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Decay estimates of a nonnegative Schrödinger heat semigroup

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1 Introduction

In this note we consider the decay estimate (L^p-L^q estimate) of a solution to the Cauchy problem for the heat equation with a potential,

$$\text{(1.1)} \qquad \begin{cases} \partial_t u = \Delta u - V(|x|)u & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $\partial_t = \partial/\partial t$, $N \geq 3$, $\phi \in L^2(\mathbf{R}^N)$, and $V = V(|x|)$ is a smooth, nonpositive, and radially symmetric function satisfying

$$\text{(1.2)} \qquad V(x) = \omega|x|^{-2}(1+o(1)) \quad \text{as} \quad |x| \to \infty,$$

where $\omega \in (-\omega_*, 0]$ and $\omega_* = (N-2)^2/4$. In the context of the Schrödinger semigroup, the "positivity" and "negativity" are discussed in the form $-(\Delta - V) = -\Delta + V$. We say that the operator $H := -\Delta + V$ is nonnegative on $L^2(\mathbf{R}^N)$ (we write $H \geq 0$), if and only if

$$\int_{\mathbf{R}^N} \left\{|\nabla\varphi|^2 + V(|x|)\varphi^2\right\} dx \geq 0$$

holds for any $\varphi \in C_0^\infty(\mathbf{R}^N)$. Here, we discuss the decay rate of a solution under the condition that $-\Delta + V$ is nonnegative. Due to the Hardy inequality, we easily see that $-\Delta + V$ is not nonnegative if $\omega < -\omega_*$.

The behavior of the solution u of (1.1) heavily depends on the behavior of the potential V, in particular, the constant ω in (1.2). It is of interest to study the relationship between the large time behavior of u and the constant ω, and we descuss the decay rate of $L^q(\mathbf{R}^N)$-norm ($q \geq 2$) of the solution u as $t \to \infty$.

Let $H := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$, where $V \in L^p_{\text{loc}}(\mathbf{R}^N)$ with $p > N/2$.We itroduce the notion of criticality of the operator H.

(i) H is said to be subcritical if for any $W \in C_0^\infty(\mathbf{R}^N)$, there holds $H - \epsilon W \geq 0$ for small enough $\epsilon > 0$.

(ii) H is said to be strongly subcritical if $H - \epsilon V_- \geq 0$ for small enough $\epsilon > 0$, where $V_- = \max\{-V, 0\}$.

(iii) H is said to be critical if H is nonnegative and is not subcritical.

(iv) H is said to be supercritical if H is not nonnegative.

The subcriticality and the criticality of the Schrödinger operator H as well as the structure and the behavior of harmonic functions for the operator H have been studied intensively in many papers and in various directions, for examples, see [2], [11], [12], [15]–[17], [19]–[21], [22], [23], [24], [25], and references therein. Among others, Davies and Simon [2] considered the subcritical Schrödinger operator $H = -\Delta + V$ satisfying (1.2), and study the decay rates of $\|e^{-tH}\|_{q,p}$ as $t \to \infty$ by using the positive harmonic functions for the operator H. Here $\|e^{-tH}\|_{q,p}$ is the norm of the operator e^{-tH} from $L^p(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$, where $1 \leq p \leq q \leq \infty$. In particular, if H is strongly subcritical and if H has a positive harmonic function η satisfying

$$C_1^{-1}(1+|x|)^\alpha \leq \eta(x) \leq C_1(1+|x|)^\alpha \quad \text{with} \quad -\frac{N-2}{2} < \alpha < 0$$

as $|x| \to \infty$ for some constant C_1, then, Davies and Simon proved that there holds

$$C_2^{-1} t^{-\frac{N}{4}-\frac{\alpha}{2}-\epsilon} \leq \|e^{-tH}\|_{\infty,2} \leq C_2 t^{-\frac{N}{4}-\frac{\alpha}{2}+\epsilon} \tag{1.3}$$

as $t \to \infty$ for any ϵ and some constant $C_2 > 0$ (see [2, Theorem 14]). As far as we know, it is open whether (1.3) holds with $\epsilon = 0$ or not and there are no results giving exact power decay rates of $\|e^{-tH}\|_{q,p}$ as $t \to \infty$ for the critical case.

In this paper we consider a nonnegative Schrödinger operator $H = -\Delta + V$, where V is a radially symmetric nonpositive function satisfying (1.2), and give the decay rate of $\|e^{-tH}\|_{q,2}$ with $q \geq 2$ as $t \to \infty$ for both of the subcritical case and the critical case. In particular, for the subcritical case, we prove that (1.3) holds with $\epsilon = 0$. Upper estimates of $\|e^{-tH}\phi\|_{q,2}$ are given by the use of the behavior of positive harmonic functions for the operator H at the space infinity and the comparison principle. Lower estimates of $\|e^{-tH}\phi\|_{q,2}$ are given as byproducts of the study of the large time behavior of the solution of (1.1).

It is of independent interest to study the large time behavior of hot spots (the set of maxi,um points of u at time t) for the solution of (1.1). The movement of hot spots for the heat equation in unbounded domains was first studied by Chavel and Karp [1]. They studied the movement of hot spots for the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds. Subsequently the movement of hot spots has been studied in several papers, see [10], [3], [4], and [6]–[8]. Among others, in [6]–[8] the authors of this paper studied the movement of hots spots of the solution of the heat equation (1.1) with a potential V for the case where V is a nonnegative function satisfying (1.2) with $\omega \geq 0$. In this case the hot spots move to the space infinity as $t \to \infty$, and they gave the rate and the direction for hot spots to tend to the space infinity. In this case, we can also discuss the behavior of hot spots, however, we will not investigate that in this paper. Full discussions on this topic will appear in our forthcoming paper [9].

From now on, in addition to (1.2), we further assume the following:

$$(V)\quad \begin{cases} \text{(i)} \quad V = V(r) \in C([0,\infty)) \text{ and } V \le 0 \ (\not\equiv 0) \text{ on } [0,\infty); \\ \text{(ii)} \quad \text{there exist constants } \omega \in (-\omega_*, 0] \text{ and } \theta > 0 \text{ such that} \\ \qquad\qquad V(r) = \omega r^{-2} + O(r^{-2-\theta}) \quad \text{as } r \to \infty. \end{cases}$$

Let $\alpha_N(\omega)$ and $\beta_N(\omega)$ be the roots of the algebraic equation $x(x+N-2) = \omega$ such that $\beta_N(\omega) < \alpha_N(\omega)$, that is,

$$(1.4)\quad \alpha_N(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2}, \quad \beta_N(\omega) = \frac{-(N-2) - \sqrt{(N-2)^2 + 4\omega}}{2}.$$

We remark that

$$(1.5)\qquad -(N-2) \le \beta_N(\omega) < -\frac{N-2}{2} < \alpha_N(\omega) \le 0$$

and that the functions $r^{\alpha_N(\omega)}$ and $r^{\beta_N(\omega)}$ are solutions of the ordinary differential equation

$$(1.6)\qquad U'' + \frac{N-1}{r}U' - \frac{\omega}{r^2}U = 0 \quad \text{in} \quad (0,\infty).$$

Furthermore we remark that

$$(1.7)\qquad \alpha_{N+2k}(\omega) + k = \alpha_N(\omega + \omega_k)$$

holds for all $k = 0, 1, 2, \ldots$.

