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Chain-connected component decomposition of the canonical cycle

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Abstract

We study the chain-connected component decomposition of canonical cycles of numerically Gorenstein surface singularities, and determine it for singularities of fundamental genus 2.

Introduction

Let (V, o) be a normal surface singular point and $\pi : X \to V$ the minimal resolution. We denote by Z the fundamental cycle on the exceptional set $\pi^{-1}(o)$. We call the arithmetic genus of Z the *fundamental genus* of (V, o) and denote it by $p_f(V, o)$. The *arithmetic genus* and the *geometric genus* of (V, o) are respectively defined by $p_a(V, o) = \max\{p_a(D) | 0 \prec D,\ \mathrm{Supp}(D) \subseteq \pi^{-1}(o)\}$ and $p_g(V, o) = \dim(R^1\pi_*\mathcal{O}_X)_o$. It is known that $p_f(V, o) \le p_a(V, o) \le p_g(V, o)$. See [14]. Since the intersection form is negative definite on the exceptional set $\pi^{-1}(o)$, there is a $\mathbb{Q}$-divisor Z_K, called the *canonical cycle*, such that $-Z_K$ is numerically equivalent to K_X. If it is a $\mathbb{Z}$-divisor, then we say that (V, o) a *numerically Gorenstein* singularity. Note that (V, o) is Gorenstein i.e., $\mathcal{O}_{V,o}$ is a Gorenstein local ring, if and only if $-Z_K$ is linearly equivalent to K_X.

Suppose that (V, o) is numerically Gorenstein. When (V, o) is an elliptic singular point, that is, $p_f(V, o) = 1$, Yau's elliptic sequence [15] computing Z_K has played a very important role in the study (see, e.g., [15], [8], [9]). In [3], we generalized it and introduced a similar decomposition of Z_K by its chain-connected subcurves also for $p_f(V, o) > 1$. In fact, it was shown that our decomposition is nothing more than the elliptic sequence when $p_f(V, o) = 1$. One of the main results in [3] is the upper bound of the geometric genus via the topological data, i.e., the number of chain-connected curves appearing in the decomposition and the fundamental genus. Though we also exhibited some naive properties of the decomposition there, it needs more systematic investigations for further applications.

In this report, we continue the study to extract more information. In §1, we try to determine the "leading term" of the decomposition, consisting of those chain-components whose respective arithmetic genus equals the fundamental genus, under a certain uniform condition (see, Proposition 1.6).

We apply it to singularities with $p_f(V,o) = 2$. When (V,o) is numerically Gorenstein, we describe in Theorem 2.1 possible types of the chain-connected component decomposition of Z_K. It suggests that subcurves obtained by gluing two or three successive chain-connected components are essential pieces. In §5, we compute the geometric genus, the multiplicity and the embedding dimension for Gorenstein singularities with $p_f = 2$, $Z^2 = -1$ and $Z_K = 3Z$. Such singular points fall into two classes according to the geometric genus. See, Theorem 3.1 for the precise statement. Another fundamental class of $p_f = 2$, that is, those with $Z^2 = -2$ and $Z_K = 2Z$ can be found in [5].

Notation. Throughout the paper, a curve means a non-zero effective divisor (with compact irreducible components) on a non-singular surface. A curve D is chain-connected if $\mathcal{O}_{D-\Gamma}(-\Gamma)$ is not nef for any proper subcurve $0 \prec \Gamma \prec D$. One of the remarkable features of a chain-connected curve D is that, if $\mathcal{O}_D(-C)$ is nef for a curve C, then either $D \preceq C$ or $\mathrm{Supp}(C) \cap \mathrm{Supp}(D) = \emptyset$. If $p_a(D) > 0$, then there uniquely exists a chain-connected subcurve $D_{\min}$ of D such that $p_a(D_{\min}) = p_a(D)$ and $K_{D_{\min}}$ is nef. We call $D_{\min}$ the minimal model of D. Then we have

$$\begin{aligned} D_{\min} &= \min\{\Gamma \mid 0 \prec \Gamma \preceq D,\ p_a(\Gamma) = p_a(D)\} \\ &= \max\{\Gamma \mid 0 \prec \Gamma \preceq D,\ K_\Gamma \text{ is nef}\}. \end{aligned}$$

Every curve C decomposes into a sum of chain-connected curves C_i in such a way that $\mathcal{O}_{C_j}(-C_i)$ is nef for $i < j$ and C_i is a maximal chain-connected subcurve of $C - \sum_{j<i} C_j$. We call it a chain-connected component decomposition (a CCC-decomposition for short) of C. See [3] for further properties.

1 CCC-decompositions of the canonical cycle.

In this section, we show some properties of a CCC-decomposition of the canonical cycle of a numerically Gorenstein surface singularity, in order to supplement [3].

Let (V,o) be a normal 2-dimensional singularity. We usually denote by $\pi : X \to V$ the minimal resolution and let Z be the fundamental cycle on $\pi^{-1}(o)$. The arithmetic genus of Z is called the fundamental genus of (V,o). We denote it by $p_f(V,o)$ and assume $p_f(V,o) > 0$ in what follows. We say that (V,o) is numerically Gorenstein, if there is a curve Z_K with support in $\pi^{-1}(o)$ such that $-Z_K$ is numerically equivalent to K_X. The curve Z_K is called the canonical cycle.

Let $Z_K = \Gamma_1 + \cdots + \Gamma_n$ be a CCC-decomposition, that is, each Γ_i is a maximal chain-connected subcurve of $Z_K - \sum_{j<i}\Gamma_j$ and $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is nef for $i < j$. Such an ordered decomposition exists and is unique up to permutations reserving the second property. When $p_f(V,o) > 0$, we showed in [3] the following:

- $\Gamma_1 = Z$ is the fundamental cycle and, if $n \geq 2$,
- $\Gamma_2 = \gcd(\Gamma_1, Z_K - \Gamma_1)$, $p_a(\Gamma_2) = p_f(V,o)$ and $\mathrm{Supp}(\Gamma_1 - \Gamma_2) \cap \mathrm{Supp}(Z_K - \Gamma_1 - \Gamma_2) = \emptyset$,
- $p_a(\Gamma_i) > 0$ and $\Gamma_i \preceq \Gamma_2$ for any $i \geq 3$,
- for $i < j$, either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$,
- the dualizing sheaf of every minimal curve in $\{\Gamma_i\}_{i=1}^n$ is nef.

