数理解析研究所講究録
第 1747 巻 2011 年 1-16

RECENT RESULTS ON AMENABLE L^2-THEORETIC METHODS FOR HOMOLOGY COBORDISM AND KNOT CONCORDANCE

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This note is an executive summary of recent results on new L^2-theoretic methods, which use Cheeger-Gromov ρ-invariants associated to certain amenable groups to study knot concordance and homology cobordism of 3-manifolds. Many results are joint with Kent Orr. This work is related to several areas, including topological 4-manifolds, surgery theory, knot theory, functional analysis and operator algebra, amenable groups, and homological algebra.

The main aim of this note is to deliver the present snapshot of our on-going development. We will not deal with thorough details in this note—essentially this note is an extended abstract of results in [COb, Chaa, COa], in which more details can be found, plus some basic backgrounds.

In Section 1, we give a brief review of necessary backgrounds on the definition of L^2-signatures and Cheeger-Gromov invariants, from an algebraic and topological viewpoint. It is written for readers not familiar with L^2-theory and Cheeger-Gromov invariants. Other readers may skip Section 1.

In Section 2, we discuss main results of the paper [COb], which first introduces the fundamental ideas of our new L^2-theoretic methods for amenable groups and gives new homology cobordism invariants from Cheeger-Gromov invariants. Some applications are also given.

In Section 3, we overview new obstructions to a knot being slice and to admitting a Whitney tower of given height, which is obtained from Cheeger-Gromov invariants associated to certain amenable groups [Chaa]. We also discuss the author's results on knots which do not admit a Whitney tower of given height (and so not slice) but are not detected by any prior methods including the invariants and obstructions of Levine, Casson-Gordon, and Cochran-Orr-Teichner and subsequent works.

In Section 4, we discuss new notions of "torsion" in 3-manifolds groups, which are first introduced in terms of the fundamental group of 4-dimensional homology cobordism in [COa]. We outline results in [COa] which shows that our new notion of torsion often gives homology cobordism classes of (even hyperbolic) 3-manifolds not detected by prior methods. This illustrates the significance of our new method regarding torsion.

I remark that this note is a vastly extended version of one of the two talks that I gave in 2010 RIMS Seminar, *Twisted topological invariants and topology of low-dimensional manifolds*, which was held at Akita, Japan, during September 13–17, 2010. I appreciate the warm hospitality of the organizers, Takayuki Morifuji, Masaaki Suzuki, and Teruaki Kitano.

Received March 4, 2011.

1. L^2-SIGNATURES AND CHEEGER-GROMOV ρ-INVARIANTS

In this section we give a quick review of L^2-signatures of 4-manifolds and Cheeger-Gromov ρ-invariants of 3-manifolds, as a preliminary to later sections. In this note manifolds are always oriented topological manifolds, unless stated otherwise.

Essentially the L^2-signature of a 4-manifold is defined from the Poincaré duality, or equivalently the intersection form, with coefficients in the *group von Neumann algebra.* Our treatment of Cheeger-Gromov invariants of manifolds is as topological as possible, without using any differential operators. Indeed we will regard the Cheeger-Gromov invariant as an L^2-signature defect of a certain bounding 4-manifold, based on a topological index theoretic approach due to Weinberger.

For our purpose, we need two key properties of the group von Neumann algebra, namely a spectral theorem for hermitian forms and Lück's L^2-dimension theory, both of which are discusses in Section 1.1 below. We give a brief treatment for readers unfamiliar with these results, without giving detailed proofs. A nice reference on L^2-dimension theory is Lück's book [Lüc02], as well as his original paper [Lüc98]. The spectral theorem we state in this section is not new and must be regarded as folklore, while I could not find a written proof in the literature.

In my manuscript in preparation [Chab], one can find thorough detailed elementary treatments of the topics of this section, including the spectral theorem, L^2-dimension theory, and L^2-signatures and Cheeger-Gromov invariants of manifolds, which are accessible to readers without any substantial preliminaries.

1.1. **Group von Neumann algebra.** We begin with the definition of the group von Neumann algebra. For a countable group G, the *group von Neumann algebra* $\mathcal{N}G$ is defined as follows. First we consider the Hilbert space $\ell^2 G$ generated by (the elements of) G as an orthonormal basis. Namely,

$$\ell^2 G = \Big\{ \sum_{g\in G} z_g g \;\Big|\; z_g \in \mathbb{C},\ \sum_{g\in G} |z_g|^2 < \infty \Big\}.$$

The inner product is given by

$$\Big\langle \sum_{g\in G} z_g g, \sum_{g\in G} w_g g \Big\rangle = \sum_{g\in G} z_g \overline{w_g}.$$

Let $B(\ell^2 G)$ be the algebra of bounded linear operators $a\colon \ell^2 G \to \ell^2 G$. (The multiplication is defined to be composition.) As a convention, operators acts on the left of $\ell^2 G$. A group element $g \in G$ can be regarded as an operator R_g in $B(\ell^2 G)$ via right multiplication, i.e., $R_g(\sum_{h\in G} z_h h) = \sum_{h\in G} z_h (hg)$. Now the group von Neumann algebra $\mathcal{N}G$ of G is defined by

$$\mathcal{N}G = \{a \in B(\ell^2 G) \mid aR_g = R_g a \text{ for any } g \in G\}.$$

Spectral decomposition. To state a spectral decomposition theorem which we will use to define the L^2-signature, we need a standard algebraic formulation of "positivity" of an operator. Recall that the adjoint operator a^* of an operator a is defined by the requirement $\langle a(x), y\rangle = \langle x, a^*(y)\rangle$. For an element $a \in B(\ell^2 G)$, we write $a \geq 0$ if $a = b^* b$ for some $b \in B(\ell^2 G)$. (If $a \in \mathcal{N}G$ and $a \geq 0$, it can be shown that $a = b^* b$ for some $b \in \mathcal{N}G$.) We write $a > 0$ if $a \geq 0$ and $a \neq 0$, and write $a \leq 0$ (resp. $a < 0$) if $-a \geq 0$ (resp. $-a < 0$).

The following innocent-looking statement is true: if $a \leq 0$ and $a \geq 0$, then $a = 0$. A proof of this is a good exercise of the use of the polarization identity over complex scalar.