We introduce some notation. For $1 \le p \le \infty$, we denote by $\|\cdot\|_p$ the norm of the $L^p(\mathbf{R}^N)$ space. We also denote by $\|\cdot\|$ the norm of the $L^2(\mathbf{R}^N)$ space with weight $e^{|x|^2/4}$, that is, $L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$. Let $B(0,R) = \{x \in \mathbf{R}^N : |x| < R\}$ for $R > 0$. Let $\Delta_{\mathbf{S}^{N-1}}$ be the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$ and $\{\omega_k\}_{k=0}^{\infty}$ the eigenvalues of

$$(1.8)\qquad -\Delta_{\mathbf{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \mathbf{S}^{N-1}, \qquad Q \in L^2(\mathbf{S}^{N-1}),$$

that is,

$$(1.9)\qquad \omega_k = k(N+k-2), \qquad k = 0, 1, 2, \ldots.$$

Furthermore let $\{Q_{k,i}\}_{i=1}^{l_k}$ and l_k be the orthonormal system and the dimension of the eigenspace corresponding to ω_k, respectively. In particular, $l_0 = 1$, $l_1 = N$, and we may write

$$(1.10)\qquad Q_{0,1}\left(\frac{x}{|x|}\right) = \kappa_0, \qquad Q_{1,i}\left(\frac{x}{|x|}\right) = \kappa_1 \frac{x_i}{|x|}, \quad i = 1, \ldots, N,$$

where κ_0 and κ_1 are positive constants.

As be pointed in [2], the behavior of $\|e^{-tH}\|_{q,2}$ depends on the behavior of positive harmonic functions at the space infinity. Before stating our results on the decay rate of $\|e^{-tH}\|_{q,2}$, we first give one theorem on the behavior of positive harmonic functions for the operator $H := -\Delta + V$.

Theorem 1.1 *Let $N \geq 3$. Assume condition (V), and let $H := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$. Then, for any $k = 0, 1, 2, \dots$, there exists a positive solution $U_{N,k}$ of*

$$(1.11) \qquad U'' + \frac{N-1}{r}U' - \left(V(r) + \frac{\omega_k}{r^2}\right)U = 0 \quad in \quad (0, \infty)$$

such that

$$(1.12) \qquad U_{N,k}(r) = d_{N,k} r^k (1 + o(1)) \quad as \quad r \to 0,$$

$$(1.13) \qquad U_{N,k}(r) = r^{A_{N,k}}(1 + o(1)) \quad as \quad r \to \infty,$$

where $d_{N,k}$ is a positive constant and

$$A_{N,k} = \begin{cases} \alpha_N(\omega) & \text{if } k = 0 \text{ and } H \text{ is subcritical,} \\ \beta_N(\omega) & \text{if } k = 0 \text{ and } H \text{ is critical,} \\ \alpha_N(\omega + \omega_k) & \text{if } k \geq 1. \end{cases}$$

Furthermore $r^{-k}U_k(r)$ is monotone decreasing in $[0, \infty)$ and

$$(1.14) \qquad U'_{N,k}(r) = \begin{cases} O(r) & as \quad r \to 0 \quad if \quad k = 0, \\ O(r^{k-1}) & as \quad r \to 0 \quad if \quad k \geq 1, \end{cases}$$

$$(1.15) \qquad U'_{N,k}(r) = (A_{N,k} + o(1))r^{A_{N,k}-1} \quad as \quad r \to \infty.$$

We remark that the function $U_{N,0} = U_{N,0}(|x|)$ is a positive harmonic function for the operator $H = -\Delta + V$ on $L^2(\mathbf{R}^N)$ and plays an important role in our study.

Remark 1.1 *Murata* [17] *investigated the structure and the behavior of positive harmonic functions for nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$. His results are applicable to problem* (1.1) *under assumption (V), and the existence of positive solutions of* (1.11) *satisfying* (1.12) *and* (1.13) *for the case $k = 0$ can be proved as a direct consequence of* [17, Theorem 5.7]. *See also* [11, Remark 2.2] *for the case where $\omega = 0$ and $k = 0$.*

Remark 1.2 *By* (1.11) *we see that*

$$(1.16) \qquad U_{N,k}(r) = r^k U_{N+2k,0}(r)$$

holds for all $r \geq 0$, where $k = 1, 2, \dots$. See also Lemma 2.1 *and* (1.7).

Also, for any $k = 0, 1, 2, \dots$ and $i = 1, \dots, l_k$, the function

$$\mathcal{U}_{k,i}(x) := U_k(|x|)Q_{k,i}(x/|x|)$$

is a harmonic function for the operator H. Furthermore, for the solution u of (1.1),

$$\frac{d}{dt}\int_{\mathbf{R}^N} u(x,t)\mathcal{U}_{k,i}(x)dx = 0$$

holds for all $t > 0$ under a suitable integrability condition on the solution u.

In what follows, we use

$$\alpha(\omega) = \alpha_N(\omega),\ \beta(\omega) = \beta_N(\omega),\ U_k(r) = U_{N,k}(r),\ A = A_{N,0}.$$

For any sets Λ and Σ, let $f = f(\lambda, \sigma)$ and $h = h(\lambda, \sigma)$ be maps from $\Lambda \times \Sigma$ to $(0, \infty)$. Then we say

$$f(\lambda, \sigma) \preceq h(\lambda, \sigma)$$

for all $\lambda \in \Lambda$ if, for any $\sigma \in \Sigma$, there exists a positive constant C such that $f(\lambda, \sigma) \leq Ch(\lambda, \sigma)$ for all $\lambda \in \Lambda$. In addition, we say $f(\lambda, \sigma) \asymp h(\lambda, \sigma)$ for all $\lambda \in \Lambda$ if $f(\lambda, \sigma) \preceq h(\lambda, \sigma)$ and $f(\lambda, \sigma) \succeq h(\lambda, \sigma)$ for all $\lambda \in \Lambda$.

Now we give a result on the decay rate of $\|e^{-tH}\|_{q,2}$ $(q \geq 2)$ as $t \to \infty$. We recall that

$$\sup_{t>0} \|e^{-tH}\|_{2,2} \leq 1. \tag{1.17}$$

holds if H is nonnegative.

Theorem 1.2 *Let $N \geq 3$. Assume condition (V), and let $H := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$. Let $A := A_{N,0}$ be the constant given in Theorem 1.1. Then there holds the following:*

(i) *if $A > -N/2$, then*

$$\|e^{-tH}\|_{q,2} \preceq \begin{cases} t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} & \text{if } \ qA + N > 0, \\ t^{-\frac{N}{4}-\frac{A}{2}}(\log t)^{\frac{1}{q}} & \text{if } \ qA + N = 0, \\ t^{-\frac{N}{4}-\frac{A}{2}} & \text{if } \ qA + N < 0, \end{cases} \tag{1.18}$$

for all sufficiently large t;

(ii) *if $A \leq -N/2$, then*

$$\|e^{-tH}\|_{q,2} \preceq t^{-\frac{N+2A}{2-N-2A}(\frac{1}{2}-\frac{1}{q})} \tag{1.19}$$

for all sufficiently large t.

We remark that, if H is subcritical, then $A = \alpha(\omega) > -N/2$ and (1.18) holds with $A = \alpha(\omega)$. In the following theorem we assume

$$(V') \quad \begin{cases} \text{(i)} & V = V(r) \in C^1([0,\infty)) \text{ and } V \text{ satisfies condition } (V) \text{ for some} \\ & \text{constants } \omega \in (-\omega_*, 0] \text{ and } \theta > 0; \\ \text{(ii)} & \sup_{r \geq 1} \left| r^3 V'(r) \right| < \infty, \end{cases}$$

instead of (V), and prove that if H is subcritical, then the decay rate (1.18) is optimal.

Theorem 1.3 *Let $N \geq 3$. Assume condition (V'), and let $H := -\Delta + V$ be a subcritical Schrödinger operator on $L^2(\mathbf{R}^N)$. Then, for any $q \in [2, \infty]$, there holds*

$$\|e^{-tH}\|_{q,2} \asymp \begin{cases} t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} & \text{if } \ q\alpha(\omega) + N > 0, \\ t^{-\frac{N}{4}-\frac{\alpha(\omega)}{2}}(\log t)^{\frac{1}{q}} & \text{if } \ q\alpha(\omega) + N = 0, \\ t^{-\frac{N}{4}-\frac{\alpha(\omega)}{2}} & \text{if } \ q\alpha(\omega) + N < 0, \end{cases} \tag{1.20}$$

for all $t \geq 1$.

Theorem 1.3 with $q = \infty$ implies that, under assumption (V'), inequality (1.3) holds with $\epsilon = 0$.