Lemma 1.1. *Assume that $p_f(V,o) > 1$. Then $n \geq 2$ and $2 - 2p_f(V,o) \leq \Gamma_1\Gamma_2 \leq -1$.*

Proof. If $n = 1$, then $Z_K = Z$ and $1 = p_a(Z_K) = p_a(Z) = p_f(V,o) > 1$, a contradiction. Hence $n \geq 2$. We have $2p_a(\Gamma_1) - 2 = \Gamma_1(K_X + \Gamma_1) = -\Gamma_1(Z_K - \Gamma_1)$. This implies that there exists an index $i \geq 2$ with $-\Gamma_1\Gamma_i > 0$, if $p_a(\Gamma_1) > 1$. Since $\Gamma_i \preceq \Gamma_2$ and $-\Gamma_1$ is nef, we get $-\Gamma_1\Gamma_2 > 0$. We have $\Gamma_1\Gamma_2 \geq \Gamma_1(Z_K - \Gamma_1) = 2 - 2p_f$. □

In fact, when $p_f(V,o) > 0$, we have $n = 1$ if and only if (V,o) is a minimally elliptic singularity ([7], [10]).

Lemma 1.2. *Assume that $i < j$, $\Gamma_j \preceq \Gamma_i$ and $p_a(\Gamma_i) = p_a(\Gamma_j)$. Then $\Gamma_i^2 \leq \Gamma_j^2$ with equality holding only when, either $\Gamma_i = \Gamma_j$ or $\Gamma_i - \Gamma_j$ consists of (-2)-curves.*

Proof. We have $2p_a(\Gamma_i) - 2 = -Z_K\Gamma_i + \Gamma_i^2$. Hence $\Gamma_j^2 - \Gamma_i^2 = 2(p_a(\Gamma_j) - p_a(\Gamma_i)) - Z_K(\Gamma_i - \Gamma_j) = -Z_K(\Gamma_i - \Gamma_j)) \geq 0$, since $-Z_K \equiv K_X$ is nef. □

In particular, we get $\Gamma_1^2 \leq \Gamma_2^2$.

Lemma 1.3. *Assume that $\Gamma_{i+1} \preceq \Gamma_i$ and $\mathcal{O}_{\Gamma_i - \Gamma_{i+1}}(-\sum_{j<i}\Gamma_j)$ is numerically trivial. Then the following hold.*

(1) $\Gamma_{i+1} = \gcd(\Gamma_i, Z_K - \sum_{j\leq i}\Gamma_j)$, $p_a(\Gamma_{i+1}) = p_a(\Gamma_i)$ *and* $\mathrm{Supp}(\Gamma_i - \Gamma_{i+1}) \cap \mathrm{Supp}(Z_K - \sum_{j\leq i+1}\Gamma_j) = \emptyset$.

(2) $\Gamma_i^2 \leq \Gamma_{i+1}^2$. *Furthermore, $\Gamma_{i+1} = \Gamma_i$ holds if and only if $\Gamma_i(\Gamma_i - \Gamma_{i+1}) = 0$.*

Proof. (1): Put $G = \gcd(\Gamma_i, Z_K - \sum_{j\leq i}\Gamma_j)$. Then, since $\Gamma_{i+1} \preceq G \preceq \Gamma_i$,

$$\begin{aligned}2p_a(G) - 2 &= -G(Z_K - G)\\ &= -\Gamma_i(Z_K - \Gamma_i) + (\Gamma_i - G)(Z_K - G - \sum_{j\leq i}\Gamma_j) + (\Gamma_i - G)\sum_{j<i}\Gamma_j\\ &= 2p_a(\Gamma_i) - 2 + (\Gamma_i - G)(Z_K - G - \sum_{j\leq i}\Gamma_j).\end{aligned}$$

By the choice of G, $\Gamma_i - G$ has no common components with $Z_K - G - \sum_{j\leq i}\Gamma_j$. Hence $(\Gamma_i - G)(Z_K - G - \sum_{j\leq i}\Gamma_j) \geq 0$ and we get $p_a(G) \geq p_a(\Gamma_i)$. Since Γ_i is chain-connected, $p_a(G) \leq h^1(G, \mathcal{O}_G) \leq h^1(\Gamma_i, \mathcal{O}_{\Gamma_i}) = p_a(\Gamma_i)$. In sum, we get $p_a(G) = p_a(\Gamma_i)$ and $\mathrm{Supp}(\Gamma_i - G) \cap \mathrm{Supp}(Z_K - G - \sum_{j\leq i}\Gamma_j) = \emptyset$. Note that G is chain-connected, since $p_a(G) = p_a(\Gamma_i) > 0$ (see, [3]). We have $G - \Gamma_{i+1} \preceq Z_K - \sum_{j\leq i+1}\Gamma_j$. So, $\mathcal{O}_{G-\Gamma_{i+1}}(-\Gamma_{i+1})$ is nef. Since G is chain-connected, we must have $\Gamma_{i+1} = G$.

(2): The first assertion follows from Lemma 1.2. To show the last equivalence, we only have to show the converse. Since $\mathcal{O}_{\Gamma_i - \Gamma_{i+1}}(-\sum_{j<i}\Gamma_j)$ is numerically trivial, we have $(\Gamma_i + \Gamma_{i+1})(\Gamma_i - \Gamma_{i+1}) = Z_K(\Gamma_i - \Gamma_{i+1}) - (\Gamma_i - \Gamma_{i+1})\sum_{j<i}\Gamma_j - (\Gamma_i - \Gamma_{i+1})(Z_K - \sum_{j\leq i+1}\Gamma_j) = Z_K(\Gamma_i - \Gamma_{i+1})$ by (1). If $\Gamma_i(\Gamma_i - \Gamma_{i+1}) = 0$, then $0 \geq (\Gamma_i - \Gamma_{i+1})^2 = -(\Gamma_i + \Gamma_{i+1})(\Gamma_i - \Gamma_{i+1}) = -Z_K(\Gamma_i - \Gamma_{i+1}) \geq 0$. Hence $(\Gamma_i - \Gamma_{i+1})^2 = 0$ and it follows $\Gamma_{i+1} = \Gamma_i$, since the intersection form is negative definite on $\pi^{-1}(o)$. □

We turn our attention to minimal chain-connected components.