Now we think of hermitian forms over $\mathcal{N}G$. As a convention, all modules are left modules. For an $\mathcal{N}G$-module M, we denote $M^* = \operatorname{Hom}_{\mathcal{N}G}(M, \mathcal{N}G)$. To make it a left $\mathcal{N}G$-module, the scalar multiplication on M^* is defined by $(r \cdot f)(x) = f(x) \cdot \overline{r}$ for $r \in \mathcal{N}G$, $f\colon M \to \mathcal{N}G$, $x \in M$, where $r \to \overline{r}$ is the involution on $\mathcal{N}G$ induced by the group inversion $g \to g^{-1}$.

Definition 1.1. (1) An $\mathcal{N}G$-module homomorphism $\phi\colon M \to M^*$ is called a *hermitian form* if M is finitely generated over $\mathcal{N}G$ and $\phi(x)(y) = \overline{\phi(y)(x)}$ for any $x, y \in M$.

(2) A hermitian form $\phi\colon M \to M^*$ is said to be *positive definite* on a submodule $N \subset M$ if for any nonzero $x \in N$, $\phi(x)(x) > 0$ in $\mathcal{N}G$.

As usual, we often view a hermitian form ϕ as $\phi\colon M \times M \to \mathcal{N}G$, sending (x, y) to $\phi(x)(y)$. We say an inner direct sum $M = \bigoplus_i M_i$ is an *orthogonal sum* with respect to ϕ if $\phi(x, y) = 0$ whenever $x \in M_i$, $y \in M_j$, $i \neq j$.

Theorem 1.2 (Spectral Decomposition). *Suppose $\phi\colon M \to M^*$ is a hermitian form on a finitely generated $\mathcal{N}G$-module M. Then there is an orthogonal sum decomposition $M = P_+ \oplus P_- \oplus M_0$ with respect to ϕ such that P_+, P_- are finitely generated $\mathcal{N}G$-projective modules and ϕ is positive definite, negative definite, and zero on P_+, P_-, and M_0, respectively.*

A proof can be found in a manuscript of the author, in preparation [Chab].

L^2-dimension. The von Neumann trace defined for operators in $\mathcal{N}G$ is used to define L^2-betti numbers and L^2-signatures in the earlier works of Atiyah and Cheeger-Gromov, in place of ordinary complex dimension. Motivated from these works, in his work [Lüc98, Lüc02] Lück gives a beautiful and elegant algebraic formulation of L^2-dimension theory. The key statements we need are summarized as the following theorem.

Theorem 1.3 (L^2-dimension [Lüc98, Lüc02]). *There is a function*

$$\dim^{(2)}\colon \{\text{(isomorphism classes of) } \mathcal{N}G\text{-modules}\} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

satisfying the following:

(1) $\dim^{(2)} M < \infty$ *if M is finitely generated over $\mathcal{N}G$.*
(2) $\dim^{(2)} 0 = 0$ *and* $\dim^{(2)} \mathcal{N}G = 1$.
(3) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{N}G$-modules, then* $\dim^{(2)} M = \dim^{(2)} M' + \dim^{(2)} M''$.

In (3) above, we adopt the convention that $\infty + r = \infty$ for $r \geq 0$.

A proof of Theorem 1.3 is given in Lück's book [Lüc02] (see also his original paper [Lüc98]). An elementary treatment of the L^2-dimension theory, including the proof Theorem 1.3, can be found in my manuscript in preparation [Chab].

1.2. **L^2-signatures of hermitian forms over $\mathcal{N}G$.** Now we can define the L^2-signature of a hermitian form over $\mathcal{N}G$, exactly in the same way as the finite dimensional signature.

Definition 1.4. The L^2-signature of a hermitian form $\phi\colon M \to M^*$ is defined by

$$\operatorname{sign}^{(2)} \phi = \dim^{(2)} P_+ - \dim^{(2)} P_-$$

where $M = P_+ \oplus P_- \oplus M_0$ is a direct sum decomposition as in Theorem 1.2, and $\dim^{(2)}$ designates the L^2-dimension function in Theorem 1.3.

The well-definedness is shown by the same argument as that of finite dimensional case. We give a proof below, since it illustrates the usefulness of the formulations given in Theorem 1.2 and 1.3.

Lemma 1.5. *$\operatorname{sign}^{(2)} \phi$ is well-defined, independent of the choice of the decomposition $M = P_+ \oplus P_- \oplus M_0$.*

Proof. Suppose $M = P_+ \oplus P_- \oplus M_0 = P'_+ \oplus P'_- \oplus M'_0$ are two decompositions satisfying the conclusion of Theorem 1.2. Since $\phi(x) \neq 0$ in M^* for $x \in P_+ \oplus P_-$, $\operatorname{Ker}\phi \cap (P_+ \oplus P_-) = 0$. It follows that $M_0 = \operatorname{Ker}\phi$. Similarly $M'_0 = \operatorname{Ker}\phi$. Applying Theorem 1.3, it follows that

$$\begin{aligned}\dim^{(2)} P_+ + \dim^{(2)} P_- &= \dim^{(2)} M - \dim^{(2)} M_0 \\ &= \dim^{(2)} M - \dim^{(2)} M'_0 = \dim^{(2)} P'_+ + \dim^{(2)} P'_-.\end{aligned}$$

Suppose $\dim^{(2)} P_+ > \dim^{(2)} P'_+$. Then

$$\begin{aligned}\dim^{(2)}(P_+ \cap (P'_- \oplus M'_0)) &= \dim^{(2)} P_+ + \dim^{(2)}(P'_- \oplus M'_0) - \dim^{(2)}(P_+ + (P'_- \oplus M'_0)) \\ &> \dim^{(2)} P'_+ + \dim^{(2)} P'_- + \dim^{(2)} M'_0 - \dim^{(2)} M \\ &= 0.\end{aligned}$$

Therefore there is a nonzero element x in $P_+ \cap (P'_- \oplus M'_0)$. This contradicts our choice of the decompositions of M, since no element $a \in \mathcal{N}G$ can satisfy both $a > 0$ and $a \leq 0$. (See the previous subsection.) It follows that $\dim^{(2)} P_+ \leq \dim^{(2)} P'_+$. Switching the roles of the decompositions, we obtain $\dim^{(2)} P_+ \geq \dim^{(2)} P'_+$. Therefore $\dim^{(2)} P_+ = \dim^{(2)} P'_+$. By a similar argument, or by observing that the L^2-dimension of P_- is determined by those of M, P_+, M_0, it follows that $\dim^{(2)} P_- = \dim^{(2)} P'_-$. □