Next, we state a result on the large time behavior of the solution u of (1.1). Assume condition (V') and let $U_k := U_{N,k}$ $(k = 0, 1, 2, \dots)$ be a solution of (1.11), having the properties described in Theorem 1.1. Put

$$M_0 = \int_{\mathbf{R}^N} \phi(x) U_0(|x|) dx \tag{1.21}$$

Furthermore, for any $k = 0, 1, 2, \dots$, since $\alpha(\omega + \omega_k) > -N/2$, we can define $\varphi_{N,k}$ by

$$\varphi_{N,k}(y) = c_{N,k} |y|^{\alpha_N(\omega+\omega_k)} e^{-|y|^2/4},$$

where $c_{N,k}$ is a positive constant such that $\|\varphi_{N,k}\| = 1$. Here, by (1.7) we have

$$c_{N,k} = c_{N+2k,0}, \qquad \varphi_{N,k}(y) = |y|^k \varphi_{N+2k,0}(y). \tag{1.22}$$

We write $\varphi_k = \varphi_{N,k}$ and $c_k = c_{N,k}$ for simplicity. We give a result on the large time behavior of solution of (1.1).

Theorem 1.4 *Let $N \geq 3$. Assume condition (V'), and let $H := -\Delta + V$ be a subcritical Schrödinger operator on $L^2(\mathbf{R}^N)$. Let u be a solution of* (1.1) *with the initial function $\phi(x) \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$. Then there exists a constant C such that*

$$\|u(t)\|_2 \leq C t^{-\frac{N}{4} - \frac{\alpha(\omega)}{2}} \|\phi\|_{L^2(\mathbf{R}^N, e^{|x|^2/4}dx)}, \qquad t \geq 1. \tag{1.23}$$

Furthermore there hold

$$\lim_{t\to\infty} \sup_{x \in B(0,L)} \left| t^{\frac{N}{2}+\alpha(\omega)} u(x,t) - c_0^2 M_0 U_0(x,t) \right| = 0, \quad L > 0, \tag{1.24}$$

and

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u\left((1+t)^{\frac{1}{2}} y, t\right) = c_0 M_0 \varphi_0(y) \quad \text{in} \quad L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N, e^{|y|^2/4}dy). \tag{1.25}$$

The rest of this paper is organized as follows. In Section 2 we study some properties of the functions U_k of the ordinary differential equation (1.11), and prove Theorem 1.1. In Section 3 we prove Theorem 1.2 by using some supersolutions of (1.1), which are constructed by the function U_0. In Section 4 we study the large time behavior of radially symmetric solutions of (1.1), and in Section 5 we prove Theorems 1.3 and 1.4. We will not give proofs to all the lemmas. Their proofs will appear in the forthcoming paper [9].

2 Positive harmonic functions

In this section we study the behavior of the positive harmonic functions for nonnegative Schrödinger operators, and prove Theorem 1.1.

Assume $V \in C([0,\infty))$ and consider the Schrödinger operator $H := -\Delta + V(|x|)$. We first recall some properties for the operator H, see [17]. By the standard arguments for ordinary differential equations, we see that there exists a unique solution of U of

$$\text{(O)} \qquad U'' + \frac{N-1}{r}U' - V(r)U = 0 \quad \text{in} \quad (0,\infty)$$

with

$$\text{(2.1)} \qquad \lim_{r\to 0} U(r) = 1.$$

Then we have the following properties:

(P_1) $H \geq 0$, that is, $H = -\Delta + V(|x|)$ is a nonnegative operator on $L^2(\mathbf{R}^N)$ if and only if $U(r) > 0$ on $[0,\infty)$;

(P_2) Assume $H \geq 0$. Then there exists a positive constant c such that

$$U(r) = cr^A(1 + o(1))$$

as $r \to \infty$, where $A = \alpha(\omega)$ if H is subcritical and $A = \beta(\omega)$ if H is critical;

(P_3) Assume that H is critical. Then, for any nonnegative function $W \in C_0([0,\infty))$ with $W \not\equiv 0$, there holds

$$\begin{cases} H + W \text{ is subcritical,} \\ H - W \text{ is supercritical.} \end{cases}$$

See Theorems 2.5, 3.1, and 5.7 in [17]. In addition, by the same argument as in [6] we have

(P_4) for any solution $\tilde{U}$ of (O) satisfying $\limsup_{r\to 0} |\tilde{U}(r)| < \infty$, there exists a constant c' such that $\tilde{U}(r) = c'U(r)$ on $[0,\infty)$.

Under the assumptions of Theorem 1.1, properties (P_1) and (P_2) ensure the existence of the function $U_0 = U_0(r)$ satisfying (1.12) and (1.13) for the case $k = 0$.

Next we assume condition (V), and study the behavior of $U_0'(r)$ as $r \to 0$ and $r \to \infty$. Let $k = 0$. Let $d_0 := d_{N,0}$ be the constant given in (1.12). Since the function

$$U_0(0) + \int_0^r s^{1-N}\left(\int_0^s \tau^{N-1}V(\tau)U_0(\tau)d\tau\right)dr$$

is also a solution of (O). Property (P_4) implies

$$\text{(2.2)} \qquad U_0(r) = U_0(0) + \int_0^r s^{1-N}\left(\int_0^s \tau^{N-1}V(\tau)U_0(\tau)d\tau\right)dr \quad \text{on} \quad [0,\infty).$$

Then we have

$$\text{(2.3)} \qquad U_0'(r) = r^{1-N}\int_0^r \tau^{N-1}V(\tau)U_0(\tau)d\tau \leq (\not\equiv)\, 0 \quad \text{on} \quad [0,\infty),$$

$$\text{(2.4)} \qquad U_0'(r) = \frac{V(0)U_0(0)}{N}r(1 + o(1)) \quad \text{as} \quad r \to 0.$$

In particular, (2.4) yields (1.14) with $k=0$. Furthermore, by (V) (i), (P_1), and (2.3) we see that, if $H=-\Delta+V$ is nonnegative, then $U_0'(r)\le 0$ in $[0,\infty)$. Next we prove (1.15) with $k=0$.

Proof of (1.15) **with** $k=0$. The function

$$v(r):=\frac{-\beta(\omega)U_0(1)+U_0'(1)}{\alpha(\omega)-\beta(\omega)}r^{\alpha(\omega)}+\frac{\alpha(\omega)U_0(1)-U_0'(1)}{\alpha(\omega)-\beta(\omega)}r^{\beta(\omega)}$$

is also a solution of (1.6) such that $v(1)=U_0(1)$ and $v'(1)=U_0'(1)$. On the other hand, the function

$$G(r):=r^{\beta(\omega)}\int_1^r s^{1-N-2\beta(\omega)}\left(\int_1^s \tau^{N-1+\beta(\omega)}\left(V(\tau)-\frac{\omega}{\tau^2}\right)U_0(\tau)d\tau\right)ds. \tag{2.5}$$

satisfies

$$G''+\frac{N-1}{r}G'-\frac{\omega}{r^2}G=\left(V(r)-\frac{\omega}{r^2}\right)U_0(r)\quad\text{in}\quad(0,\infty),\quad G(1)=G'(1)=0.$$

Then the uniqueness theorem for ordinary differential equations implies

$$U_0(r)=v(r)+G(r),\qquad t\ge 1. \tag{2.6}$$

Furthermore, since $U_0(r)=O(r^{\alpha(\omega)})$ as $r\to\infty$, by condition (V) (ii) and (1.4) we have

$$\begin{aligned}&\int_s^\infty \tau^{N-1+\beta(\omega)}\left|V(\tau)-\frac{\omega}{\tau^2}\right|U_0(\tau)d\tau\\&\le C_1\int_s^\infty \tau^{N-1+\beta(\omega)}\tau^{-2-\theta}\tau^{\alpha(\omega)}d\tau=C_1\int_s^\infty\tau^{-1-\theta}d\tau\le C_2s^{-\theta}\le C_2\end{aligned} \tag{2.7}$$

for all $s\ge 1$, where C_1 and C_2 are constants. Then, since $2-N-\beta(\omega)=\alpha(\omega)$, by (2.5), (2.6), and (2.7) we have

$$U_0(r)=a_2r^{\alpha(\omega)}+b_2r^{\beta(\omega)}+O(r^{\alpha(\omega)-\theta})+O(r^{\beta(\omega)}) \tag{2.8}$$

as $r\to\infty$ for some constants a_1, a_2, b_1, and b_2.