Lemma 1.4. *Assume that $p_f(V, o) > 0$. Then Γ_i contains at most $p_a(\Gamma_i)$ distinct minimal elements in $\{\Gamma_j\}_{j=1}^n$. In particular, $\{\Gamma_i\}_{i=1}^n$ has at most $p_f(V, o)$ distinct minimal elements.*

Proof. Recall that $p_a(\Gamma_j) > 0$ for any j and that any two distinct minimal elements are disjoint. Take $i \in \{1, 2, \ldots, n\}$. If Γ denotes the sum of all distinct minimal elements in $\{\Gamma_j\}_{j=1}^n$ such that $\Gamma_j \preceq \Gamma_i$, then Γ is a subcurve of Γ_i such that $h^1(\Gamma, \mathcal{O}_\Gamma)$ equals the sum of the arithmetic genera of the minimal elements in Γ. Since $h^1(\Gamma, \mathcal{O}_\Gamma) \leq h^1(\Gamma_i, \mathcal{O}_{\Gamma_i}) = p_a(\Gamma_i)$, we get $p_a(\Gamma_i) \geq \sum_{\nu=1}^{\mu} p_a(\Gamma_{i_\nu}) \geq \mu$, if we put $\Gamma = \Gamma_{i_1} + \cdots + \Gamma_{i_\mu}$. Applying the above argument to $i = 1$, we see that $\{\Gamma_j\}_{j=1}^n$ has at most $p_a(\Gamma_1) = p_f(V, o)$ minimal elements. □

The upper bound in Lemma 1.4 is sharp, as the following example shows.

Example 1.5. Let p be a positive integer. Let A_0 be a non-singular rational curve with $A_0^2 = -p - 1$, A_i a non-singular elliptic curve with $A_i^2 = -1$ and $A_0 A_i = 1$ for

$i = 1, \dots, p$. Assume that $A_iA_j = 0$ for $1 \le i < j \le p$ and put $Z = \sum_{i=0}^{p} A_i$. Then Z is the fundamental cycle on its support and $Z^2 = -1$, $p_a(Z) = p$. The canonical cycle is written as $Z_K = (2p-1)A_0 + 2p(A_1 + \cdots + A_p) = (2p-1)Z + A_1 + \cdots + A_p$. Hence a CCC-decomposition of Z_K is given by putting $\Gamma_i = Z$ for $1 \le i \le 2p-1$ and $\Gamma_{2p-1+i} = A_i$ for $1 \le i \le p$. So, there are exactly p distinct minimal elements in $\{\Gamma_i\}_{i=1}^{3p-1}$.

Proposition 1.6. *Assume that $p_f(V, o) > 1$ and write $2p_f - 2 = ab$ with two positive integers a, b. If there exist exactly b indices $i \ge 2$ satisfying $-\Gamma_1\Gamma_i = a$, then the following hold.*

(1) $\Gamma_{i+1} = \gcd(\Gamma_i, Z_K - \sum_{j \le i} \Gamma_j)$ *and* $p_a(\Gamma_{i+1}) = p_f(V, o)$ *for* $i \in \{1, 2, \dots, b\}$.

(2) $\Gamma_{b+1} \preceq \Gamma_b \preceq \cdots \preceq \Gamma_2 \preceq \Gamma_1$ *and* $\Gamma_1^2 \le \Gamma_2^2 \le \cdots \le \Gamma_{b+1}^2$.

(3) *For* $1 \le i < j \le b+1$, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ *is nef of degree* a.

(4) *For* $1 \le i < j < k \le b+1$, $\operatorname{Supp}(\Gamma_i - \Gamma_j) \cap \operatorname{Supp}(\Gamma_k) = \emptyset$.

In particular, $p_a(\Delta) = 1$ and $Z_K - \Delta$ is numerically equivalent to $-K_X$ on its support, where $\Delta = \sum_{i=1}^{b+1} \Gamma_i$.

Proof. We have $-\Gamma_1\Gamma_i \in \{a, 0\}$ for $i \ge 2$ by the choice of a, b, since $-\Gamma_1(Z_K - \Gamma_1) = 2p_f - 2$. We have $-\Gamma_1\Gamma_i \ge -\Gamma_1\Gamma_j$ when $\Gamma_j \preceq \Gamma_i$. Since $\Gamma_i \preceq \Gamma_2$ for $i \ge 3$, we have $-\Gamma_1\Gamma_2 = a$.

Let i_0 be the smallest index with $i_0 \ge 3$ and $-\Gamma_1\Gamma_{i_0} = a$. Then Γ_{i_0} is a maximal element in $\{\Gamma_i\}_{i=3}^{n}$. So, we can assume that $i_0 = 3$ after re-numbering if necessary. Since $\mathcal{O}_{\Gamma_2 - \Gamma_3}(-\Gamma_1)$ is numerically trivial, it follows from Lemma 1.3 that $\Gamma_3 = \gcd(\Gamma_2, Z_K - \Gamma_1 - \Gamma_2)$, $p_a(\Gamma_3) = p_a(\Gamma_2)$, $\Gamma_2^2 \le \Gamma_3^2$ and $\operatorname{Supp}(\Gamma_2 - \Gamma_3, Z_K - \Gamma_1 - \Gamma_2 - \Gamma_3) = \emptyset$. Note that the last condition implies that Γ_1, Γ_2 and Γ_3 are linearly equivalent on $Z_K - \Gamma_1 - \Gamma_2 - \Gamma_3$.

We claim that $\Gamma_i \preceq \Gamma_3$ for $i \ge 3$. If not, then Γ_3 and Γ_i are disjoint. Then $\Gamma_3 + \Gamma_i \preceq \Gamma_2$ and we get $p_a(\Gamma_3) + p_a(\Gamma_i) = h^1(\Gamma_3 + \Gamma_i, \mathcal{O}) \le h^1(\Gamma_2, \mathcal{O}_{\Gamma_2}) = p_a(\Gamma_2)$. This is impossible, since $p_a(\Gamma_3) = p_a(\Gamma_2)$ and $p_a(\Gamma_i) > 0$. Therefore, $\Gamma_i \preceq \Gamma_3$ for $i \ge 3$.

Now, the obvious induction shows the assertions (1)–(4). The rest may be clear. $\square$

2 Singularities of fundamental genus two

From now on, we concentrate on the case that $p_f(V, o) = 2$.

We denote by $\pi : X \to V$ the minimal resolution and work on X. We also assume that (V, o) is numerically Gorenstein and consider the canonical cycle.

