of CDMA system [10, 11, 12, 13]. For asynchronous synchronous systems, total square asynchronous correlation (TSAC) is defined as

$$\begin{aligned}&\mathrm{TSAC}(S,\boldsymbol{\tau})\\&=\sum_{j=0}^{K-1}\sum_{k=0}^{K-1}\left\{C_{j,k}(\tau_{j,k})^2+C_{j,k}(\tau_{j,k}-T)^2\right\},\end{aligned}\tag{23}$$

where $\boldsymbol{\tau}=(\tau_0,\ldots,\tau_{K-1})$ is a delay profile, $S=[\boldsymbol{s}_0,\ldots,\boldsymbol{s}_{K-1}]$, and $C_{j,k}(\tau)$ is defined by Eq.(4) When the system is symbol-synchronous, i.e., $\tau_{j,k}=0$ for $j,k=0,1,\ldots,K-1$, TSAC is equal to a quantity called total square correlations (TSC) [10]. If $K\leq N$, we can select $s_{n,j}$ so that $\boldsymbol{s}_j=(s_{0,j},s_{1,j},\ldots,s_{N-1,j})^T$ $(j=0,1,\ldots,K-1)$ are orthonormal. Thus it is trivial that the minimum value of TSC is K if $K\leq N$. In this case there is no interference between users, i.e., $C_{j,k}(0)=0$ for $j\neq k$.

Giving the minimum of TSC for $K>N$ is non-trivial. In 1974, Welch [14] gave a lower bound on TSC. Welch proved[2] that TSC is lower bounded by K^2/N. The bound is actually attainable. Massey and Mittelholzer [10] showed an interesting property that the bound is attained if $(s_{n,0},s_{n,1},\ldots,s_{n,K-1})$ $(n=0,1,\ldots,N-1)$, i.e., row vectors of matrix $S=[\boldsymbol{s}_0,\boldsymbol{s}_1,\ldots,\boldsymbol{s}_{K-1}]$, are orthogonal.

[2]Welch considered a more general case of a bound on $(\sum_{n=0}^{N-1}s_{n,j}s_{n,k})^{2q}$ for $q\geq 1$, but square correlation ($q=1$) is of practical importance because it is a variance of interference and is related to bit error probability. A set of sequences which achieves the lower bound on TSC with $q=1$ is called Welch bound equality (WBE) sequences.

Analysis of interference in asynchronous CDMA systems is more difficult than synchronous one. In order to make the TSAC minimization problem tractable, it is assumed time delays are restricted to $\tau_j = \ell_j T_c$ with integer ℓ_j. Such a system with integer time delay is called chip-synchronous CDMA system. The minimum value of Eq.(23) for chip-synchronous system is given by Ulukus and Yates[11]. It was shown that lower bound of TSAC was the same as that of TSC, i.e.,

$$\min_{S,\|s_k\|^2=1} \mathrm{TSAC}(S,\boldsymbol{\tau}) = \begin{cases} K & \text{if } K \le N \\ K^2/N & \text{if } K > N \end{cases}$$
$$\text{for any } \tau_k = \ell_k T_c \tag{24}$$

TSAC minimization[3] for real-valued τ_j was discussed in in [15, 13], where user's signature waveform is defined as $\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s_n v(t-nT_c)$, where $v(t)$ is a chip waveform chosen from a list of known waveforms and $T_c = T/N$ is a chip duration. Chip-asynchronous case is important since minimum TSAC of chip-asynchronous system is smaller than that of chip-synchronous system, and therefore one can obtain a benefit from chip-asynchrony of CDMA system. It should be noted that minimum TSAC depends on the delay profile. For $N = 1$, TSAC is minimized when the delay profile is $\boldsymbol{\tau} = (0, \frac{1}{K}, \frac{2}{K}, \ldots, \frac{K-1}{K})$ [13]. For $N \ge 2$, let $\tau_k = (\ell_k + \epsilon_k) T_c$ where ℓ_k is integer and ϵ_k is a fraction. It is recommended to put $\epsilon_k = \frac{k}{K}$ for $k = 0, 1, \ldots, K-1$ [13].

The TSAC improvement ratio depends on the chip waveform, even if its energy is normalized. Hombs and Lehnert [13] defined a quantity called 'effective dimension' as

$$\left(\int_{-\infty}^{\infty} |V(f)|^4 df \right)^{-1}. \tag{25}$$

The values of the effective dimension is listed in [13] for several famous Nyquist waveforms. The importance of this quantity has been pointed out in [16].