Assume that $H=-\Delta+V$ is subcritical. Then (1.13) implies $a_2=1$. Furthermore, by (2.7) and (2.8) we have

$$U_0'(r)=\alpha(\omega)r^{\alpha(\omega)-1}+b_2\beta(\omega)r^{\beta(\omega)-1}+O(r^{\alpha(\omega)-\theta-1})$$

as $r\to\infty$. So we have (1.15) with $k=0$.

Next assume that $H=-\Delta+V$ is critical. Then, since $U_0(r)=O(r^{\beta(\omega)})$ as $r\to\infty$, by (V) (ii) we have

$$\int_s^\infty \tau^{N-1+\beta(\omega)}\left|V(\tau)-\frac{\omega}{\tau^2}\right|U_0(\tau)d\tau=O(s^{N-2-\theta+2\beta(\omega)})\quad\text{as}\quad s\to\infty,$$

instead of (2.7). Therefore, by (2.8) we have

$$U_0(r)=a\alpha(\omega)r^{\alpha(\omega)-1}+b'\beta(\omega)r^{\beta(\omega)-1}+O(r^{\beta(\omega)-1})$$

as $r\to\infty$ for some constant b'. Then, by (1.13) we have $a=0$ and $b'=1$, and obtain (1.15) with $k=0$. Therefore the proof of (1.15) with $k=0$ is complete. □

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1 *Assume condition* (V) *and let* $H_N := -\Delta + V(|x|)$ *be nonnegative operator on* $L^2(\mathbf{R}^N)$. *Then, for any* $k = 1, 2, \ldots$, *the operator*

$$H_{N+k} := -\Delta_{N+k} + V(|x|)$$

is subcritical as an operator on $L^2(\mathbf{R}^{N+k})$, *where* Δ_{N+k} *is the* $(N+k)$*-dimensional Laplacian.*

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to prove Theorem 1.1 for the case $k \geq 1$. Let $k = 1, 2, \ldots$. Since $H_{N+2k} := -\Delta_{N+2k} + V$ is subcritical by Lemma 2.1, Theorem 1.1 with $k = 0$ implies the existence of the positive function $U_{N+2k,0}(r)$ satisfying (1.11)–(1.15) with k, N, and $A_{N,k}$ replaced by 0, $N+2k$, and $\alpha_{N+2k}(\omega)$, respectively. Furthermore the function $U_{N+2k,0}(r)$ is monotone decreasing in $[0, \infty)$.

Put $U_k(r) = r^k U_{N+2k,0}(r)$. Then we easily see that U_k satisfies (1.11), (1.12), and (1.14) and that $r^{-k}U_k(r)$ is monotone decreasing in $[0, \infty)$. Furthermore, by (1.7) we also see that U_k satisfies (1.12) and (1.15). Thus the proof of Theorem 1.1 for the case $k \geq 1$ is complete, and Theorem 1.1 follows. □

At the end of this section we give two lemmas, which are used in the proof of Theorem 1.4.

Lemma 2.2 *Assume condition* (V) *and let* $H := -\Delta + V$ *be nonnegative operator on* $L^2(\mathbf{R}^N)$. *Let* $f \in C([0, \infty))$ *and* v *be a solution of*

$$U'' + \frac{N-1}{r}U' - V(r)U = f \quad \textit{in} \quad (0, \infty), \tag{2.9}$$

such that $\limsup_{r \to 0} |v(r)| < \infty$. *Then there exists a constant* c *such that*

$$v(r) = cU_0(r) + F[f](r), \qquad r \geq 0, \tag{2.10}$$

where

$$F[f](r) = U_0(r) \int_0^r s^{1-N}[U_0(s)]^{-2} \left(\int_0^s \tau^{N-1} U_0(\tau) f(\tau) d\tau \right) ds.$$

Lemma 2.3 *Let* $\omega < 0$ *and* $k = 1, 2, \ldots$. *Then*

$$\alpha(\omega + \omega_k) > \alpha(\omega) + k.$$

3 Proof of Theorem 1.2

Assume condition (V) and let $H := -\Delta + V$ be nonnegative operator on $L^2(\mathbf{R}^N)$. In this section we construct supersolutions of (1.1), and give upper bounds of the solution of (1.1). Furthermore we prove Theorem 1.2.

For any $\epsilon > 0$, let $h_\epsilon = h_\epsilon(t)$ be a function in $[0, \infty)$ defined by

$$\epsilon(1+t) = \int_0^{h_\epsilon(t)} s^{1-N}[U_0(s)]^{-2} \left(\int_0^s \tau^{N-1} U_0(\tau)^2 d\tau \right) ds. \tag{3.1}$$

Then $h_\epsilon(t)$ is a positive and increasing function in $[0,\infty)$, and by Theorem 1.1 we have

$$(3.2)\qquad h_\epsilon(t) \asymp \begin{cases} (\epsilon(1+t))^{1/2} & \text{if} \quad A > -N/2, \\ (\epsilon(1+t))^{1/2}(\log(2+t))^{-1/2} & \text{if} \quad A = -N/2, \\ (\epsilon(1+t))^{1/(2-N-2A)} & \text{if} \quad A < -N/2, \end{cases}$$

for all $t > 0$. For any $\epsilon > 0$ and $T \ge 0$, let

$$\begin{aligned} D_\epsilon(T) &= \left\{(x,t) \in \mathbf{R}^N \times (T,\infty) \,:\, |x| < h_\epsilon(t)\right\}, \\ \Gamma_\epsilon(T) &= \left\{(x,t) \in \mathbf{R}^N \times (T,\infty) \,:\, |x| = h_\epsilon(t)\right\} \\ &\qquad \cup \left\{(x,T) \in \mathbf{R}^N \times \{T\} \,:\, |x| < h_\epsilon(T)\right\}. \end{aligned}$$

In this section we construct a supersolution W of problem (1.1), and prove Theorem 1.2.

We first construct supersolutions of problem (1.1).

Lemma 3.1 *Assume condition* (V) *and let* $H := -\Delta + V(|x|)$ *be nonnegative operator on* $L^2(\mathbf{R}^N)$. *Let* $\gamma_1 \ge 0$, $\gamma_2 \ge 0$, *and* A *be the constant given in Theorem* 1.1. *Put*

$$\zeta(t) = \begin{cases} (1+t)^{-\gamma_1-\frac{A}{2}}[\log(2+t)]^{-\gamma_2} & \textit{if} \quad A > -N/2, \\ (1+t)^{-\gamma_1-\frac{A}{2}}[\log(2+t)]^{-\gamma_2+\frac{A}{2}} & \textit{if} \quad A = -N/2, \\ (1+t)^{-\gamma_1-\frac{A}{2-N-2A}}[\log(2+t)]^{-\gamma_2} & \textit{if} \quad A < -N/2. \end{cases}$$

Then, for any sufficiently small $\epsilon > 0$ *and any* $T > 0$, *there exist a constant* C *and a function* $W(x,t)$ *such that*

$$(3.3)\qquad \partial_t W \ge \Delta W - V(|x|)W \quad \textit{in} \quad \mathbf{R}^N \times (0,\infty),$$

$$(3.4)\qquad W(x,t) \le C\zeta(t)U_0(|x|) \quad \textit{in} \quad D_\epsilon(T),$$

$$(3.5)\qquad W(x,t) \ge (1+t)^{-\gamma_1}[\log(2+t)]^{-\gamma_2} \quad \textit{in} \quad \Gamma_\epsilon(T).$$

Proof. Let $T > 0$, $\gamma_1 \ge 0$, and $\gamma_2 \ge 0$. Let κ be a positive constant such that

$$(3.6)\qquad |\zeta'(t)| \le \kappa(1+t)^{-1}\zeta(t), \qquad t > 0.$$

Let $F[U_0](|x|)$ be the function given in Lemma 2.2. By (3.1) we can take a sufficiently small $\epsilon_0 > 0$ so that

$$(3.7)\qquad 0 \le \frac{\kappa}{1+t}F[U_0](|x|) \le \epsilon\kappa U_0(|x|) \le \frac{1}{2}U_0(|x|)$$

for all $(x,t) \in D_\epsilon(T)$ and $0 < \epsilon < \epsilon_0$. Let

$$(3.8)\qquad W(x,t) = C\zeta(t)\left[U_0(|x|) - \kappa(1+t)^{-1}F[U_0](|x|)\right],$$

where C is a constant to be chosen suitably. Then $W(x,t)$ is the desired supersolution. □

Next we give a lemma on pointwise estimates of the solution $u = e^{-tH}\phi$ of (1.1) by use of Lemma 3.1.