Theorem 2.1. *Let Z_K be the canonical cycle on the minimal resolution of an isolated numerically Gorenstein surface singular point with $p_f(V, o) = 2$. Then Z_K decomposes as*

$$Z_K = \Delta_1 + \cdots + \Delta_m + E,$$

where the Δ_i's and E are curves satisfying the following conditions.

(1) Δ_i is a curve with $p_a(\Delta_i) = 1$ and $\mathcal{O}_{\Delta_j}(-\Delta_i)$ is numerically trivial when $i < j$. In particular, for any $i \in \{1, \dots, m\}$, $Z_K - \sum_{j=1}^{i} \Delta_j$ is the canonical cycle on its support.

(2) For any $i \in \{1, \dots, m\}$, the CCC-decomposition of Δ_i is one of the following types:

(a) $\Delta_i = \Gamma_{i,1} + \Gamma_{i,2} + \Gamma_{i,3}$, $\Gamma_{i,3} \preceq \Gamma_{i,2} \preceq \Gamma_{i,1}$, $\Gamma_{i,1}^2 \leq \Gamma_{i,2}^2 \leq \Gamma_{i,3}^2$ and $\mathcal{O}_{\Gamma_{i,\nu}}(-\Gamma_{i,\mu})$ is nef of degree 1 for $\mu < \nu$.

(b) $\Delta_i = \Gamma_{i,1} + \Gamma_{i,2}$, $\Gamma_{i,2} \preceq \Gamma_{i,1}$, $\Gamma_{i,1}^2 \leq \Gamma_{i,2}^2$ and $\mathcal{O}_{\Gamma_{i,2}}(-\Gamma_{i,1})$ is nef of degree 2.

Furthermore, $p_a(\Gamma_{i,\nu}) = 2$, $\Gamma_{i,1}$ is the fundamental cycle on its support and, when $i < j$, $\mathcal{O}_{\Gamma_{j,\nu}}(-\Gamma_{i,\mu})$ is numerically trivial and $\Gamma_{j,\nu} \prec \Gamma_{i,\mu}$, $\Gamma_{i,\mu}^2 \leq \Gamma_{j,\nu}^2$ for any μ, ν; $\mathcal{O}_{\Delta_j}(-\Delta_i) \simeq \mathcal{O}_{\Delta_j}(-3\Gamma_{i,1})$ or $\mathcal{O}_{\Delta_j}(-\Delta_i) \simeq \mathcal{O}_{\Delta_j}(-2\Gamma_{i,1})$ according to whether Δ_i is as in (a) or (b).

(3) If $E \neq 0$, then either E is the canonical cycle of a numerically Gorenstein elliptic singular point, or it is the sum of two disjoint canonical cycles of numerically Gorenstein elliptic singular points. Every $\Gamma_{i,\mu}$ $(i \leq m)$ as in (2) is numerically trivial on E.

(4) When $E = 0$, the smallest chain-component $\Gamma^* := \Gamma_{m,\mu}$, where $\mu = 3$ or 2 according to the types of Δ_m as in (2), is the minimal model of the fundamental cycle $Z = \Gamma_{1,1}$ for (V, o).

Proof. Let $Z_K = \sum_{i=1}^{n} \Gamma_i$ be a CCC-decomposition. We have $2 = 2p_a(\Gamma_1) - 2 = -\Gamma_1(Z_K - \Gamma_1) = -\Gamma_1 \sum_{i=2}^{n} \Gamma_i$. Since $\mathcal{O}_{\Gamma_i}(-\Gamma_1)$ is nef, we have $\Gamma_1\Gamma_2 = -1, -2$ and, in any case, the hypothesis of Theorem 1.6 is satisfied.

We put $\Delta_1 = \Gamma_1 + \Gamma_2$ when $-\Gamma_1\Gamma_2 = 2$, and $\Delta_1 = \Gamma_1 + \Gamma_2 + \Gamma_3$ when $-\Gamma_1\Gamma_2 = 1$. Then $p_a(\Delta_1) = 1$ and $Z_K - \Delta_1$ is the canonical cycle on its support by Theorem 1.6. If $Z_K - \Delta_1 = 0$, then we stop with $m = 1$ and $E = 0$. Assume that $Z_K - \Delta_1 \neq 0$. If the support of $Z_K - \Delta_1$ is not connected, then, by Lemma 1.4, it is a sum of two canonical cycles of elliptic singularities, and we stop with $m = 1$ and $E = Z_K - \Delta_1$. Assume that the support of $Z_K - \Delta_1$ is connected. If it is the canonical cycle of an elliptic singularity, then we stop with $m = 1$ and $E = Z_K - \Delta_1$. So, we may assume that $Z_K - \Delta_1$ is the canonical cycle of a singular point of $p_f = 2$. Then, we can repeat the above argument to find Δ_2 consisting of two or three chain-connected curves of arithmetic genus 2 from $Z_K - \Delta_1$.

Now, the obvious induction shows the assertions (1)–(4). □

We say that Δ_i, $1 \leq i \leq m$, is of type (a) or (b) according to whether it decomposes as in (a) or (b) in (2) of Theorem 2.1. The curve E will be sometimes referred to as the *elliptic remainder*.

Example 2.2. Let A_i $(0 \leq i \leq 4)$ be non-singular projective curves with $A_i^2 = -2$. Suppose that the dual graph of $\mathcal{A} = \bigcup_{i=0}^4 A_i$ is of type $(\mathbf{D}_5)$ as in Figure 2.1. We denote by (V, o) the singularity obtained by contracting $\mathcal{A}$. Then $Z = A_0 + A_1 + 2A_2 + 2A_3 + A_4$ is the fundamental cycle on $\mathcal{A}$ and we have $Z^2 = -2$.

(1) This example shows that both types (a) and (b) actually occur. Assume that A_0 is of genus two and $A_i \simeq \mathbb{P}^1$ for $1 \leq i \leq 4$. Then $p_f(V, o) = 2$ and $Z_K = 5A_0 + 3A_1 + 6A_2 + 4A_3 + 2A_4$ is the canonical cycle. It is easy to see that Z_K has five chain-components $\Gamma_1 = \Gamma_2 = Z$, $\Gamma_3 = A_0 + A_1 + A_2$, $\Gamma_4 = A_0 + A_2$ and $\Gamma_5 = A_0$. We have $\Gamma_1\Gamma_2 = -2$ and $\Gamma_i\Gamma_j = -1$ for $3 \leq i < j \leq 5$. Put $\Delta_1 = \Gamma_1 + \Gamma_2$, $\Delta_2 = \Gamma_3 + \Gamma_4 + \Gamma_5$. Then $Z_K = \Delta_1 + \Delta_2$ is the decomposition as in Theorem 2.1 with $m = 2$, Δ_2 is of type (a) while Δ_1 is of type (b). We have $p_a(V, o) = 3$, because Γ_3 is the arithmetic subcycle of Z and $\Gamma_3 A_0 = -1 < 0$.