1.3. ***L^2*-signatures of 4-manifolds.** We begin by defining $\mathcal{N}G$-coefficient homology. Let X be a finite CW complex, and $\phi\colon \pi_1(X) \to G$ be a group homomorphism where G is countable. (For convenience, when X is not connected, we often regard $\pi_1(X)$ as the free product of the fundamental groups of the components of X.) Following notations used in many papers in the literature, we often omit ϕ in the notation, even when it depends on ϕ as well as the group G. Let X^G be the regular cover of X which is determined by ϕ. Lifting the cell structure of X to X^G, we have a natural cell structure of X^G, and the group G acts on (the left of) X^G cellularly as the covering transformation. This makes the cellular chain complex $C_*(X^G)$ a free $\mathbb{Z}G$-module, where $\mathbb{Z}G$ is the integral group ring of the group G.

We define the $\mathcal{N}G$-coefficient chain complex of (X, ϕ) by

$$C_*(X; \mathcal{N}G) = \mathcal{N}G \otimes_{\mathbb{Z}G} C_*(X^G)$$

and the $\mathcal{N}G$-coefficient homology of (X, ϕ) by

$$H_*(X; \mathcal{N}G) = H_*(C_*(X; \mathcal{N}G)).$$

Note that $C_*(X;\mathcal{N}G)$ is finitely generated and free over $\mathcal{N}G$. It follows that $H_*(X;\mathcal{N}G)$ is finitely generated, since X is semihereditary, namely any finitely generated submodule of a projective $\mathcal{N}G$-module is $\mathcal{N}G$-projective.

The cochain complex and cohomology modules are defined similarly:

$$C^*(X;\mathcal{N}G) = \mathrm{Hom}_{\mathcal{N}G}(C_*(X;\mathcal{N}G),\mathcal{N}G) = C_*(X;\mathcal{N}G)^*$$

and

$$H^*(X;\mathcal{N}G) = H^*(C^*(X;\mathcal{N}G)).$$

Homology and cohomology of pairs with coefficients in $\mathcal{N}G$ are defined similarly. Once we have the above definition, the L^2-Betti number can be defined immediately:

Definition 1.6. The *k-th L^2-Betti number* of (X,ϕ) is defined by

$$b_k^{(2)}(X;\phi) = \dim^{(2)} H_k(X;\mathcal{N}G).$$

Now to define the L^2-signature, suppose W is a compact 4-manifold endowed with $\phi\colon \pi_1(W) \to G$. By Poincaré duality with $\mathcal{N}G$-coefficients, we have an isomorphism

$$H_*(W,\partial W;\mathcal{N}G) \cong H^{4-*}(W;\mathcal{N}G)$$

This, together with the Kronecker evaluation map, gives rise to the $\mathcal{N}G$-coefficient intersection form on the middle dimension:

$$\begin{aligned}\lambda_W\colon H_2(W;\mathcal{N}G) \longrightarrow H_2(W,\partial W;\mathcal{N}G) &\xrightarrow{\cong} H^2(W;\mathcal{N}G)\\ &\longrightarrow \mathrm{Hom}_{\mathcal{N}G}(H_2(W;\mathcal{N}G),\mathcal{N}G) = H_2(W;\mathcal{N}G)^*\end{aligned}$$

It is a standard fact that λ_W satisfies $\lambda_W(x)(y) = \overline{\lambda_W(y)(x)}$. Since $H_2(W;\mathcal{N}G)$ is finitely generated over $\mathcal{N}G$ as discussed above, it follows that λ_W is a hermitian form on $H_2(W;\mathcal{N}G)$.

Definition 1.7. The L^2-signature of (W,ϕ) is defined by

$$\mathrm{sign}_G^{(2)}(W) = \mathrm{sign}^{(2)}\{\lambda_W\colon H_2(W;\mathcal{N}G) \longrightarrow H_2(W;\mathcal{N}G)^*\}.$$

It is easily seen that the ordinary signature $\mathrm{sign}(W)$, namely the signature of the ordinary intersection form on $H_2(W;\mathbb{Q})$, is identical with the L^2-signature of W associated to the homomorphism into a trivial group.

Theorem 1.8 (Topological Atiyah-type theorem [Ati76, LS03, CW03])**.** *Suppose W is a closed 4-manifold, and $\phi\colon \pi_1(W) \to G$ is a homomorphism. Then $\mathrm{sign}_G^{(2)}(W) = \mathrm{sign}(W)$.*

When W is smooth, Theorem 1.8 was first shown by Atiyah [Ati76]. A direct proof of the topological version stated above is given by Lück and Schick [LS03]. Alternatively, one can obtain the topological version from the smooth version by using the two facts that $\mathrm{sign}_G^{(2)}$ is invariant under bordism over G, and that the natural map $\Omega_*^{\mathrm{smooth}}(G)\otimes\mathbb{Q} \to \Omega_*^{\mathrm{top}}(G)\otimes\mathbb{Q}$ is an isomorphism. In the appendix of work of Chang and Weinberger [CW03], a short and elegant bordism theoretic proof using embeddings of groups into acyclic groups is given. See also [Chab].

The following is a very useful property which is perculiar to L^2-signatures (in contrast to the finite dimensional Atiyah-Singer-Patodi signatures). As mentioned in [COT03, Proposition 5.13], essentially this is a consequence of its analogue on L^2-dimension function (e.g., see [Lüc02, Section 6.3]). For more details see, e.g., [Chab].

Theorem 1.9 (L^2-induction). *If G is a subgroup of H, then for $(W, \phi\colon \pi_1(W) \to G)$ as above,* $\operatorname{sign}^{(2)}_G(W) = \operatorname{sign}^{(2)}_H(W)$.

We remark that one can define the L^2-signature of a $4k$-manifold W endowed with $\pi_1(W) \to G$ exactly in the same way. Theorem 1.8 and 1.9 hold for $4k$-manifolds as well.

