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ON METABELIAN REIDEMEISTER TORSION

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1. INTRODUCTION

Building on ideas of Cochran, Orr and Teichner [2], non-abelian generalizations of the classical Alexander polynomial which are called higher-order Alexander polynomials were introduced for knots by Cochran [1] and extended to 3-manifolds by Harvey [8] and Turaev [18]. The polynomials have coefficients in certain skew fields and are known by Friedl [3] to be essentially equal to Reidemeister torsion over the functional fields of the skew fields. In particular, several properties and applications of the degrees of such polynomials, which are called Cochran-Harvey invariants, were investigated also in [4], [5], [9], [14] and [15].

Let M be a compact connected oriented 3-manifold with empty or toroidal boundary and $b_1(M) > 0$, and let $\psi\colon \pi_1 M \to \mathbb{Z}$ be an epimorphism. The aim of this article is to introduce and study a combinatorially computable invariant $c(\psi)$ which can be regarded as the highest degree coefficient of a 'metabelian higher-order Alexander polynomial' associated to ψ. In the construction of $c(\psi)$ we use Reidemeister torsion because of its smaller indeterminacy than higher-order Alexander polynomials. We give a fiberedness obstruction on $c(\psi)$ and show that there are infinitely many non-fibered knots with same Alexander polynomials as fibered knots of same genus such that the non-fiberedness can be detected by the obstruction. (See Theorems 3.6 and 3.8.)

By comparing the definitions, we can check from [6, Theorem 5.4] and [7, Theorem 3.8] that the obstruction is essentially equal to that by Goda and Sakasai [6, Theorem 4.6] for *homologically fibered links.* Note that they considered not only 'metabelian coefficients' but more general non-commutative ones and also gave an obstruction on Magnus representations of the complementary homology cylinder of a minimal genus Seifert surface. One advantage of using $c(\psi)$ is that we do not need to find such a Thurston norm minimizing surface in computations.

This work was intended as an attempt to extract another kind of information from a higher-order Alexander polynomial than the degree, and more general results and computational examples are to be provided in [12].

In this paper all homology groups and cohomology groups are with respect to integral coefficients unless specifically noted.

2. METABELIAN REIDEMEISTER TORSION

We begin with the definition of Reidemeister torsion over a skew field $\mathbb{K}$. See [13], [16] and [17] for more details.

For a matrix over $\mathbb{K}$, we mean by an elementary row operation the addition of a left multiple of one row to another row. After elementary row operations we can turn any

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matrix $A \in GL_k(\mathbb{K})$ into a diagonal matrix $(d_{i,j})$. Then the *Dieudonné determinant* $\det A$ is defined to be $[\prod_{i=1}^{k} d_{i,i}] \in \mathbb{K}_{ab}^{\times} := \mathbb{K}^{\times}/[\mathbb{K}^{\times}, \mathbb{K}^{\times}]$.

Let $C_* = (C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0)$ be an acyclic chain complex of finite dimensional right $\mathbb{K}$-vector spaces. If we have a basis b_{i-1} of $\operatorname{Im} \partial_i$ for $i = 0, 1, \ldots, n$, picking a lift of b_{i-1} in C_i and combining it with b_i, we can obtain a basis $b_i b_{i-1}$ of C_i for $i = 0, 1, \ldots, n$.

Definition 2.1. For a given basis $\boldsymbol{c} = \{c_i\}$ of C_*, we choose a basis $\{b_i\}$ of $\operatorname{Im} \partial_*$ and define

$$\tau(C_*, \boldsymbol{c}) := \prod_{i=0}^{n} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{K}_{ab}^{\times},$$

where $[b_i b_{i-1}/c_i]$ is the Dieudonné determinant of the base change matrix from c_i to $b_i b_{i-1}$.

It can be easily checked that $\tau(C_*, \boldsymbol{c})$ does not depend on the choices of b_i and $b_i b_{i-1}$. Torsion has the following multiplicative property. Let

$$0 \to C'_* \to C_* \to C''_* \to 0$$

be a short exact sequence of acyclic finite chain complexes of finite dimensional right $\mathbb{K}$-vector spaces and let $\boldsymbol{c} = \{c_i\}, \boldsymbol{c}' = \{c'_i\}, \boldsymbol{c}'' = \{c''_i\}$ be bases of C_*, C'_*, C''_*. Picking a lift of c''_i in C_i and combining it with the image of c'_i in C_i, we obtain a basis $c'_i c''_i$ of C_i.

Lemma 2.2. *([13, Theorem 3. 1]) If $[c'_i c''_i/c_i] = 1$ for all i, then*

$$\tau(C_*, \boldsymbol{c}) = \tau(C'_*, \boldsymbol{c}')\tau(C''_*, \boldsymbol{c}'').$$

The following lemma is a certain non-commutative version of [16, Theorem 2.2]. Turaev's proof can be easily applied to this setting.