Lemma 3.2 *Assume condition* (V). *Let* ϵ *be a sufficiently small positive constant and* $T>0$. *Let* $u=e^{-tH}\phi$ *be a solution of* (1.1) *such that*

$$\|u(t)\|_2 \le C_1(1+t)^{-d}\|\phi\|_2, \qquad t>0, \tag{3.9}$$

for some constant $C_1>0$ *and* $d\ge 0$. *Then there exists a constant* C_2 *such that*

$$|u(x,t)| \le C_2\|\phi\|_2 \times \begin{cases} (1+t)^{-d-\frac{N}{4}} & \text{if } A>-N/2, \\ (1+t)^{-d-\frac{N}{4}}[\log(2+t)]^{\frac{N}{4}} & \text{if } A=-N/2, \\ (1+t)^{-d-\frac{N}{2(2-N-2A)}} & \text{if } A<-N/2, \end{cases} \tag{3.10}$$

for all $x\in\mathbf{R}^N$ *and* $t>0$ *with* $|x|\ge h_\epsilon(t)$. *Furthermore, for any* $T>0$, *there exists a constant* C_3 *such that*

$$|u(x,t)| \le C_3\|\phi\|_2 U_0(|x|) \times \begin{cases} (1+t)^{-d-\frac{N}{4}-\frac{A}{2}} & \text{if } A>-N/2, \\ (1+t)^{-d-\frac{N+2A}{2(2-N-2A)}} & \text{if } A\le -N/2, \end{cases} \tag{3.11}$$

for all $(x,t)\in D_\epsilon(T)$.

Proof. Let ϵ be a sufficiently small positive constant. We first prove (3.10). Then we can assume, without loss of generality, that

$$h_\epsilon(t)^2 < t/2$$

for all $t>0$. Let $(x,t)\in\mathbf{R}^N\times(0,\infty)$ with $|x|\ge h_\epsilon(t)$. For any $(z,\tau)\in B(0,1)\times(-1,0)$, we put

$$\tilde{u}(z,\tau) = u(\eta z + x, \eta^2\tau + t), \qquad \eta = h_\epsilon(t)/2. \tag{3.12}$$

Then $\tilde{u}$ satisfies

$$\partial_\tau \tilde{u} = \Delta_z \tilde{u} + \eta^2 V(|\eta z + x|)\tilde{u} \quad \text{in} \quad B(0,1)\times(-1,0). \tag{3.13}$$

We can find constants C_3 and C_4, independent of ϵ, satisfying

$$\eta^2 V(|\eta z + x|) \le \frac{C_3\eta^2}{|\eta z + x|^2} \le C_4, \qquad z\in B(0,1) \tag{3.14}$$

to obtain

$$|\tilde{u}(0,0)| \le C_5 \sup_{-1<\tau<0} \|\tilde{u}(\tau)\|_{L^2(B(0,1))}$$

for some constant C_5. This together with (3.2) and (3.9) implies

$$\begin{aligned} |u(x,t)| &= |\tilde{u}(0,0)| \preceq \eta^{-\frac{N}{2}} \sup_{t-\eta^2<s<t} \|u(s)\|_{L^2(B(x,\eta))} \\ &\preceq h_\epsilon(t)^{-\frac{N}{2}} \sup_{t/2<\tau<t} \|u(\tau)\|_2 \preceq (1+t)^{-d} h_\epsilon(t)^{-\frac{N}{2}} \|\phi\|_2 \end{aligned} \tag{3.15}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$ with $|x| \geq h_\epsilon(t)$. Then, by (3.2) we have (3.10).

Next we prove (3.11). Let $T > 0$. Put

$$(\gamma_1, \gamma_2) = \begin{cases} (d + N/4, 0) & \text{if} \quad A > -N/2, \\ (d + N/4, -N/4) & \text{if} \quad A = -N/2, \\ (d + N/2(2 - N - 2A), 0) & \text{if} \quad A < -N/2. \end{cases}$$

Let W be the function given in Lemma 3.1. We take a sufficiently large constant C_6, and put

$$\overline{u}(x,t) = C_6 \|\phi\|_2 W(x,t)$$

Then, by Lemma 3.1 and (3.10) we apply the comparison principle to obtain

$$\begin{aligned} |u(x,t)| &\leq \overline{u}(x,t) \\ &\leq C_7 \|\phi\|_2 U_0(|x|) \times \begin{cases} (1+t)^{-d-\frac{N}{4}-\frac{A}{2}} & \text{if} \quad A > -N/2, \\ (1+t)^{-d} & \text{if} \quad A = -N/2, \\ (1+t)^{-d-\frac{N}{2(2-N-2A)}-\frac{A}{2-N-2A}} & \text{if} \quad A < -N/2 \end{cases} \end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$, where C_7 is a constant. Thus we obtain (3.11), and and the proof of Lemma 3.2 is complete. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Put $u = e^{-tH}\phi$. By (1.17) we have

$$\|u(t)\|_2 \leq \|\phi\|_2, \qquad t > 0. \tag{3.16}$$

Let ϵ be a sufficiently small positive constant.

We first consider the case $A \leq -N/2$. By (3.16) we apply Lemma 3.2 with $d = 0$ to obtain

$$\|u(t)\|_\infty \preceq t^{-\frac{N+2A}{2(2-N-2A)}} \|\phi\|_2$$

Therefore, by (3.16) we have

$$\|u(t)\|_q \leq \|u(t)\|_\infty^{1-\frac{2}{q}} \|u(t)\|_2^{\frac{2}{q}} \preceq t^{-\frac{N+2A}{2-N-2A}(\frac{1}{2}-\frac{1}{q})} \|\phi\|_2$$

for all sufficiently large t, where $q \in [2,\infty]$. So we have (1.19), and the proof of Theorem 1.2 for the case $A \leq -N/2$ is complete.