Remark: The minimum value of TSAC is still open for $N \ge 2$. In [13], for $N \ge 2$, TSAC is averaged over i.i.d. codes to make the problem tractable. However, Eq. (22) suggests that we should use Markov codes rather than i.i.d. codes to find minimum value of TSAC.

5 Bandlimited vs Timelimited waveforms

The channel is assumed to be bandlimited in most of the communication systems, but a rectangular waveform used in Section 3 is not bandlimited. The channel capacity, or Shannon capacity, of a single-user bandlimited channel, increases as its bandwidth increases. Hence, the discussion in Section 3 was not fair.

[3] In [13], The same quantity is called asynchronous total square correlations (ATSC)

We must consider a signature waveform is bandlimited as well as time-limited. Before considering such a both band and time-limited signal, we should give a remark that if a waveform is bandlimited but not time-limited, optimal chip waveform is a sinc pulse, which cannot be realizable. Let the bandwidth of a signature waveform be W. Then the Welch bound discussed in the previous section can be easily extended to the chip-asynchronous system as follows:

Lemma 5 *Let T be a symbol duration and W be a bandwidth of signature waveforms. Assume $u_k(t)$, $k = 0, 1, \ldots, K-1$ are bandlimited to W Hz but not timelimited. If $N' = 2WT$ is an integer*[4]*, then TSAC is lower bounded by*[5] *[17]*

$$\sum_{j=0}^{K-1} \mathbf{E}_D[I_{j,p}^2] \geq \begin{cases} \frac{K^2}{N'} - K & \text{if } K > N' \\ 0 & \text{if } K < N' \end{cases} \tag{27}$$

Proof: Since signature waveforms are bandlimited to W Hz, we can express them as $u_j(t) = \sum_{n=-\infty}^{\infty} v_{n,j} \mathrm{sinc}(2W(t+\tau_j) - n)$, giving $C_{j,k}(\tau_{j,k} - pT) = \int u_j(t-\tau_j) u_k(t - \tau_k - pT)dt = \sum_{n=-\infty}^{\infty} v_{n,j} v_{n-pN',k}$. Assume $v_{n,j} = 0$ for $n < 0$ and $n \geq N$. Then

$$C_{j,k}(\tau_{j,k} - pT) = \begin{cases} \sum_{n=0}^{N-1} v_{n,j} v_{n,k} & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

which implies that cross-correlation of bandlimited signature waveforms is equivalent to that of discrete-time signals. Eq.(27) is satisfied with equality if $v_{n,j}$s are WBE sequences. In this case, signature waveform is $u_j(t) = \sum_{n=0}^{N-1} v_{n,j} \mathrm{sinc}(2W(t+\tau_j) - n)$. □

The above lemma shows that if $u_j(t)$ is bandlimited but not timelimited, we get undesirable

We have assumed that signature waveforms $u_j(t)$ are bandlimited but not-timelimited. The lack of restriction brought us an undesirable answer for the TSAC minimization problem, i.e., we can design $u_j(t - \tau_j)$ as a low pass filtered WBE sequences, which implies the energy of the signal is mainly distributed in $[\tau_j, \tau_j + T]$ for a given τ_j. Of

[4]We use N' for bandlimited CDMA systems to distinguish from the spreading factor in a timelimited CDMA system denoted by $N = T/T_c$.

[5]Rigorously speaking, $I_{k,p}$ defined in Eq. (3) must be replaced by

$$I_{k,p} = \sum_{j \neq k} \sum_{q=-\infty}^{\infty} d_{p+q}^{(j)} C_{j,k}(\tau_{j,k} - qT), \tag{26}$$

since the waveform $u_j(t - \tau_j)$ is now supposed to be bandlimited and thus not time-limited. The signature waveform is overlapped with $u_k(t - \tau_k)$ as well as $u_k(t - \tau_k + pT)$ with $p = \pm 1, \pm 2, \ldots$. However, practically it is sufficient to consider the effect from $q = 0$ and $q = 1$, because we may consider $u_j(t)$ is almost time-limited to T second as well as bandlimited to W Hz.

course, this is not our desired answer. In order to design a signature waveform, we should assume $u_j(t)$s are timelimited as well as bandlimited.