(2) Let A_2 be an elliptic curve, and $A_i \simeq \mathbb{P}^1$ for $i \neq 2$. Then $p_f(V, o) = 2$ and the canonical cycle is $Z_K = 3A_0 + 3A_1 + 6A_2 + 4A_3 + 2A_4$ which has four chain-components: $\Gamma_1 = \Gamma_2 = Z$, $\Gamma_3 = A_0 + A_1 + A_2$ and $\Gamma_4 = A_2$. If we put $\Delta_1 = \Gamma_1 + \Gamma_2$ and $E = \Gamma_3 + \Gamma_4$, then $Z_K = \Delta_1 + E$ is the decomposition as in Theorem 2.1 with $m = 1$, Δ_1 is of type (b). The elliptic remainder E is the canonical cycle of an elliptic singularity with fundamental cycle Γ_3. We get $p_a(V, o) = 2$, because $Z_{\min} = A_0 + A_1 + 2A_2 + A_3$ and $Z_{\min} Z < 0$. Therefore, if the elliptic remainder appears, the smallest chain-component of arithmetic genus 2 of Z_K is not necessarily the minimal model of the fundamental cycle. The hypersurface singularity defined by $x^2 + y^7 + z^{10} = 0$ also enjoys such a property, as pointed out in [13, Example 2.5].

As to the elliptic remainder, we have the following:

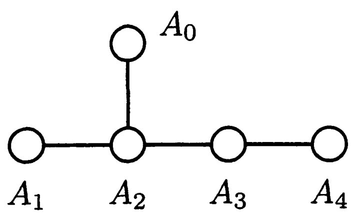

Fig. 2.1

Lemma 2.3. *Let the notation be as in Theorem* 2.1. *Let* Γ^* *be the smallest chain-component of* Δ_m. *If* $\bar{A}$ *denotes the smallest subcurve of* Γ^* *such that* $\mathcal{O}_{\Gamma^*-\bar{A}}(-\Delta_m)$ *is numerically trivial. Then* $\mathcal{O}_{\Gamma^*-\bar{A}}(-E)$ *is nef. If* $E \neq 0$, *then every maximal chain-component of* $\Gamma^* - \bar{A}$ *that is not the fundamental cycle of a rational double point is the fundamental cycle on a connected component of* E, *and vice versa. In particular,* $E = 0$ *if and only if* $\Gamma^* - \bar{A}$ *consists of (at most)* (-2)*-curves.*

Now, we collect some applications of Theorem 2.1. A singular point of positive fundamental genus is sometimes called a *minimal* singularity, when the fundamental cycle coincides with its minimal model, i.e., the dualizing sheaf is nef.

Corollary 2.4. *Let* (V, o) *be an isolated numerically Gorenstein surface singular point of* $p_f(V, o) = 2$. *Let* Z *be the fundamental cycle on the minimal resolution and assume that* K_Z *is nef. Then* $p_a(V, o) = 2$, $m = 1$ *and* $Z^2 = -1, -2$; $\Delta_1 = 3Z$ *or* $2Z$ *according to whether* $Z^2 = -1$ *or* -2.

Proof. Since $Z_{\min} = Z$, we have $Z_{\min}Z < 0$ and it follows $p_a(V, o) = 2$. Then we have $m = 1$ in Theorem 2.1. Since Z is minimal, every chain-component of Δ_1 equals Z. So, $Z^2 = \Gamma_1\Gamma_2 = -1, -2$. □

Example 2.5. The elliptic remainder appears even when K_Z is nef or Z is 2-connected.

(1) Let A_0, A_1 be two elliptic curve such that $A_0^2 = -1$, $A_1^2 = -a$ and $A_0A_1 = 1$, where $a = 2, 3$. Then $Z = A_0 + A_1$ is the fundamental cycle on $A_0 \cup A_1$. It is clear that $p_a(Z) = 2$ and K_Z is nef. We have $Z^2 = 1 - a$. As to the canonical cycle, we have $Z_K = \frac{a+1}{a-1}Z + A_0$.

(2) A_0 be an elliptic curve with $A_0^2 = -2$. A_1 be a $(-b)$-curve such that $A_0A_1 = 2$, where $b = 3, 4$. Put $Z = A_0 + A_1$. Then $Z^2 = 2 - b$ and $Z_K = \frac{2(b-1)}{b-2}A_0 + \frac{b}{b-2}A_1 = \frac{b}{b-2}Z + A_0$.

It is well-known that any Gorenstein singularity with $p_g = 2$ is elliptic. So, the next target may be those with $p_g = 3$. The following provides useful information about Gorenstein singularities with $p_g = 3$.

Corollary 2.6. *Let (V,o) be an isolated Gorenstein surface singular point with $p_g(V,o) = 3$ that is not an elliptic singular point. Then $p_f(V,o) = p_a(V,o) = 2$ and the canonical cycle on the minimal resolution decomposes as in Theorem 2.1 with $m = 1$: $Z_K = \Delta_1 + E$. If $E \neq 0$, then $h^1(\Delta_1, \mathcal{O}_{\Delta_1}) = 2$ and the total sum of the geometric genera of numerically Gorenstein elliptic singularities corresponding to E is at most 2.*

Proof. Since (V,o) is Gorenstein but not elliptic, we have $2 \leq p_f(V,o) \leq p_a(V,o) < p_g(V,o)$ (see, [12]). Then, from $p_g(V,o) = 3$, we get $p_f(V,o) = p_a(V,o) = 2$. By Theorem 2.1, we get $m = 1$ and $Z_K = \Delta_1 + E$. Since (V,o) is Gorenstein, the canonical cycle is the cohomological cycle. Hence $h^1(E, \mathcal{O}_E) < h^1(Z_K, \mathcal{O}_{Z_K}) = 3$. For the same reasoning, we have $h^1(\Delta_1, \mathcal{O}_{\Delta_1}) < 3$ when $E \neq 0$. Since $h^1(\Delta_1, \mathcal{O}_{\Delta_1}) \geq h^1(Z, \mathcal{O}_Z) = 2$, we get $h^1(\Delta_1, \mathcal{O}_{\Delta_1}) = 2$. $\square$

Let the situation be as above and $E \neq 0$. Recall that

$$\mathcal{O}_E(K_X + E) \simeq \mathcal{O}_E(-\Delta_1) \simeq \begin{cases} \mathcal{O}_E(-3Z) & \text{if } \Delta_1 \text{ is of type (a),} \\ \mathcal{O}_E(-2Z) & \text{if } \Delta_1 \text{ is of type (b).} \end{cases}$$

Therefore, the singular point obtained by contracting E may not be Gorenstein, even when (V,o) is Gorenstein.