1.4. **Cheeger-Gromov ρ-invariants of 3-manifolds.** In this subsection we give a topological definition of the Cheeger-Gromov ρ-invariant. Suppose M is a closed 3-manifold and $\phi\colon \pi_1(M) \to \Gamma$ is a group homomorphism (with Γ countable as usual). It is known that there is a pair (ι, W) of a monomorphism $\iota\colon \Gamma \hookrightarrow G$ into a countable group G and a compact 4-manifold W such that $\partial W = M$ and the composition $\iota \circ \phi\colon \pi_1(M) \to \Gamma \to G$ factors through $\pi_1(W)$. We note that a proof for the special case of $\Gamma = \pi_1(M)$ and $\phi = \mathrm{id}_{\pi_1(M)}$ is given in [CW03]. Their argument easily generalizes to our case; e.g., see [Har08], [Chab].

Definition 1.10. The *Cheeger-Gromov invariant* of (M, ϕ) is defined to be the following signature defect of a bounding 4-manifold W as above:

$$\rho^{(2)}(M, \phi) = \operatorname{sign}^{(2)}_G(W) - \operatorname{sign}(W).$$

When $\phi = \mathrm{id}_{\pi_1(M)}$, the above definition specializes to that of [CW03]. According to [LS03], it is known that this signature defect definition is equivalent to the original L^2 Atiyah-Patodi-Singer style definition of the ρ-invariant due to Cheeger and Gromov [CG85].

We remark that it is shown that $\rho^{(2)}(M, \phi)$ is well-defined by a standard argument using the Novikov additivity, L^2-induction, and the topological Atiyah-type theorem.

We remark that $\rho^{(2)}(M, \phi)$ can be defined similarly for $(4k-1)$-manifolds M. In this case, we need to allow several copies of M as the boundary of the $4k$-manifold W used in the definition, namely $\partial W = rM$ for some $r > 0$. Then $\rho^{(2)}(M, \phi)$ is defined to be $(\operatorname{sign}^{(2)}_G(W) - \operatorname{sign}(W))/r$.

2. Homology cobordism and amenable Cheeger-Gromov invariants

Regarding the relationship of knot theory with 4-dimensional topology, concordance of knots and links play an essential role. This also naturally leads us to study homology cobordism of 3-manifolds; recall that two closed 3-manifolds M and M' are (topologically) *homology cobordant* if there is a 4-dimensional topological cobordism W between M and M' satisfying $H_*(W, M) = 0 = H_*(W, M')$. All known obstructions to being topologically concordant are known to be indeed obstructions to being homology cobordant.

Recently the Cheeger-Gromov ρ-invariants have been used as a key ingredient of several interesting results, since the landmark work of Cochran-Orr-Teichner [COT03].

In a joint work with Kent Orr [COb], we developed a new L^2-theoretic method aiming at the study of concordance of knots and links and homology cobordism of 3-manifolds. This result is significant in two aspects: firstly, it gives us far more general construction of new invariants that reveal deeper structures invisible via prior tools, and secondly, it provides new techniques that essentially make an homological use of analytic properties of amenable groups. Our technique is anticipated to produce further remarkable applications.

The main question we address in [COb], which lies at the core of the recent results on L^2-signatures, concordance and homology cobordism, is as follows. In general, we consider the category of manifolds over a fixed group Γ and homology with coefficients in the group ring $R\Gamma$, where R is a commutative ring with unity. Suppose W is a $R\Gamma$-homology cobordism between two closed manifolds M and M' over Γ. Then, *for which groups G and commutative diagrams like below, are the Cheeger-Gromov invariants $\rho^{(2)}(M,\phi)$ and $\rho^{(2)}(M',\phi')$ equal?*

$$\begin{tikzcd} \pi_1(M) \arrow[d] \arrow[dr, "\phi"] & & \\ \pi_1(W) \arrow[r] & G \arrow[r] & \Gamma \\ \pi_1(M') \arrow[u] \arrow[ur, "\phi'"'] & & \end{tikzcd}$$

One may also ask an analogue question for Atiyah-Singer G-signatures and Atiyah-Patodi-Singer η-invariants. For all of these signature invariants, the only previously known useful case for which an affirmative result is available is when (Γ is the trivial group and) G is either a p-group or *poly-torsion-free-abelian (PTFA)* group.

We recall that a group G is *PTFA* if there is a subnormal series $G = G_0 \supset G_1 \supset \cdots \supset G_n \supset G_{n+1} = \{e\}$ for which each G_i/G_{i+1} is torsion-free and abelian. A group G is a *p-group* (p prime) if it is a finite group whose order is a power of p.

We remark that p-groups are nilpotent and PTFA groups are solvable. PTFA groups are the only previously known useful ones which may reveal information from solvable groups beyond nilpotent groups. A notable drawback of PTFA technology is that information related torsion (finite order) elements is invisible. We will discuss more about this later.

Known results on invariants related to p-groups are traced back to Gilmer [Gil81], Gilmer-Livingston [GL83], Ruberman [Rub84], Cappell-Ruberman [CR88], Levine [Lev94], Cha-Ko [CK99], and Friedl [Fri05]. The PTFA case is due to the ground-breaking work of Cochran, Orr, and Teichner [COT03] in the context of knot concordance and L^2-signatures. The homology cobordism invariance statement for PTFA groups first appeared in the work of Harvey [Har08], using results in [COT03]. Recent known applications of L^2-signatures to concordance and homology cobordism depend on the PTFA case.

2.1. **Invariance of amenable Cheeger-Gromov invariants under homology cobordism.** One of our main result provides a generalized positive answer to the above question, beyond p-groups and PTFA groups.

We recall that a (discrete) group G is called *amenable* if there is a finitely additive left G-invariant measure on G. There are several other equivalent definitions. For more information about amenable groups, see, e.g., [Pat88].

Following [Str74], a group G is said to be in Strebel's class $D(R)$, where R is a commutative ring with unity, if a homomorphism $\alpha\colon P \to Q$ on RG-projective modules P and Q is injective whenever the induced homomorphism $1_R \otimes \alpha\colon R \otimes_{RG} P \to R \otimes_{RG} Q$ is injective.

In this note, R always assumed to be either a finite cyclic ring or a subring of $\mathbb{Q}$, unless stated otherwise.