Instead of selecting a chip-waveform from a list of famous ones and then optimizing spreading codes, we may directly optimize signature waveforms.

use Slepian's notion of time-limited as well as band-limited signals [18, 19]. We formulate the signature waveform design as follows:

Let W and T be a bandwidth and symbol duration of a CDMA system. Let τ_k be time delay for k-th user. For given $0 < \alpha^2 \leq 1$, and $0 < \beta^2 \leq 1$, we would like to design K signature waveforms, $u_k(t)$ $(k = 0, 1, \ldots, K-1)$, minimizing

$$\sum_{k=0}^{K-1} \mathbf{E}_D[I_{k,p}^2] \tag{28}$$

under the condition that

$$\int_{-\infty}^{\infty} u_j(t)^2 dt = 1, \text{(unit energy)} \tag{29}$$

$$\int_0^T |u_j(t)|^2 dt \geq \alpha^2, \text{(time-limitedness)} \tag{30}$$

$$\int_{-W}^{W} |U_j(f)|^2 df \geq \beta^2. \text{(band-limitedness)} \tag{31}$$

Remark: It is strongly suggested from [13] that minimum value of Eq.(28) depends on the delay profile, and TSAC with uniformly distributed $\boldsymbol{\tau}$ is smaller than the one with $\tau_k = 0$, i.e., asynchronous CDMA system is better than synchronous one.

In the above formulation, we assume that the signature waveforms can be optimized after time delays τ_j are given. However, it is more reasonable to suppose that we design a signature waveforms for unknown time delays. Assume time delays τ_j are independently and uniformly distributed in $[0, T]$. Then, the objective function Eq. (28) is replaced by

$$\sum_{k=0}^{K-1} \mathbf{E}_{\boldsymbol{\tau}}[\mathbf{E}_D[I_{k,p}^2]]. \tag{32}$$

6 Concluding Remarks

Negatively correlated Markov codes minimize the variance of interferences in asynchronous CDMA systems. The auto-correlation function of Markov codes is exponentially vanishing with alternative signs. The cross-correlation with continuous-time valued relative time delay depends on the shape of chip waveforms. Optimal eigenvalue λ for given chip waveform $v(t)$ is found by numerical optimization. The next question was what is the optimal chip waveform.

If a signature waveform is assumed to be bandlimited but not time-limited, sinc waveform is optimal. However, sinc waveform is not realizable. In order to discuss realizability of a waveform, we used Slepian's notion of both band- and time-limited waveforms. We have proposed to employ Gaussian chip-waveform and its associated Markov codes [20]. Its performances in terms of BER and time-frequency energy concentrations were very close to optimal.

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A Bit Error Rate

In this Appendix, we explain the relation between interference and the bit error error (BER) of CDMA systems. The channel noise is assumed to be additive white Gaussian. We start with a review of one-to-one communication through a bandlimited channel with bandwidth W, which helps us to understand the problem in many-to-many communications.

The received signal in a single-user system is given by $r(t) = \sum_{p=-\infty}^{\infty} X_p u(t - pT - \tau) + n(t)$, where X_p is a data of p-th period, τ is a propagation delay, $T = 1/(2W)$ is a symbol duration, $n(t)$ is a white Gaussian noise, and $u(t) = \frac{\sin(\pi W t)}{\pi W t}$ is a sinc function or a sampling function with bandwidth W. The time delay τ is unknown to the receiver, and must be estimated. Denote the estimated time delay by $\hat{\tau}$. The receiver output is given by $Y_p = \int_{-\infty}^{\infty} r(t)u(t - pT - \hat{\tau})dt$.

If the receiver is completely synchronized, i.e., if $\tau = \hat{\tau}$, then inter-symbol interference (ISI) is zero, i.e., $\int_{-\infty}^{\infty} u(t - pT)u(t - qT)dt = \delta_{p,q}$, where $\delta_{p,q}$ is the Kronecker's delta. Based on this assumption, a discrete-time one-to-one communication is often described by

$$Y_p = X_p + Z_p, \tag{33}$$

where $Z_p = \int_{-\infty}^{\infty} n(t)u(t - pT - \hat{\tau})dt$. Let $\mathbf{E}[X_p^2] = P$, $\mathbf{E}[Z_p^2] = \frac{N_0}{2}$, where $\mathbf{E}[X]$ denotes the expectation of a random variable X. The capacity of a Gaussian noise channel is expressed by the celebrated Shannon's formula: $C = \frac{1}{2}\log_2(1 + \frac{2P}{N_0})$ (bit per symbol). The Shannon capacity is of theoretical importance in the sense that it characterized as a maximum rate of data transmission with an arbitrary small bit error rate (BER). On the other hand, it is practically important to mention the BER of binary phase shift keying (BPSK) system. Let X_p take values in $\{\sqrt{P}, -\sqrt{P}\}$. Suppose $X_p = -\sqrt{P}$ is transmitted. Then bit error occurs if Z_p is greater than $\sqrt{P}$. Hence, BER is $P_e = Q\left(\sqrt{\frac{2P}{N_0}}\right)$, where $Q(x) = (2\pi)^{-1/2}\int_x^{\infty} \exp\left(-u^2/2\right) du$.