3 Certain singularities with $Z^2 = -1$, $p_f = 2$

In this section, we study a special singular point of fundamental genus 2. We denote by $\mathfrak{m}$ the ideal sheaf of $o \in V$. Let $Z_{\mathfrak{m}}$ be the maximal ideal cycle, that is, the divisorial fixed part of the linear system $|\mathfrak{m}\mathcal{O}_X|$ with support in $\pi^{-1}(o)$. Then $-Z_{\mathfrak{m}}$ is nef on $\pi^{-1}(o)$ and $Z \preceq Z_{\mathfrak{m}}$.

The purpose of the section is to show the following:

Theorem 3.1. *Let (V,o) be a Gorenstein surface singularity with $p_f(V,o) = 2$ such that $Z^2 = -1$ and $Z_K = 3Z$ hold on the minimal resolution. Then $p_a(V,o) = 2$ and there are the following two cases.*

(1) *$p_g(V,o) = 4$, $Z_{\mathfrak{m}} = Z$, $\mathfrak{m}\mathcal{O}_X \simeq \mathfrak{m}_x\mathcal{O}_X(-Z)$ with a non-singular point $x \in Z$,* $\mathrm{mult}(V,o) = 2$ *and* $\mathrm{embdim}(V,o) = 3$.

(2) $p_g(V,o)=3$, $Z_{\mathfrak{m}}=2Z$, $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-2Z)$, $\operatorname{mult}(V,o)=4$ *and* $\operatorname{embdim}(V,o)=4$.

Let the situation be as in Theorem 3.1. We first remark that Z is 2-connected and K_Z is free. This can be seen as follows. Since $Z^2=-1$, Z is at least 1-connected by [2, Lemma 2.1]. If C is a proper subcurve of Z, then $\deg K_Z|_C = \deg K_C + C(Z-C)$. Hence $C(Z-C) = -2CZ+2-2p_a(C)$ is even. This is sufficient to imply that Z is 2-connected. Then it is known that $|K_Z|$ is free from base points (see, e.g., [1, Proposition (A.7)]).

Let A be the irreducible component of Z with $-AZ=1$. Then $Z-A$ consists of (-2)-curves at most, because $K_X(Z-A)=-3Z(Z-A)=0$. We have $0\le p_a(A)\le 2$ and $A^2=2p_a(A)-5$. Hence, $(p_a(A),A^2)=(2,-1),(1,-3)$ or $(0,-5)$, and we have $Z=A$ when $p_a(A)=2$; $Z-A$ is the fundamental cycle of a rational double point with $A(Z-A)=2$ when $p_a(A)=1$; and $Z-A$ consists of two fundamental cycles C_1, C_2 of rational double points with $\mathcal{O}_{C_2}(-C_1)\simeq \mathcal{O}_{C_2}$ and $AC_1=AC_2=2$ when $p_a(A)=0$.

We know that $\mathcal{O}_Z(-Z)$ is nef of degree one and $p_a(V,o)=2$. Consider the cohomology long exact sequence for

$$0\to \mathcal{O}_X(-(i+1)Z)\to \mathcal{O}_X(-iZ)\to \mathcal{O}_Z(-iZ)\to 0.$$

Since $-3Z$ is the canonical cycle, we have $H^1(X,-(i+1)Z)=0$ for $i\ge 2$ by the vanishing theorem. Hence $H^0(X,-iZ)\to H^0(Z,-iZ)$ is surjective when $i\ge 2$.

Case 1. We first assume that $H^0(X,-2Z)\to H^0(X,-Z)$ is an isomorphism. Then $2Z\preceq Z_{\mathfrak{m}}$. Since $|K_Z|=|\mathcal{O}_Z(-2Z)|$ is free from base points, $|\mathcal{O}_X(-2Z)|$ is π-free. Hence $Z_{\mathfrak{m}}=2Z$ and $\operatorname{mult}(V,o)=-Z_{\mathfrak{m}}^2=4$.

Lemma 3.2. $H^0(Z,-Z)=H^1(Z,-Z)=0$ *and* $p_g(V,o)=3$.

Proof. To compute $p_g(V,o)$, we consider the cohomology long exact sequence for

$$0\to \mathcal{O}_{2Z}(-Z)\to \mathcal{O}_{Z_K}\to \mathcal{O}_Z\to 0.$$

Since the restriction map $H^0(Z_K,\mathcal{O}_{Z_K})\to H^0(Z,\mathcal{O}_Z)$ is surjective, we get $p_g(V,o)=h^0(Z_K,\mathcal{O}_{Z_K})=h^0(2Z,-Z)+1$. Consider

$$0\to \mathcal{O}_Z(-2Z)\to \mathcal{O}_{2Z}(-Z)\to \mathcal{O}_Z(-Z)\to 0.$$

The restriction map $H^0(X,-Z)\to H^0(2Z,-Z)$ is surjective by the fact that $H^1(X,-3Z)=0$, while $H^0(X,-Z)\to H^0(Z,-Z)$ is zero by the assumption. It follows that $H^0(2Z,-Z) \to H^0(Z,-Z)$ is also zero. Then $h^0(2Z,-Z) = h^0(Z,-2Z) = 2$ and we get $p_g(V,o) = 3$.