Next we consider the case $A > -N/2$. By (3.16) we apply (3.10) with $d = 0$ to obtain

$$\|u(t)\|_{L^\infty(\{|x|>h_\epsilon(t)\})} \preceq t^{-\frac{N}{4}} \|\phi\|_2$$

for all sufficiently large t. Therefore, by (3.16) we have

$$\|u(t)\|_{L^q(\{|x|>h_\epsilon(t)\})} \leq \|u(t)\|_{L^\infty(\{|x|>h_\epsilon(t)\})}^{1-\frac{2}{q}} \|u(t)\|_2^{\frac{2}{q}} \preceq t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{q})} \|\phi\|_2 \tag{3.17}$$

for all sufficiently large t, where $q \in [2,\infty]$. Furthermore, by Theorem 1.1, (3.2), and (3.11) with $d=0$ we have

$$\|u(t)\|_{L^q(\{x<h_\epsilon(t)\})} \preceq t^{-\frac{N}{4}-\frac{A}{2}}\|\phi\|_2 \left(\int_{\{|x|<C(\epsilon t)^{1/2}\}} (1+|x|)^{Aq}dx\right)^{1/q}$$
$$\preceq \begin{cases} t^{-\frac{N}{2}(1-\frac{1}{q})}\|\phi\| & \text{if} \quad qA+N>0, \\ t^{-\frac{N}{4}-\frac{A}{2}}(\log t)^{\frac{1}{q}}\|\phi\| & \text{if} \quad qA+N=0, \\ t^{-\frac{N}{4}-\frac{A}{2}}\|\phi\| & \text{if} \quad qA+N<0, \end{cases}$$

for all sufficiently large t, where C is a positive constant. By (3.17) and (3.18) we have (1.18), and the proof of Theorem 1.2 for the case $A>-N/2$ is complete. Thus Theorem 1.2 follows. □

4 Decay estimates of radial solutions

In this section we assume condition (V') and that the $H := -\Delta + V(|x|)$ is a subcritical operator on $L^2(\mathbf{R}^N)$, and consider the radial solutions of problem (1.1). In this case we have

$$A=\alpha(\omega), \qquad h_\epsilon(t) \asymp \epsilon^{1/2}(1+t)^{1/2} \quad \text{for} \quad t>0.$$

We have the following decay astimates. (see also Lemma 3.2 in [6]).

Lemma 4.1 *Assume condition* (V) *and let* $H := -\Delta + V(|x|)$ *be a subcritical operator on* $L^2(\mathbf{R}^N)$. *Let* ϕ *be a radial function such that* $\phi \in L^2(\mathbf{R}^N)$ *and put* $v(t) = e^{-tH}\phi$. *Assume that there exist positive constants* C_* *and* d *such that*

$$\|v(t)\|_2 \le C_* t^{-d}\|\phi\|_2, \qquad t>0. \tag{4.1}$$

Then, for any $j=0,1,2,\dots$, *there exists a constant* C_1 *such that*

$$\|\partial_t^j v(t)\|_2 \le C_1 t^{-d-j}\|\phi\|_2, \qquad t>0. \tag{4.2}$$

Furthermore, for any $T>0$ *and any sufficiently small* $\epsilon>0$, *there exists a constant* C_2 *such that*

$$|(\partial_t^j v)(x,t)| \le C_2 t^{-d-\frac{N}{4}-\frac{\alpha(\omega)}{2}-j} U_0(|x|)\|\phi\|_2, \tag{4.3}$$
$$|(\partial_r v)(x,t)| \le C_2 t^{-d-\frac{N}{4}-\frac{\alpha(\omega)}{2}} \left[|U_0'(|x|)| + t^{-1}|x|U_0(|x|)\right]\|\phi\|_2, \tag{4.4}$$

for all $(x,t) \in D_\epsilon(T)$.

Next we assume $\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$, and study the large time behavior of the solution $v=e^{-tH}\phi$ via the self-similar transformation. Put

$$w(y,s) = (1+t)^{\frac{N}{2}}v(x,t), \quad y=(1+t)^{-1/2}x, \quad s=\log(1+t). \tag{4.5}$$

Since $V \in L^\infty([0,\infty))$, we apply the comparison principle to obtain

$$0 \le |v(y,s)| \le e^{t\|V\|_{L^\infty([0,\infty))}} e^{t\Delta}|\phi|, \qquad (x,t) \in \mathbf{R}^N \times (0,\infty).$$

This together with $\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$ implies that

$$\sup_{0<\tau<s} \|w(\tau)\| < \infty \tag{4.6}$$

for any $s > 0$. Furthermore the function w satisfies

$$\partial_s w = Lw \quad \text{in} \quad \mathbf{R}^N \times (0,\infty), \quad w(y,0) = \phi(|y|) \quad \text{in} \quad \mathbf{R}^N, \tag{L}$$

where

$$Lw = L^* w - \left[e^s V(e^{-\frac{s}{2}} y) - \frac{\omega}{|y|^2} \right] w, \quad L^* w = \frac{1}{\rho} \operatorname{div}(\rho \nabla_y w) - \frac{\omega}{|y|^2} w + \frac{N}{2} w.$$

We next recall the following lemma on the eigenvalue problem for the operator L^*,

$$\begin{cases} L^*\varphi = -\lambda\varphi & \text{in} \quad \mathbf{R}^N, \\ \varphi \text{ is a radial function in } \mathbf{R}^N \text{ with respect to } 0, \\ \varphi \in H^1(\mathbf{R}^N, \rho dy). \end{cases} \tag{E}$$

Lemma 4.2 *Let $\omega \in (-\omega_*, 0]$. Let $\{\lambda_i\}_{i=0}^\infty$ be the eigenvalues of (E) such that $\lambda_0 < \lambda_1 < \dots$. Then all eigenvalues are simple and*

$$\lambda_i = \frac{\alpha(\omega)}{2} + i. \tag{4.7}$$

Furthermore the eigenfunction corresponding to λ_0 is given by the function φ_0 given in Section 1.2.

This lemma is obtained by the same argument as in the proof of Lemma 2.2 in [14].

Next we apply the arguments in Section 3 in [6] with Lemma 4.2, and obtain the following proposition. Let φ_0 and c_0 be the function and the constant given in Section 1, respectively.

Proposition 4.1 *Assume condition (V') and let $H := -\Delta + V(|x|)$ be a subcritical operator on $L^2(\mathbf{R}^N)$. Let ϕ be a radial function such that $\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$ and put $v(t) = e^{-tH}\phi$. Then there holds the following:*

(i) *There exists a constant C such that*

$$\begin{aligned} &\|w(s)\| \le Ce^{-\frac{\alpha(\omega)}{2}s}\|\phi\|, && s > 0, \\ &\|v(t)\|_{L^2(\mathbf{R}^N, \rho_t dx)} \le C(1+t)^{-\frac{\alpha(\omega)}{2}}, && t > 0, \end{aligned} \tag{4.8}$$

where $\rho_t(x) = (1+t)^{N/2}\exp(|x|^2/4(1+t))$;

(ii) *There holds*

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} v\left((1+t)^{\frac{1}{2}}y, t\right) = a(\phi)\varphi_0(y) \quad in \quad L^2(\mathbf{R}^N, e^{|y|^2/4}dy) \tag{4.9}$$

and for any $L > 0$ *and* $l \in \{0, 1, 2\}$,

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)+l}{2}} (\nabla_x^l v)\left((1+t)^{\frac{1}{2}}y, t\right) = a(\phi)(\nabla_y^l \varphi_0)(y) \tag{4.10}$$

in $C(\{L^{-1} \le |y| \le L\})$, *where*

$$a(\phi) = c_0 \int_{\mathbf{R}^N} \phi(x) U_0(|x|) dx. \tag{4.11}$$

In particular, if $a(\phi) = 0$, *for any* $L > 0$, *there exists a constant* C_2 *such that*

$$(1+t)^{\frac{N+\alpha(\omega)}{2}} \left| v\left((1+t)^{\frac{1}{2}}y, t\right)\right| \le C_2(1+t)^{-1} \tag{4.12}$$

for all $L^{-1} \le |y| \le L$ *and* $t \ge 1$;
(iii) *There exists a function* $c(t)$ *in* $(0,\infty)$ *satisfying*

$$v(x,t) = c(t)U_0(|x|) + F[(\partial_t v)(\cdot, t)](|x|) \quad in \quad \mathbf{R}^N \times (0,\infty). \tag{4.13}$$

such that

$$t^{\frac{N}{2}+\alpha(\omega)} c(t) = c_0 a(v)(1 + o(1)) + O(t^{-1}) \quad as \quad t \to \infty. \tag{4.14}$$

Furthermore there exists a function $d(t)$ *in* $(0,\infty)$ *satisfying*

$$t^{\frac{N}{2}+\alpha(\omega)+1} d(t) = -c_0 a(v) \left(\frac{N}{2} + \alpha(\omega)\right)(1 + o(1)) \quad as \quad t \to \infty \tag{4.15}$$

such that, for any sufficiently small $\epsilon > 0$ *and* $l \in \{0, 1, 2\}$,

$$\begin{aligned} & t^{\frac{N}{2}+\alpha(\omega)} \partial_r^l F[(\partial_t v)(\cdot, t)](|x|) \\ & = t^{\frac{N}{2}+\alpha(\omega)} d(t)(\partial_r^l F[U_0])(|x|) + O(t^{-2}(|x|+1)^{4-l} U_0(|x|)) \\ & = O(t^{-1}(|x|+1)^{2-l} U_0(|x|)) \end{aligned} \tag{4.16}$$

for all $(x,t) \in D_\epsilon(1)$.