Lemma 2.3. *If we find a decomposition $C_* = C'_* \oplus C''_*$ such that C'_i and C''_i are spanned by subbases of c_i and the induced map $pr_{C''_{i-1}} \circ \partial_i|_{C'_i} \colon C'_i \to C''_{i-1}$ is an isomorphism for each i, then*

$$\tau(C_*, \boldsymbol{c}) = \pm \prod_{i=0}^{n} (\det pr_{C''_{i-1}} \circ \partial|_{C'_i})^{(-1)^i}.$$

Let X be a connected finite CW-complex and let $\varphi \colon \mathbb{Z}[\pi_1 X] \to \mathbb{K}$ be a ring homomorphism. We define the twisted homology group associated to φ as follows:

$$H_i^{\varphi}(X; \mathbb{K}) := H_i(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{K}),$$

where $\widetilde{X}$ is the universal covering of X.

Definition 2.4. If $H_*^{\varphi}(X; \mathbb{K}) = 0$, then we define the *Reidemeister torsion* $\tau_{\varphi}(X)$ associated to φ as follows. We choose a lift $\tilde{e}$ in $\widetilde{X}$ for each cell e. Then

$$\tau_{\varphi}(X) := [\tau(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{K}, \langle \tilde{e} \otimes 1 \rangle_e)] \in \mathbb{K}_{ab}^{\times}/ \pm \varphi(\pi_1 X).$$

We can check that $\tau_{\varphi}(X)$ does not depend on the choice of $\tilde{e}$. It is known that Reidemeister torsion is a simple homotopy invariant of a finite CW-complex.

Now we define a metabelian torsion invariant of the pair (M, ψ) as an element of a functional filed.

We denote by A the quotient group of $\operatorname{Ker}\psi/[\operatorname{Ker}\psi, \operatorname{Ker}\psi]$ by the torsion subgroup and by $\mathbb{Q}(A)$ the quotient field of $\mathbb{Z}[A]$. We pick $\mu \in \pi_1 M/[\operatorname{Ker}\psi, \operatorname{Ker}\psi]$ such that $\psi(\mu) = 1$

and let $\theta\colon \mathbb{Q}(A) \to \mathbb{Q}(A)$ be the automorphism given by $\theta(x) = \mu x \mu^{-1}$ for $x \in \mathbb{Q}(A)$. Now the functional field $\mathbb{Q}(A)(t)$ is defined as the quotient (skew) field of the Laurent polynomial ring $\mathbb{Q}(A)[t, t^{-1}]$ whose multiplication is given by the rule $tx = \theta(x)t$. Note that the isomorphism type of $\mathbb{Q}(A)(t)$ does not depend on the choice of μ. We consider the homomorphism $\rho\colon \mathbb{Z}[\pi_1 M] \to \mathbb{Q}(A)(t)$ defined by

$$\sum_{\gamma \in \pi_1 M} a_\gamma\, \gamma \mapsto \sum_{\gamma \in \pi_1 M} a_\gamma \gamma \mu^{-\psi(\gamma)} t^{\psi(\gamma)}.$$

If $H_*^\rho(M; \mathbb{Q}(A)(t)) = 0$, then we have $\tau_\rho(M) \in \mathbb{Q}(A)(t)^\times_{ab}/ \pm A \cdot \langle t \rangle$.

Let $\bar{\cdot}\colon \mathbb{Q}(A)(t)^\times_{ab}/ \pm A \cdot \langle t \rangle \to \mathbb{Q}(A)(t)^\times_{ab}/ \pm A \cdot \langle t \rangle$ be the involution induced by the involution $a \cdot t \mapsto t^{-1} \cdot a^{-1}$ for $a \in A$. The torsion has the following duality. We refer the reader to [5, Theorem 5.4].

Lemma 2.5. *If $H_*^\rho(M; \mathbb{Q}(A)(t)) = 0$, then*

$$\overline{\tau_\rho(M)} = \tau_\rho(M).$$

For $f = \sum_{i=m}^n a_i t^i \in \mathbb{Q}(A)[t, t^{-1}]$ with $a_m a_n \neq 0$, we write $\deg f := n - m$. Setting $\deg fg^{-1} := \deg f - \deg g$, we can extend $\deg\colon \mathbb{Q}(A)[t, t^{-1}] \setminus 0 \to \mathbb{Z}$ to a homomorphism $\deg\colon \mathbb{Q}(A)(t)^\times \to \mathbb{Z}$, which in turn induces a homomorphism $\deg\colon \mathbb{Q}(A)(t)^\times_{ab} \to \mathbb{Z}$.

Definition 2.6. If $H_*^\rho(M; \mathbb{Q}(A)(t)) = 0$, then we define

$$\delta(\psi) := \deg \tau_\rho(M) \in \mathbb{Z}.$$

Remark 2.7. The invariant $\delta(\psi)$ is essentially equal to the *Cochran-Harvey invariant* associated to the pair $(\pi_1 M \to \pi_1 M/[\operatorname{Ker}\psi, \operatorname{Ker}\psi], \psi)$. See [3] and [4] for the correspondence.

3. The highest degree coefficient

First we introduce the highest degree coefficient $c(\psi)$ of $\tau_\rho(M)$.