It remains to show that $h^0(Z,-Z) = 0$. Since $h^1(2Z,-Z) = 1$, we get $h^0(2Z,\mathcal{O}_{2Z}) = 1$ by the duality theorem. Then, since $H^0(2Z,\mathcal{O}_{2Z}) \to H^0(Z,\mathcal{O}_Z)$ is an isomorphism, it follows from the cohomology long exact sequence for

$$0 \to \mathcal{O}_Z(-Z) \to \mathcal{O}_{2Z} \to \mathcal{O}_Z \to 0$$

that $H^0(Z,-Z) = H^1(Z,-Z) = 0$. □

We compute the embedding dimension. Before going in detail, we remark that $|\mathcal{O}_Z(-3Z)|$ is free from base points. This can be seen as follows. If it has a base point x, then, by [2, Proposition 5.1], there exists a subcurve Δ of Z such that $\Delta^2 = -1$, x is a non-singular point of Δ and $\mathcal{O}_\Delta(-3Z) \simeq \omega_\Delta \otimes \mathcal{O}_\Delta(x)$. Since $\Delta^2 = -1$, Δ is 1-connected. By $Z\Delta = 0, -1$ and $\deg \omega_\Delta = 2p_a(\Delta) - 2$, the possible case is only: $Z\Delta = -1$ and $p_a(\Delta) = 2$. This implies that $\Delta = Z$, since Z is its own minimal model. Then we get $\mathcal{O}_Z(-Z) \simeq \mathcal{O}_Z(x)$, contradicting that $H^0(Z,-Z) = 0$.

We study the graded ring $R(Z,-Z) = \bigoplus_{i \geq 0} H^0(Z,-iZ)$. We have $h^0(Z,-2Z) = 2$ and $h^0(Z,-iZ) = i-1$ for $i \geq 3$. By the free-pencil trick, $\mu_i : H^0(Z,-iZ) \otimes H^0(Z,-2Z) \to H^0(Z,-(i+2)Z)$ is surjective for $i \geq 2$, $i \neq 4$. This is because $H^1(Z,-(i-2)Z) = 0$ when $i = 3$ or $i \geq 5$, while we get it by dimension count when $i = 2$. Therefore, $R(Z,-Z)$ is generated in degrees at most 6. Let $\{x_0, x_1\}$ be a basis for $H^0(Z,-2Z)$. Then $H^0(Z,-4Z)$ is generated by x_0^2, x_0x_1, x_1^2. Let $\{y_0, y_1\}$ be a basis for $H^0(Z,-3Z)$. Then $H^0(Z,-5Z)$ is generated by $x_0y_0, x_0y_1, x_1y_0, x_1y_1$. We consider $H^0(Z,-6Z)$. Here, we have four elements $x_0^j x_1^{3-j}$ $(0 \leq j \leq 3)$ which generate a subspace V_1 of codimension one. Recall that $|\mathcal{O}_Z(-3Z)|$ is free from base points. By the free-pencil-trick, one can show that $\mathrm{Sym}^2 H^0(Z,-3Z) \to H^0(Z,-6Z)$ is injective, and the image $V_2 = \langle y_0^2, y_0y_1, y_1^2 \rangle$ is a subspace of dimension three. We claim that $H^0(Z,-6Z) = V_1 + V_2$. Assume not. Then $V_2 \subset V_1$ and we have three relations: $y_0^2 = c_1(x)$, $y_0y_1 = c_2(x)$ and $y_1^2 = c_3(x)$, where c_1, c_2, c_3 are cubic forms in x_0, x_1. It follows $y_1/y_0 = c_2(x)/c_1(x)$. This implies that the morphism defined by $|\mathcal{O}_Z(-3Z)|$ is the composite of the morphism defined by $|\mathcal{O}_Z(-2Z)|$ and the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by c_2/c_1, which is impossible, because $-3Z^2 = 3$ and $-2Z^2 = 2$. Therefore, $V_2 \not\subset V_1$. For the same reasoning, we may assume that $y_0^2, y_1^2 \in V_1$ and $y_0y_1 \notin V_1$. Now, we have two relations: $y_0^2 = \varphi_0(x_0, x_1)$, $y_1^2 = \varphi_1(x_0, x_1)$, where φ_0, φ_1 are cubic forms. It is not hard to confirm that there are no further relations in $R(Z,-Z)$. Therefore, $R(Z,-Z) \simeq \mathbb{C}[X_0, X_1, Y_0, Y_1]/(Y_0^2 - \varphi_0(X_0, X_1), Y_1^2 - \varphi_1(X_0, X_1))$ as graded

$\mathbb{C}$-algebras, where $\deg X_0 = \deg X_1 = 2$ and $\deg Y_0 = \deg Y_1 = 3$.

Let $\bar{x}_i$ and $\bar{y}_i$ $(i = 0, 1)$ be preimages of x_i in $H^0(X, -2Z)$ and y_i in $H^0(X, -3Z)$, respectively. Then $\bar{y}_0, \bar{y}_1$ generate $H^0(X, -3Z)/H^0(X, \mathfrak{m}^2\mathcal{O}_X)$. Hence

$$\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \frac{H^0(X, \mathfrak{m}\mathcal{O}_X)}{H^0(X, \mathfrak{m}^2\mathcal{O}_X)} = \dim \frac{H^0(X, -2Z)}{H^0(X, -3Z)} + 2 = h^0(Z, -2Z) + 2$$

and we get $\operatorname{embdim}(V, o) = 4$ as wished.

Case 2. We assume that $H^0(X, -2Z) \to H^0(X, -Z)$ is not surjective. Let $s \in H^0(Z, -Z)$ be a non-zero element coming from $H^0(X, -Z)$. It follows from [1, (A.5) Proposition] that s does not vanish identically on any components of Z, since Z is 2-connected and $-Z^2 = 1$. Hence s vanishes at only one point $x \in Z$ which should be a non-singular point of Z. We have $\mathcal{O}_Z(-Z) \simeq \mathcal{O}_Z(x)$. By the Δ-inequality [3], we have $h^0(Z, -Z) \leq 2$ and, if $h^0(Z, -Z) = 2$, the component A containing x is $\mathbb{P}^1$ and $h^0(Z-A, \omega_{Z-A}) = 2$. But the last equality means that A is a fixed component of $|K_Z|$, which is inadequate. So, we conclude that $h^0(Z, -Z) = 1$ and that $|\mathcal{O}_X(-Z)|$ has no fixed components but has x as a base point. Hence $Z_\mathfrak{m} = Z$ but $\mathfrak{m}\mathcal{O}_X \not\simeq \mathcal{O}_X(-Z)$. We can compute $h^1(Z_K, \mathcal{O}_{Z_K})$ as in the proof of Lemma 3.2 to get $p_g(V, o) = 4$.

Let $\rho : \tilde{X} \to X$ be the blowing-up at x and put $E = \rho^{-1}(x)$. We denote by Z' the proper transform of Z. Then $Z' \in |\rho^*Z - E|$ and ρ gives an isomorphism $Z' \to Z$.