Theorem 2.1 ([COb]). *For a given diagram as above, if G is amenable and the kernel of $G \to \Gamma$ lies in Strebel's class $D(R)$, then the i-th L^2-betti numbers $b_i^{(2)}(M;\phi)$ and*

$b_i^{(2)}(M';\phi')$ *are equal for any* i. *In addition, if* M *is of dimension* $4k-1$, *then the Cheeger-Gromov invariants* $\rho^{(2)}(M,\phi)$ *and* $\rho^{(2)}(M',\phi')$ *are equal.*

An immediate consequence is the following:

Corollary 2.2. *Suppose* G *is an amenable group lying in* $D(R)$. *If* M *and* M' *are* R*-homology cobordant and* $\phi\colon \pi_1(M)\to G$, $\phi'\colon \pi_1(M')\to G$ *are homomorphisms which have a common extension to* $\pi_1(W)$, *then* $b_i^{(2)}(M;\phi)$ *and* $b_i^{(2)}(M';\phi')$ *are equal for any* i. *In addition, if* M *is of dimension* $4k-1$, *then* $\rho^{(2)}(M,\phi)=\rho^{(2)}(M',\phi')$.

Although its description may look technical, we would like to emphasize that the class of groups G in Theorem 2.1 and Corollary 2.2 is large enough to contain useful groups. For example, when Γ is trivial (i.e., untwisted R-homology cobordism case as in Corollary 2.2), our class of groups subsumes the PTFA case. More important, our class of groups contains non-PTFA groups in general, including several interesting infinite/finite groups with *torsion* when $R=\mathbb{Z}_p$. As a special case p-groups are subsumed. Note that in this case we still have integral homology cobordism invariance as well as $\mathbb{Z}_p$-homology. More discussions are found in [COb].

Our new technique not only extends the prior results but also introduces a small shift of paradigm. The prior techniques which are known to be effective for p- and PTFA groups are essentially algebraic. In particular the Cochran-Orr-Teichner result and subsequent ones depend on the following algebraic fact: if G is PTFA, then the group ring $\mathbb{Z}G$ embeds into a skew quotient field, say $\mathcal{K}$, which is flat over $\mathbb{Z}G$.

For the groups G we consider, $\mathbb{Z}G$ may not embed in a skew field, requiring an entirely new approach. We employ directly L^2-methods with coefficients in the group von Neumann algebra $\mathcal{N}G$ by using results of Lück [Lüc02]. This new technique can be used to control the L^2-dimension of homology with coefficients in $\mathcal{N}G$.

A key homological result we use in the proof of Theorem 2.1 is the following.

Theorem 2.3 ([COb]). *Suppose* G *is amenable and the kernel of* $G\to\Gamma$ *is in Strebel's class* $D(R)$. *If* C_* *is a chain complex over* $\mathbb{Z}G$ *which is finitely generated and free in dimension* $\le n$, *and if* $H_i(R\Gamma\otimes_{\mathbb{Z}G}C_*)=0$ *for* $i\le n$, *then* $H_i(\mathcal{N}G\otimes_{\mathbb{Z}G}C_*)$ *has* L^2*-dimension zero over* $\mathcal{N}G$.

More applications are discussed in later sections of this note. We anticipate further applications of this result beyond these.

2.2. **Local derived series and homology cobordism.** Another tool we develop and use for applications in [COb] is a new commutator-type series of groups. A special case of our series is analogous to Harvey's torsion-free derived series of a group [CH05], but ours is often smaller and allow us to reveal more information from quotients.

We consider the category $\mathcal{G}_\Gamma$ of groups π over a fixed group Γ, i.e., π is endowed with a homomorphism $\pi\to\Gamma$, and morphisms are homomorphisms $\pi\to G$ making the following diagram commute.

$$\begin{tikzcd} \pi \arrow[rr] \arrow[dr] & & G \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

Note that in the special case $\Gamma = \{e\}$, the category $\mathcal{G}_\Gamma$ is canonically identified with the category of groups. For a given coefficient ring R, we define a new series

$$\pi \supset \pi^{(0)} \supset \pi^{(1)} \supset \cdots \supset \pi^{(n)} \supset \cdots$$

of normal subgroups $\pi^{(n)}$ for each $\pi \in \mathcal{G}_\Gamma$ in terms of the Vogel $R\Gamma$-homology localization of groups and Cohn localization of rings. We call this series $\{\pi^{(n)}\}$ *Vogel-Cohn $R\Gamma$-local derived series* to emphasize that the series is a functor on the category $\mathcal{G}_\Gamma$. Indeed the series can be defined in a more general situation; for more details see [COb, Section 3 and 4].

Similarly to Harvey's series, our series admits an injectivity theorem which enables us to make applications to L^2-signatures. On the other hand, in contrast to Harvey's, our series is functorial with respect to any morphisms in $\mathcal{G}_\Gamma$. We recall that a morphism $\pi \to G$ in $\mathcal{G}_\Gamma$ is said to be *2-connected on $H_*(-;R\Gamma)$* if the induced map $H_i(\pi;R\Gamma) \to H_i(G;R\Gamma)$ is an isomorphism for $i = 1$ and an epimorphism for $i = 2$.

Theorem 2.4 ([COb]). *Let $\{\pi^{(n)}\}$ be the Vogel-Cohn $R\Gamma$-local derived series for π in $\mathcal{G}_\Gamma$.*

(1) *(Functoriality) For any morphism $\pi \to G$ in $\mathcal{G}_\Gamma$, there are induced homomorphisms $\pi^{(n)} \to G^{(n)}$ and $\pi/\pi^{(n)} \to G/G^{(n)}$ for any n.*
(2) *(Injectivity) If $\pi \to G$ is a group homomorphism which is 2-connected on $H_*(-;R\Gamma)$ with π finitely generated, G finitely presented, then the induced map $\pi/\pi^{(n)} \to G/G^{(n)}$ is injective for any n.*

We remark that we provide a general construction of such series, which gives the above Vogel-Cohn local derived series as a special case. In particular, we also give the Bousfield analogue.

By applying the Vogel-Cohn local derived series to fundamental groups over amenable groups, it turns out that one obtains groups over Γ which satisfy the hypothesis of Theorem 2.1. In the statement below, we denote by $\mathbb{Z}_{(p)}$ the classical localization of $\mathbb{Z}$ away from p. Namely $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid b \text{ is relatively prime to } p\}$.