Proof. Since $\alpha(\omega) + (N-2)/2 > 0$ by (1.5) we can apply the same argument as in the proof of [6, Proposition 3.1] (see also [6, Theorem 1.1]) to obtain assertion (i). Furthermore, by the same argument as in the proof of [6, Proposition 3.2, Proposition 3.3] we have assertions (ii) and (iii), respectively. □.

5 Proofs of Theorems 1.3 and 1.4

In this section we study the large time behavior of solution of (1.1) by using the results in the previous sections, and prove Theorems 1.3 and 1.4.

Put

$$H_N = -\Delta_N + V(|x|), \quad H_{N,k} = -\Delta_N + V(|x|) + \frac{\omega_k}{|x|^2}, \quad \rho_{N,t}(x) = (1+t)^{\frac{N}{2}} e^{\frac{|x|^2}{4(1+t)}},$$

where $k = 1, 2, \ldots$. Let $u = e^{-tH_N}\phi$ be the solution of (1.1). Then there exists a family of radially symmetric functions $\{\phi_{k,i}\} \subset L^2(\mathbf{R}^N, \rho dx)$ such that

$$\phi = \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} \phi_{k,i}(|x|) Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\mathbf{R}^N, \rho dx). \tag{5.1}$$

For any $k = 0, 1, 2, \ldots$ and $i = 1, \ldots, l_k$, let

$$\Phi_{k,i}(x) := \phi_{k,i}(|x|) Q_{k,i}\left(\frac{x}{|x|}\right), \quad u_{k,i}(x,t) := (e^{-tH_N}\Phi_{k,i})(x), \quad v_{k,i}(x,t) := (e^{-tH_{N,k}}\phi_{k,i})(x).$$

Then we have

$$u_{k,i}(x,t) = v_{k,i}(x,t) Q_{k,i}\left(\frac{x}{|x|}\right). \tag{5.2}$$

Furthermore, putting

$$\tilde{\phi}_{k,i}(x) = |x|^{-k}\phi_{k,i}(x) \in L^2(\mathbf{R}^{N+2k}, \rho dx), \tag{5.3}$$

we have

$$v_{k,i}(x,t) = (e^{-tH_{N,k}}\phi_{k,i})(x) = |x|^k (e^{-tH_{N+2k}}\tilde{\phi}_{k,i})(x). \tag{5.4}$$

(See also Remark 1.1 (ii).) For any $m = 0, 1, 2, \ldots$, let

$$u_0(x,t) = u(x,t), \quad u_m(x,t) = \sum_{k=m}^{\infty} \sum_{i=1}^{l_k} u_{k,i}(x,t) = u(x,t) - \sum_{k=0}^{m-1} \sum_{i=1}^{l_k} u_{k,i}(x,t).$$

Then we give the following lemma, whose proof is lengthy and is omitted. A proof is left to [9].

Lemma 5.1 *Assume the same conditions as in Theorem 1.4. Let u be the solution of (1.1). Then, for any $m = 0, 1, 2, \ldots$, there exists a constant C_1 such that*

$$\|u_m(t)\|_{L^2(\mathbf{R}^N, \rho_{N,t} dx)} \le C_1 t^{-\frac{\alpha(\omega+\omega_m)}{2}} \|u_m(0)\| \le C_1 t^{-\frac{\alpha(\omega+\omega_m)}{2}} \|\phi\| \tag{5.5}$$

for all $t > 0$. Furthermore there hold the following:

(i) *For any $L_0 > 0$, there exists a positive constant C_2 such that*

$$(5.6) \qquad |u_m(x,t)| \le C_2 t^{-\frac{N+\alpha(\omega+\omega_m)}{2}} e^{-\frac{|x|^2}{C_2(1+t)}} \|\phi\|$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$ with $|x| \ge Lt^{1/2}$ and all $L \ge L_0$.

(ii) *For any any $T > 0$ and any sufficiently small $\epsilon > 0$, there exist constants C_3 and C_4 such that*

$$(5.7) \quad |u_m(x,t)| \le C_3 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)} U_m(|x|)\|\phi\| \le C_4 (t^{-\frac{N}{2}-\alpha(\omega+\omega_m)} + t^{-\frac{N}{2}-\frac{\alpha(\omega+\omega_m)}{2}})\|\phi\|$$

for all $(x,t) \in D_\epsilon(T)$. Furthermore, for any $R > 0$ and $l \in \{0,1,2\}$, there exists a constant C_5 such that

$$(5.8) \qquad |(\nabla_x^l u_m)(x,t)| \le C_5 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)} \|\phi\|$$

for all $x \in B(0,R)$ and all sufficiently large t.

Lemma 5.2 *Assume the same conditions as in Theorem 1.4. Let $i = 1,\ldots,N$. Then there hold*

$$(5.9) \qquad \lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u_{0,1}\left((1+t)^{\frac{1}{2}}y,t\right) = c_0 M_0 \varphi_0(y),$$

$$(5.10) \qquad \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_1)}{2}} u_{1,i}\left((1+t)^{\frac{1}{2}}y,t\right) = c_1 N M_i \varphi_1(y)\frac{y_i}{|y|}$$

in $C_{loc}(\mathbf{R}^N \setminus \{0\})$ and $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$. Furthermore, for any $l = 0,1,2$ and any sufficiently small $\epsilon > 0$, there hold

$$(5.11) \quad t^{\frac{N}{2}+\alpha(\omega)}(\nabla_x^l u_{0,1})(x,t) = c_0^2 M_0(1+o(1))(\nabla_x^l U_0)(x) - c_0^2 M_0\left(\frac{N}{2}+\alpha(\omega)\right) t^{-1}(1+o(1))(\nabla_x^l F[U_0])(x) + O(t^{-2}|x|^{4-l}U_0(|x|))$$

$$(5.12) \quad t^{\frac{N}{2}+\alpha(\omega+\omega_1)}(\nabla_x^l u_{1,i})(x,t) = c_1^2 N M_i (1+o(1))(\nabla_x^l Z_i)(x) + O(t^{-1}|x|^{2-l}U_1(|x|))$$

as $t \to \infty$, uniformly for all $x \in \mathbf{R}^N$ with $|x| \le \epsilon t^{1/2}$. Here

$$Z_i(x) = U_1(|x|)\frac{x_i}{|x|}.$$

Proof. By (1.10), (1.21), (5.1), and the orthonormality of $\{Q_{k,i}\}$ we have

$$a(\phi_{0,1}) = c_0 \int_{\mathbf{R}^N} \phi_{0,1}(x)U_0(|x|)dx = \frac{c_0}{\kappa_0}\int_{\mathbf{R}^N} \phi(x)U_0(|x|)dx = \frac{c_0}{\kappa_0}M_0.$$

Then, since $u_{0,1}(x,t) = \kappa_0 v_{0,1}(x,t)$, we apply Proposition 4.1 to the function $v_{0,1}(x,t)$, and we obtain (5.9) and (5.11).