Lemma 3.3. *$\mathcal{O}_{\tilde{X}}(-\rho^*Z - E)$ is π-free.*

Proof. It is easy to see that $-\rho^*Z - E$ is nef on $(\pi \circ \rho)^{-1}(o)$. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{\tilde{X}}(-3\rho^*Z + E) \to \mathcal{O}_{\tilde{X}}(-\rho^*Z - E) \to \mathcal{O}_{2Z'}(-\rho^*Z - E) \to 0.$$

We have $H^1(\tilde{X}, -3\rho^*Z + E) = H^1(\tilde{X}, K_{\tilde{X}}) = 0$. It follows that $H^0(\tilde{X}, -\rho^*Z - E) \to H^0(2Z', -\rho^*Z - E)$ is surjective. We next consider

$$0 \to \mathcal{O}_{Z'}(-2\rho^*Z) \to \mathcal{O}_{2Z'}(-\rho^*Z - E) \to \mathcal{O}_{Z'}(-\rho^*Z - E) \to 0.$$

We have $\mathcal{O}_{Z'}(-2\rho^*Z) \simeq \omega_{Z'}$ and $\mathcal{O}_{2Z'}(-\rho^*Z - E) \simeq \omega_{2Z'}$. Since $h^0(Z, \omega_Z) = 2$ and $h^0(2Z, \omega_{2Z}) = 3$, we see that $H^0(2Z', -\rho^*Z - E) \to H^0(Z', -\rho^*Z - E)$ is non-trivial. Recall that $\mathcal{O}_Z(-Z) \simeq \mathcal{O}_Z(x)$. Then $\mathcal{O}_{Z'}(-\rho^*Z - E) \simeq \mathcal{O}_{Z'}$. In sum, $H^0(\tilde{X}, -\rho^*Z - E) \to H^0(Z', -\rho^*Z - E) \simeq H^0(Z', \mathcal{O}_{Z'})$ is surjective.

By

$$0 \to \mathcal{O}_E(E) \to \mathcal{O}_{2\rho^*Z-E}(-\rho^*Z - E) \to \mathcal{O}_{2Z'}(-\rho^*Z - E) \to 0,$$

we get $H^q(2\rho^*Z - E, -\rho^*Z - E) \simeq H^q(2Z', -\rho^*Z - E)$ for $q = 0, 1$. In particular, $H^0(\tilde{X}, -\rho^*Z - E) \to H^0(2\rho^*Z - E, -\rho^*Z - E)$ is surjective and $h^0(2\rho^*Z - E, -\rho^*Z - E) = 3$.

We consider

$$0 \to \mathcal{O}_{2Z'}(-\rho^*Z - 2E) \to \mathcal{O}_{2\rho^*Z-E}(-\rho^*Z - E) \to \mathcal{O}_E(-E) \to 0$$

to see that $H^0(2\rho^*Z - E, -\rho^*Z - E) \to H^0(E, -E)$ is surjective. Since $h^0(2\rho^*Z - E, -\rho^*Z - E) = 3$, it suffices to show that $h^0(2Z', -\rho^*Z - 2E) = 1$. For this purpose, we consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{Z'}(-2\rho^*Z - E) \to \mathcal{O}_{2Z'}(-\rho^*Z - 2E) \to \mathcal{O}_{Z'}(-\rho^*Z - 2E) \to 0.$$

We have $H^0(Z', -\rho^*Z - 2E) \simeq H^0(Z', -E) = 0$ and $H^0(Z', -2\rho^*Z - E) \simeq H^0(Z', E)$ which is of dimension one. Hence $h^0(2Z', -\rho^*Z - 2E) = 1$ as wished. We have shown that $H^0(2\rho^*Z - E, -\rho^*Z - E) \to H^0(E, -E)$ is surjective, which implies that $\mathcal{O}_{\tilde{X}}(-\rho^*Z - E)$ has no base points also on E. $\square$

The maximal ideal cycle on $\tilde{X}$ is $\rho^*Z + E$ and $\mathfrak{m}\mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(-\rho^*Z - E)$. Hence $\operatorname{mult}(V, o) = -(\rho^*Z + E)^2 = 2$. One can deduce $\operatorname{embdim}(V, o) = 3$ from the general inequality: $\operatorname{embdim}(V, o) \leq \operatorname{mult}(V, o) + 1$. However, we count the embedding dimension by describing $R(Z, -Z) = \bigoplus_{i \geq 0} H^0(Z, -iZ)$. Let $s \in H^0(Z, -Z)$ be a non-zero element. As we saw above, it vanishes at a non-singular point x of Z. We have $H^0(Z, -2Z) \simeq H^0(Z, K_Z)$. Here, we have s^2 and a new element t which does not vanish at x. For $i \geq 3$, we have $h^0(Z, -iZ) = i - 1$. In $H^0(Z, -3Z)$, we have s^3 and st. In $H^0(Z, -4Z) \simeq H^0(Z, 2K_Z)$, we have s^4, s^2t, t^2. In $H^0(Z, -5Z)$, we have s^5, s^3t, st^2 and a new element u which does not vanish at x. In $H^0(Z, -6Z)$, we have s^6, s^4t, s^2t^2, t^3 and su. By the free-pencil-trick, $H^0(Z, -iZ) \otimes H^0(Z, -2Z) \to H^0(Z, -(i+2)Z)$ is surjective for $i \geq 5$. We consider $H^0(Z, -10Z)$. Here, we have 6 elements $s^{10}, s^8t, s^6t^2, s^4t^3, s^2t^4, t^5$ and 3 elements s^5u, s^3tu, st^2u. These are linearly independent. So, u^2 can be expressed as a linear combination of them, that is, we have a relation of the form $u^2 = \varphi(s, t)$ after a suitable change of coordinates, where $\varphi(s, t)$ is a linear combination of the first six elements above. Evaluating it at x, we see that the coefficient of t^5 in $\varphi(s, t)$ is non-zero. It is not so hard to see that there are no further relations among s, t, u. Therefore, $R(Z, -Z) \simeq \mathbb{C}[S, T, U]/(U^2 - T^5 - S^2\Phi(S, T))$ as graded $\mathbb{C}$-algebras, where $\deg S = 1, \deg T = 2, \deg U = 5$ and $\Phi(S, T)$ is a weighted homogeneous form of degree 8.

Using the above, we can show $\mathrm{embdim}(V, o) = 3$ as in the previous case. One finds the hypersurface singularity defined by $x^2 + y^5 + z^{10} = 0$ among typical examples.

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