Theorem 2.5 ([COb]). *Let R be either $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$, or $\mathbb{Q}$. For a closed manifold M over an amenable group Γ, view $\pi = \pi_1(M)$ as a group over Γ and denote by $\pi^{(n)}$ the Vogel-Cohn $R\Gamma$-local derived series. Let $\phi_n \colon \pi \to \pi/\pi^{(n)}$ be the quotient map. Then the L^2-betti numbers $b_i^{(2)}(M,\phi_n)$ and the L^2-signature defect $\rho^{(2)}(M,\phi_n)$ are $R\Gamma$-homology cobordism invariants of M for any n. In particular, when Γ is trivial, $b_i^{(2)}(M,\phi_n)$ and $\rho^{(2)}(M,\phi_n)$ are R-homology cobordism invariants.*

2.3. **Applications.** As an application involving non-torsion-free groups, we give a homology cobordism version of a theorem of Chang and Weinberger [CW03] on homeomorphism types of manifolds with a given homotopy type.

Theorem 2.6 ([COb]). *Suppose M is a closed $(4k-1)$-manifold with $\pi = \pi_1(M)$, $k \geq 2$. Let p be prime and $\pi^{(n)}$ be the $\mathbb{Z}_p$ or $\mathbb{Z}_{(p)}$-coefficient Vogel-Cohn local derived series of π. If π has a torsion element which remains nontrivial in $\pi/\pi^{(n)}$ for some n, then there exist infinitely many closed $(4k-1)$-manifolds $M_0 = M$, M_1, $M_2, \ldots$ such that each M_i is simple homotopy equivalent and tangentially equivalent to M but M_i and M_j are not homology cobordant for any $i \neq j$.*

In the proof, we make use of a nonvanishing property for certain L^2-signatures associated to non-torsion-free groups due to Chang and Weinberger [CW03], and apply our result to capture the invariance of these L^2-signatures under homology cobordism as well as homeomorphism.

Another application given in [COb] concerns spherical 3-space forms.

Theorem 2.7. *For any generalized quaternionic spherical 3-space form M, there are infinitely many closed 3-manifolds $M_0 = M$, M_1, $M_2, \ldots$ such that the M_i are homology equivalent to M and have identical Wall multisignatures (or equivalently Atiyah-Singer G-signatures) and Harvey L^2-signature invariants ρ_n* [Har08], *but no two of the M_i are homology cobordant.*

Our amenable L^2-signature invariants also apply to concordance of knots within a fixed homotopy class of an ambient 3-manifold, along the lines of work of Heck [Hec09].

3. New obstructions to topological knot concordance beyond Levine, Casson-Gordon, and Cochran-Orr-Teichner

In [Chaa], the author applied the new L^2-methods first initiated in the prior work joint with Kent Orr [COb] to the study of topological knot concordance. Using this, the author revealed structures of the knot concordance group which are invisible via any prior invariants based on the work of Levine, Casson-Gordon, and Cochran-Orr-Teichner.

We recall that two knots K_0, K_1 in S^3 are said to be *concordant* if there is a locally flat embedded annulus in $S^3 \times [0,1]$ bounded by $K_0 \times \{0\} \cup -K_1 \times \{1\}$. A knot K is called *slice* if K is concordant to the trivial knot, or equivalently, there is a locally flat 2-disk in the 4-disk D^4 bounded by $K \subset S^3$. The concordance classes of knots form an abelian group under connected sum, which is called the *knot concordance group.* We denote it by $\mathcal{C}$.

3.1. New obstructions to knots being slice and solvable.

As the first main result in [Chaa], the author obtained the following new obstruction to knots being slice. As in the previous section, R is always a finite cyclic ring or a subring of the rationals.

Theorem 3.1 ([Chaa]). *Suppose K is a slice knot in S^3 with zero-surgery manifold M_K, Γ is an amenable group lying in Strebel's class $D(R)$ for some R, and $\phi\colon \pi_1(M_K) \to \Gamma$ is a homomorphism extending to a slice disk exterior. Then the Cheeger-Gromov invariant $\rho^{(2)}(M_K, \phi)$ vanishes.*

In [COT03], an *(h)-solvable knot* ($h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) is defined as a knot which admits a "height h approximation" of disjoint embedded 2-spheres in certain 4-manifolds on which surgery would give a slice disk exterior, namely a *Whitney tower of height h*.

More rigorously, for a collection of framed immersed spheres in a 4-manifold, a *Whitney tower of height* 0 is those spheres themselves. Inductively, a *Whitney tower of height* $n+1$ is a Whitney tower of height n together with a collection of Whitney disks of level $n+1$, which are defined to be framed immersed Whitney disks pairing up the intersections of Whitney disks of level n (or given immersed spheres if $n = 1$) with interior disjoint from the Whitney tower of height n. We call it a Whitney tower of height $(n.5)$ if the interior of the Whitney disks of level $n+1$ are allowed to intersect Whitney disks of level n, while these are still not allowed to meet Whitney disks of lower level.

For nonnegative half integers $h = 0, 0.5, 1, 1.5, \dots$, K is said to be *(h)-solvable* if the zero-surgery manifold M_K bounds a spin 4-manifold W which has $H_1(W) \cong H_1(M_K)$ and admits a Whitney tower $\mathcal{T}$ of height h whose initial level consists of immersed 2-spheres which form a lagrangian of the ordinary intersection form of W. The 4-manifold W is called an *(h)-solution* for K, respectively.

Let $\mathcal{F}_h \subset \mathcal{C}$ be the subgroup of (the concordance classes of) (h)-solvable knots in the knot concordance group $\mathcal{C}$. The (h)-solvable filtration

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

of $\mathcal{C}$ has been playing an essential role in recent study of the topological knot concordance, providing a framework for prior works of Levine and Casson-Gordon, as well as recent results of Cochran-Orr-Teichner [COT03, COT04] and subsequent results on knot concordance that use Cheeger-Gromov invariants, including Cochran-Teichner [CT07] and Cochran-Harvey-Leidy [CHL09, CHLc, CHLa].

As another main result in [Chaa], the author gave a new obstruction to knots being $(n.5)$-solvable. In what follows n designates a nonnegative integer. In the statement below, $\Gamma^{(n+1)}$ denotes the $(n+1)$-st ordinary derived subgroup defined inductively by $\Gamma^{(0)} = \Gamma$, $\Gamma^{(k+1)} = [\Gamma^{(k)}, \Gamma^{(k)}]$.