We prove (5.10) and (5.12). Let $i = 1,\ldots,N$. By (1.16), (1.22), and (5.3) we have

$$\tilde{a}(\tilde{\phi}_{1,i}) := c_{N+2,0}\int_{\mathbf{R}^{N+2}} \tilde{\phi}_{1,i}(x)U_{N+2,0}(|x|)dx = c_1\int_{\mathbf{R}^N} \phi_{1,i}(x)U_1(|x|)dx.$$

Furthermore, by (1.10), (1.21), (5.1), and the orthonormality of $\{Q_{k,i}\}$ we have

$$\tilde{a}(\tilde{\phi}_{1,i}) = c_1 N\kappa_1^{-1}\int_{\mathbf{R}^N} \kappa_1\phi_{1,i}(x)U_1(|x|)\frac{x_i^2}{|x|^2}dx = c_1 N\kappa_1^{-1}M_i. \tag{5.13}$$

Then we apply Proposition 4.1 with the dimension N replaced by $N+2$ to the function

$$\tilde{v}_{1,i}(x,t) = (e^{-tH_{N+2}}|\tilde{\phi}_{1,i}|)(x).$$

Then we have

$$\lim_{t\to\infty} t^{\frac{N+2+\alpha_{N+2}(\omega)}{2}}\tilde{v}_{1,i}\left((1+t)^{1/2}y,t\right) = \tilde{a}(\tilde{\phi}_{1,i})\varphi_{N+2,0}(y) \tag{5.14}$$

in $C_{loc}(\mathbf{R}^{N+2}\setminus\{0\})$ and $L^2(\mathbf{R}^{N+2}, e^{|y|^2/4}dy)$. Furthermore, by (1.7), (1.22), (1.16), and (5.13) we obtain

$$\begin{aligned}(5.15)\quad (\nabla_x^l\tilde{v}_{1,i})(x,t) &= c_i(t)(\nabla_x^l U_{N+2,0})(x) + O(t^{-\frac{N+2}{2}-\alpha_{N+2}(\omega)-1}|x|^{2-l}U_{N+2,0}(|x|))\\ &= c_i(t)\nabla_x^l\left[\frac{U_1(|x|)}{|x|}\right] + O(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)-1}|x|^{2-l}|x|^{-1}U_1(|x|))\end{aligned}$$

as $t\to\infty$, uniformly for all $x\in\mathbf{R}^N$ with $|x|\le\epsilon t^{1/2}$, where

$$\begin{aligned}(5.16)\quad c_i(t) &= c_{N+2,0}\tilde{a}(\tilde{\phi}_{1,i})t^{-\frac{N+2}{2}-\alpha_{N+2}(\omega)}(1+o(1))\\ &= c_1^2 N\kappa_1^{-1}M_i t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}(1+o(1)) \quad \text{as} \quad t\to\infty.\end{aligned}$$

Therefore, since

$$u_{1,i}(x,t) = |x|\tilde{v}_{1,i}(x,t)\cdot\kappa_1\frac{x_i}{|x|} = \kappa_1 x_i\tilde{v}_{1,i}(x,t)$$

by (1.22), (5.15), and (5.16) we have

$$\begin{aligned}&\lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_1)}{2}}u_{1,i}\left((1+t)^{1/2}y,t\right)\\ &= \lim_{t\to\infty} t^{\frac{N+2+\alpha_{N+2}(\omega)}{2}}u_{1,i}\left((1+t)^{1/2}y,t\right) = c_1 NM_i\varphi_1(y)\frac{y_i}{|y|}\end{aligned}$$

in $C_{loc}(\mathbf{R}^N\setminus\{0\})$ and $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$ and

$$\begin{aligned}(\nabla_x^l u_{1,i})(x,t) = c_1^2 NM_i t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}(1+o(1))(\nabla_x^l Z_i)(x)&\\ +O(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)-1}|x|^{2-l}U_1(|x|))&\end{aligned}$$

as $t\to\infty$, uniformly for all $x\in\mathbf{R}^N$ with $|x|\le\epsilon t^{1/2}$. Therefore we have (5.10) and (5.12), and the proof of Lemma 5.2 is complete. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 5.1 we have (1.23). By (5.8) with $m=1$ and (5.11), for any $R>0$, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega)}u(x,t) = \lim_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega)}u_{0,1}(x,t) = c_0^2 MU_0(|x|)$$

in $C(B(0,R))$, and obtain (1.24). Furthermore, by (5.5), (5.7), and (5.9) we have

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u\left((1+t)^{1/2}y,t\right) = \lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u_{0,1}\left((1+t)^{1/2}y,t\right) = c_0 M_0 \varphi_0(y)$$

in $C_{loc}(\mathbf{R}^N \setminus \{0\})$ and in $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$. This implies (1.25). Thus the proof of Theorem 1.4 is complete. □

Next, by using Theorem 1.4 we prove Theorem 1.3.

Proof of Theorem 1.3. Assume that H is subcritical. Then, since $A = \alpha(\omega)$, by Theorem 1.2 we have only to prove

$$\|e^{-tH}\|_{q,2} \succeq t^{-(\frac{N}{2}+\alpha(\omega))(\frac{1}{2}-\frac{1}{q})}$$

for all sufficiently large t. Let ϕ be a radial symmetric function such that

$$\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx), \qquad M_0 = \int_{\mathbf{R}^N} \phi(x)U_0(|x|)dx > 0.$$

Put $u(t) = e^{-tH}\phi$ and let ϵ be a sufficiently small positive constant. Since $e^{-2tH}\phi = e^{-tH}e^{-tH}\phi$, we have

$$(5.17) \qquad \|e^{-tH}\|_{q,2} = \sup_{f\in L^2(\mathbf{R}^N)\setminus\{0\}} \frac{\|e^{-tH}f\|_q}{\|f\|_2} \geq \frac{\|u(2t)\|_q}{\|u(t)\|_2} \geq \frac{\|u(2t)\|_{L^q(B(0,\epsilon(1+t)^{1/2}))}}{\|u(t)\|_2}.$$

By Theorem 1.4 we have

$$(5.18) \qquad \|u(t)\|_2 \preceq t^{-\frac{N}{4}-\frac{\alpha(\omega)}{2}}$$

for all sufficiently large t. On the other hand, by the radially symmetry of ϕ we have $u(x,t) = u_{0,1}(x,t)$, and by (1.15), for any sufficiently small $\epsilon > 0$, we obtain

$$\begin{aligned} t^{\frac{N}{2}+\alpha(\omega)}u(x,t) &= c_0^2 M_0^2(1+o(1))U_0(|x|) \\ &\quad -c_0^2 M_0\left(\frac{N}{2}+\alpha(\omega)\right)t^{-1}(1+o(1))F[U_0](x) + O(t^{-2}|x|^4U_0(|x|)) \\ &= c_0^2 M_0^2(1+o(1))U_0(|x|)\left[1+O(t^{-1}|x|^2)+O(t^{-2}|x|^4)\right] \\ &= c_0^2 M_0^2(1+o(1))U_0(|x|)\left[1+O(\epsilon)+O(\epsilon^2)\right] \geq \frac{1}{2}c_0^2 M_0^2 U_0(|x|) > 0 \end{aligned}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$ with $|x| \leq (\epsilon t)^{1/2}$ and all sufficiently large t. Then, by Theorem 1.1 we have

$$(5.19) \qquad \|u(t)\|_{L^q(\{|x|\leq(\epsilon t)^{1/2}\})} \succeq t^{-\frac{N}{2}-\alpha(\omega)}\|\phi\| \left(\int_{\{|x|\leq(\epsilon t)^{1/2}\}} (1+|x|)^{q\alpha(\omega)}dx\right)^{1/q}$$

$$\succeq \begin{cases} t^{-\frac{N}{2}(1-\frac{1}{q})-\frac{\alpha(\omega)}{2}}\|\phi\| & \text{if } q\alpha(\omega)+N>0, \\ t^{-\frac{N}{2}-\alpha(\omega)}(\log(1+t))^{\frac{1}{q}}\|\phi\| & \text{if } q\alpha(\omega)+N=0, \\ t^{-\frac{N}{2}-\alpha(\omega)}\|\phi\| & \text{if } q\alpha(\omega)+N<0 \end{cases}$$

for all sufficiently large t. Therefore, by (5.17), (5.18), and (5.19) we have (1.20), and the proof of Theorem 1.3 is complete. □

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