There is a problem in using a sinc function; its impulse response decays at rate of $1/|t|$ and the truncated sinc function gives large excess band energy. This problem is practically solved by using a raised cosine waveform with decay rate $1/|t|^3$ instead of sinc waveform. Theoretically, Slepian's prolate spheroidal wave functions (PSWFs) are used instead of sinc function to analyze a signal space with is timelimited as well as bandlimited [18, 21].

If the receiver is incompletely synchronized, then BER of BPSK signal is increased. The receiver output with synchronization error $\epsilon = \hat{\tau} - \tau \neq 0$ is

$$Y'_p = a_\epsilon X_p + Z_p + I_{p,\epsilon}, \tag{34}$$

where $a_\epsilon = u * u(\epsilon) < 1$ and $I_{p,\epsilon} = \sum_{q \neq p} u * u(\epsilon + q)X_{p-q}$ denotes ISI and $u * u(t) = \int u(t')u(t' + t)dt'$. It is considered ϵ is a random variable with zero mean. The signal to interference plus noise ratio (SINR) is evaluated by $\frac{\mathbf{E}_\epsilon[a_\epsilon^2]P}{N_0/2 + \mathbf{E}_{D,\epsilon}[I_{p,\epsilon}^2]}$, while the evaluation of BER of BPSK system with incompletely synchronized receiver is slightly complicated[6].

[6]Suppose $X_p = -\sqrt{P}$ is transmitted. Then bit error occurs if Z_p is greater than $a_\epsilon\sqrt{P} - I_{p,\epsilon}$. Thus BER is given by $\tilde{P}_e = \mathbf{E}_{D,\epsilon}[Q\left(\frac{a_\epsilon\sqrt{P} - I_{p,\epsilon}}{\sqrt{N_0/2}}\right)]$.

Now we discuss a CDMA system with K users. There are K pairs of transmitters and receivers. The k-th transmitter wants to transmit its data to the k-th receiver $(k = 0, 1, \ldots, K-1)$ through a common channel. Let the time delay of k-th user be τ_k. The received signal is

$$r(t) = \sum_{k=0}^{K-1} \sum_{p=-\infty}^{\infty} X_{p,k} u_j(t - pT - \tau_k) + n(t), \tag{35}$$

where $n(t)$ is a white Gaussian noise. Assume that $D_{p,j}$ $(j \neq k)$ are i.i.d. random variables. We assume correlation receiver (or a matched filter receiver). The output of the correlation receiver of k-th user and q-th period is $Y_{q,k} = \int_{-\infty}^{\infty} r(t) u_k(t - qT - \hat{\tau}_k) dt$, where $\hat{\tau}_k$ is an estimated time delay. Assume $\hat{\tau}_k = \tau_k$. The receiver's output is expressed as

$$Y_{q,k} = X_{q,k} + Z_{q,k} + I_{q,k}, \tag{36}$$

where $Z_{q,k}$ and $I_{q,k}$ are, respectively, noise and multiple-access interference (MAI) components of $Y_{q,k}$.

The receiver gives a decision that

$$\widehat{X}_{q,k} = \begin{cases} +1 & \text{if } Y_{q,k} \geq 0 \\ -1 & \text{if } Y_{q,k} < 0 \end{cases} \tag{37}$$

Data bit is correctly (respectively, incorrectly) recovered if $\widehat{X}_{q,k} = X_{q,k}$ (respectively, $\widehat{X}_{q,k} \neq X_{q,k}$). The MAI term is denoted by $I_{q,k} = \sum_{j=0,j\neq k}^{K-1} \{X_{q,j} C_{j,k}(\tau_j - \tau_k) + X_{q+1,j} C_{j,k}(\tau_j - \tau_k - T)\}$, where $C_{j,k}(\tau)$ is defined by Eq. (4). Assume $X_{q,k} = -\sqrt{P}$ is sent. Then bit error occurs if $I_{q,k} + \eta_{q,k} > \sqrt{P}$. The variance of noise term $\eta_{q,k} = \int_{-\infty}^{\infty} n(t) u_k(t - qT - \tau_k) dt$ is $N_0/2$. Pursley [9] approximated $I_{q,k}$ as a Gaussian random variable to give a bit error rate (BER) estimation: $P_e \approx Q\left(\sqrt{\frac{P}{\frac{N_0}{2} + \sigma_k^2}}\right)$, where σ_k^2 is the variance of MAI, i.e.,

$$\sigma_k^2 = \sum_{j=0,j\neq k}^{J-1} \left\{ C_{j,k}(\tau_{j,k})^2 + C_{j,k}(\tau_{j,k} - T)^2 \right\}. \tag{38}$$