Theorem 3.2 ([Chaa]). *Suppose K is an $(n.5)$-solvable knot in S^3, R is either $\mathbb{Q}$ or $\mathbb{Z}_p$, Γ is an amenable group lying in $D(R)$ with $\Gamma^{(n+1)} = \{e\}$, and $\phi\colon \pi_1(M_K) \to \Gamma$ extends to an $(n.5)$-solution. Then the Cheeger-Gromov invariant $\rho^{(2)}(M_K, \phi)$ vanishes.*

We remark that this specializes to the result of Cochran-Orr-Teichner [COT03] when Γ is PTFA. (Recall that a PTFA group is always amenable and in $D(R)$ for any R.) Theorem 3.2 is significantly stronger than the Cochran-Orr-Teichner result—for example, a vast variety of infinite groups with torsion can be used as Γ.

In order to prove our obstruction theorem to being solvable (Theorem 3.2), we need to generalize some results about $\mathcal{N}G$-coefficient homology modules in [COb]. Among those, the following, which is a generalization of the field coefficient case of Theorem 2.3, plays a key role.

Theorem 3.3. *Suppose G is an amenable group lying in $D(R)$, where R is a field (i.e., $\mathbb{Q}$ or $\mathbb{Z}_p$). Suppose C_* is a projective chain complex over $\mathbb{Z}G$, n is fixed, and C_n finitely generated over $\mathbb{Z}G$. Then the following inequality holds:*

$$\dim^{(2)} H_n(\mathcal{N}G \otimes_{\mathbb{Z}G} C_*) \le \dim_R H_n(R \otimes_{\mathbb{Z}G} C_*)$$

3.2. **Knots with vanishing Cochran-Orr-Teichner PTFA signature obstructions.** Using the above obstruction, for each n, the author gave a large family of (n)-solvable knots which are not $(n.5)$-solvable but not detected by the PTFA L^2-signatures of Cochran-Orr-Teichner:

Definition 3.4. We say that J is *an (n)-solvable knot J with vanishing PTFA L^2-signature obstructions* if there is an (n)-solution W for J such that for any PTFA group G and for any $\phi\colon \pi_1 M(J) \to G$ extending to W, $\rho^{(2)}(M(J), \phi) = 0$.

We write $J \in \mathcal{V}_n$ if J is as above. It turns out that $\mathcal{V}_n$ is a subgroup of the knot concordance group [Chaa]. Obviously we have

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{V}_n \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{0.5} \subset \mathcal{V}_0 \subset \mathcal{F}_0 \subset \mathcal{C}$$

Theorem 3.5 ([Chaa]). *For any n, there are infinitely many (n)-solvable knots J^i ($i = 1, 2, \dots$) satisfying the following:*

(1) *Any linear combination $\#_i a_i J^i$ under connected sum is an (n)-solvable knot with vanishing PTFA L^2-signature obstructions.*
(2) *Whenever $a_i \neq 0$ for some i, $\#_i a_i J^i$ is not $(n.5)$-solvable.*

Consequently the J^i generate an infinite rank subgroup in $\mathcal{F}_n/\mathcal{F}_{n.5}$ which is invisible via PTFA L^2-signature obstructions.

An immediate consequence of Theorem 3.5 is that the quotient $\mathcal{V}_n/\mathcal{F}_n$ has infinite rank.

For any knot in $\mathcal{V}_n$, the PTFA signature obstruction of Cochran-Orr-Teichner to being $(n.5)$-solvable ([COT03, Theorem 4.2]) vanishes even for some (n)-solution W which is not necessarily an $(n.5)$-solution. Consequently, all the prior techniques using the PTFA obstructions (for example, Cochran-Orr-Teichner [COT03, COT04], Cochran-Teichner [CT07], Cochran-Harvey-Leidy [CHLb, CHL09, CHLc, CHLa]) fail to distinguish any knots in $\mathcal{V}_n$, particularly our examples in Theorem 3.5, from $(n.5)$-solvable knots up to concordance.

The invariants of Levine and Casson-Gordon also vanish for knots in $\mathcal{V}_n$ for $n \geq 2$. Therefore, our examples are not detected by any prior invariants of Levine, Casson-Gordon, Cochran-Orr-Teichner.

We remark that the twisted coefficient systems used in the proof of Theorem 3.5 may be viewed as a *higher-order generalization* of the Casson-Gordon metabelian setup. Recall that Casson-Gordon [CG86, CG78] extracts invariants from a p-torsion abelian cover of the infinite cyclic cover of the zero-surgery manifold $M(K)$. Generalizing this, our twisted coefficient system extracts information from a tower of covers

$$M_{n+1} \xrightarrow{p_n} M_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = \text{zero-surgery manifold } M(K) \text{ of } K$$

where p_0 is the infinite cyclic cover, $p_1, \dots, p_{n-1}$ are torsion-free abelian covers, and p_n is a p-torsion cover. When $n = 1$, this tower is the metabelian cover that Casson and Gordon considered.

This iterated covering for knots can also be compared with the iterated p-cover construction for links, which was used to extract link concordance invariants in the author's prior work [Cha10, Cha09].

The construction of the above twisted coefficient system requires other ingredients. Among these which are newly introduced in [Chaa], there are modulo p higher order Blanchfield linking pairing of 3-manifolds and mixed-coefficient commutator series of groups. For more details, see [Chaa, Sections 4, 5].

4. Hidden torsion of 3-manifolds

An important feature of the L^2-method in [COb] is that many infinite groups with torsion can be used to study homology cobordism and concordance. On the other hand, from a pure 3-dimensional perspective, one may remind the following: in case of a "generic" 3-manifolds, torsion elements rarely appear in the fundamental group. (e.g., all closed irreducible nonspherical 3-manifolds have torsion-free group by Geometrization.) However, in a joint work with Kent Orr [COa] subsequent to [COb], we showed that even for a generic 3-manifold (e.g., closed hyperbolic 3-manifold), torsion elements appear naturally in the context of homology cobordism. This illustrates that the study of the interplay of 3- and 4-dimensional topology has a very different aspect from that of 3-dimensional topology regarding the fundamental group.

Also, our result shows that there are 3-manifolds for which invariants from torsion-free groups (e.g., PTFA groups) and nilpotent groups (e.g. p-groups) are not sufficient to understand their homology cobordism classes. We illustrate that certain non-nilpotent infinite groups with torsion are necessary to understand these.

4.1. **Hidden torsion and its algebraic analogue.** We begin with the definition of hidden torsion of 3-manifolds.

Definition 4.1. For a closed 3-manifold M, an element $g \in \pi_1(M)$ is called *hidden torsion* of M if g has infinite order in $\pi_1(M)$, is not null-homotopic in any homology cobordism W of M, but for some homology cobordism W of M, the image of g in $\pi_1(W)$ has finite order.

We note that if $g \in \pi_1(M)$ and there is a homology cobordism W of M for which g has finite order in $\pi_1(W)$, then for any N homology cobordant to M, there is a homology cobordism V between M and N for which g has finite order in $\pi_1(M)$. In fact, such a cobordism V is obtained by attaching W and $-W$ to any homology cobordism between M and N. This says that even when one fixes the other end of homology cobordisms of M in the above definition, one obtains equivalent one.

To define an algebraic analogue of hidden torsion, we employ the notion of homology localization of a group, which is originally due to Vogel [Vog78] and Levine [Lev89a]. What we use for this purpose is a slightly modified version which is explicitly defined in [Cha08, COb]. Here we just mention the following only: the homology localization is a functorial association of a group $\widehat{\pi}$ and a homomorphism $\pi \to \widehat{\pi}$ to each group π with the property that (i) whenever $\pi \to G$ is a group homomorphism between finitely presented groups π and G which is 2-connected on $H_*(-;\mathbb{Z})$, the homomorphism $\pi \to \widehat{\pi}$ factors through $\pi \to G$ in a unique way, and (ii) $\pi \to \widehat{\pi}$ is universal (initial) among such functors in an appropriate sense. For more details, see, e.g., [Lev89a, Cha08, COb].

Homology localization is well-known as a fundamental machinery in homotopy theory, and also used as a key ingredient in the study of homology cobordism and concordance. An immediate consequence of the above property, which indeed plays a key role is the following: if $X \to Y$ is a map between finite complexes which induces isomorphisms on $H_*(-;\mathbb{Z})$, then $\pi_1(X) \to \pi_1(Y)$ induces an isomorphism $\widehat{\pi_1(X)} \to \widehat{\pi_1(Y)}$. It follows that any homomorphism $\widehat{\pi_1(M)} \to G$ gives rise to a coefficient system of M over G which extends automatically to the fundamental group of any homology cobordism of M.

Definition 4.2. Let G be a group, and $\widehat{G}$ the homology localization of G. An element $g \in G$ is called *local hidden torsion* of G if g has infinite order in G and its image under $G \to \widehat{G}$ has nontrivial finite order.

There are rather simple examples of closed hyperbolic 3-manifolds M which have hidden torsion that is also local hidden torsion of $\pi_1(M)$. (See [COa] for examples obtained by surgery along a knot in S^3.)

In high dimensions, it turns out that the notions of hidden torsion and local hidden torsion agree.

Theorem 4.3 ([COa]). *Suppose M is a closed n-manifold with $n > 3$. Then an element $g \in \pi_1(M)$ is hidden torsion of M if and only if g is local hidden torsion of $\pi_1(M)$.*

4.2. **Hyperbolic 3-manifolds with local hidden torsion and their homology cobordism.** In [COa], we constructed interesting examples which have local hidden torsion in a deeper part of the fundamental group. To state the result, we recall the following definition: the lower central subgroups of a group G is defined by $G_1 = G$, $G_{q+1} = [G, G_q]$, and denote the first transfinite lower central subgroup by $G_w = \bigcap_{q<\infty} G_q$. ($\omega$ designates the first infinite ordinal.)

Theorem 4.4 ([COa]). *There are closed hyperbolic 3-manifolds M which have hidden local torsion in $\pi_1(M)_\omega$.*

We remark that the local hidden torsion in Theorem 4.4 is obviously invisible in any residually nilpotent quotient of the fundamental group.

In general it is very difficult to compute the homology localization of a given group. Though, in [COa], we give a construction of certain hyperbolic 3-manifolds for which we can explicitly compute the homology localization. In addition to this, the proof of Theorem 4.4 involves several other techniques, including a construction of homology cobordism based on the equation approach to homology localization of groups which was first suggested in Levine's work [Lev89a, Lev89b] (see also [Cha08]).

The behavior of these local hidden torsion is reflected significantly to homology cobordism classes of 3-manifolds, and often plays a key role in understanding interesting subtle aspects, as illustrated in the following result:

Theorem 4.5 ([COa]). *There is a sequence of infinitely many closed hyperbolic 3-manifolds $M = M_0, M_1, M_2, \ldots$ with the following properties:*

(1) *For each i, there is a homology equivalence $f_i \colon M_i \to M$. That is, f_i induces an isomorphism on $H_*(-;\mathbb{Z})$.*
(2) *Whenever $i \neq j$, M_i and M_i are not homology cobordant.*

Furthermore, all prior known homology cobordism obstructions fail to distinguish these examples. In particular,

(3) *For any homomorphism $\phi \colon \pi_1(M) \to G$ with G torsion-free, the L^2-signature defects (= von Neumann-Cheeger-Gromov invariants)*

$$\rho^{(2)}(M, \phi) \text{ and } \rho^{(2)}(M_i, \phi \circ f_{i*})$$

are equal for each i. In particular Harvey's ρ_n-invariants [Har08] *of the M_i are the same.*
(4) *Similarly, the following homology cobordism invariants for the M_i are equal:*
    (a) *Multi-signatures (= Casson-Gordon invariants) for prime power order characters in* [Gil81, GL83, Rub84, CR88]
    (b) *Atiyah-Patodi-Singer ρ-invariants associated to representations that factor through p-groups in* [Lev94, Fri05]
    (c) *Twisted torsion invariants associated to representations that factor through p-groups in* [CF]
    (d) *Hirzebruch-type Witt-class-valued invariants from iterated p-covers in* [Cha10]

The properties of our local hidden torsion that it is invisible in any residually nilpotent group and that it is torsion in the homology localization are crucial in proving (3) and (4) of Theorem 4.5, namely that prior invariants do not distinguish our examples.

We use the main result of [COb] (Theorem 2.1 and Corollary 2.2 in this note) to detect the homology cobordism classes of the examples in Theorem 4.5. The coefficient system $\pi_1(M) \to G$ we use in order to detect our examples M is obtained from our computation of the homology localization of $\pi_1(M)$ combined with the technique of mixed-coefficient commutator series which appeared in [Chaa].

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