We denote by C the subgroup of $\mathbb{Q}(A)^\times$ generated by

$$\{\pm a \cdot \frac{\theta(p)}{p} \mid a \in A, p \in \mathbb{Q}(A)^\times\}.$$

We define a map $c\colon \mathbb{Q}(A)(t)^\times_{ab} \to \mathbb{Q}(A)^\times/C$ by

$$c([(a_m t^m + a_{m-1} t^{m-1} + \dots)(b_n t^n + b_{n-1} t^{n-1} + \dots)^{-1}]) = \left[\frac{a_m}{b_n}\right],$$

where $a_i, b_i \in \mathbb{Q}(A)$ for all i and $a_m b_n \neq 0$. The proof of the following lemma is straightforward.

Lemma 3.1. *The map $c\colon \mathbb{Q}(A)(t)^\times_{ab} \to \mathbb{Q}(A)^\times/C$ is a well-defined homomorphism.*

Definition 3.2. If $H_*^\rho(M; \mathbb{Q}(A)(t)) = 0$, then we define

$$c(\psi) := c(\tau_\rho(M)) \in \mathbb{Q}(A)^\times/C.$$

Remark 3.3. We say that irreducible $p, q \in \mathbb{Z}[A]$ are equivalent if there are $a \in A$ and $n \in \mathbb{Z}$ such that $p = \pm a\theta^n(q)$. Since $\mathbb{Z}[A]$ is a unique factorization domain, $\mathbb{Q}(A)^\times/C$ is the free abelian group generated by such equivalence classes and is, in particular, of infinite rank.

The following lemma follows immediately from Lemma 2.5.

Lemma 3.4. *The following equality holds:*

$$c(-\psi) = c(\psi).$$

The following theorem was shown for knots by Cochran [1, Proposition 9.1] and for general 3-manifolds by Harvey [8, Theorem 12.1]. The reformulation in terms of Reidemeister torsion is given by Friedl [3, Theorem 1.2].

Theorem 3.5. *If $M \neq S^1 \times D^2, S^1 \times S^2$ is fibered over S^1 and $\psi\colon \pi_1 M \to \mathbb{Z}$ is represented by the fibration, then*

$$\delta(\psi) = ||\psi||_T,$$

where $||\psi||$ is the Thurston norm of $\psi \in H^1(M)$.

The following theorem gives another fiberedness obstruction on $\tau_\rho(M)$.

Theorem 3.6. *If M is fibered over S^1 and $\psi\colon \pi_1 M \to \mathbb{Z}$ is represented by the fibration, then $c(\psi) = 1$.*

Proof. Let $\Sigma \subset M$ be a fiber surface and let $f\colon \Sigma \to \Sigma$ be a monodromy map. We take a triangulation T of Σ and a cellular approximation $g\colon (\Sigma, T) \to (\Sigma, T)$ to f. We pick a homotopy equivalence map between the mapping torus $T_g := \Sigma \times [0,1]/(x,1) \sim (g(x),0)$ and M, and identify $\pi_1 T_g$ with $\pi_1 M$. It can be checked that

$$\tau_\rho(T_g) = \tau_\rho(M).$$

(See for instance [10, Lemma 3.6] and [11, Lemma 4.2].)

A cell decomposition of T_g is given by $\{\sigma \times [0,1] \mid \sigma \in T\}$ and T. We denote by C'_* and C''_* the subcomplexes of $C_*(\widetilde{T_g}) \otimes_{\mathbb{Z}[\pi_1 T_g]} \mathbb{Q}(A)(t)$ generated by lifts of cells in $\{\sigma \times [0,1] \mid \sigma \in T\}$ and T respectively. Since $pr_{C''_{i-1}} \circ \partial_i|_{C'_i}\colon C'_i \to C''_{i-1}$ is expressed by a matrix of the form $tA_i - I$, where coefficients of A_i are all in $\mathbb{Z}[A]$, and is an isomorphism for each i, by Lemma 2.3

$$\tau_\rho(T_g) = \prod_i [\det pr_{C''_{i-1}} \circ \partial_i|_{C'_i}]^{(-1)^i}.$$

Therefore we see at once that

$$c(\overline{\tau_\rho(T_g)}) = 1.$$

Now the theorem follows from Lemma 2.5. □

For an oriented tame knot $K \subset S^3$, we denote by E_K the exterior of K. In the following we only consider the case where $M = E_K$ and $\psi\colon \pi_1 E_K \to \mathbb{Z}$ is the epimorphism which maps a meridional element compatible with the orientation to 1. We can easily check that $H^\rho_*(E_K; \mathbb{Q}(A)(t)) = 0$. Note that by Lemma 3.4 the choice of orientations is inessential in considering the value of $c(\psi)$.

It is a classical result of Neuwirth that for a fibered knot K,

$$\Delta_K \text{ is monic and } \deg \Delta_K = 2g(K). \tag{1}$$

We call (1) the Neuwirth condition.

Remark 3.7. From the monotonicity [1, Theorem 5.4], [9, Theorem 2.2], [3, Theorem 1.3] of $\delta(\psi)$ and the inequality [1, Theorem 7.1], [8, Theorem 10.1], [3, Theorem 1.2] between $\delta(\psi)$ and $||\psi||_T$ we have $\delta(\psi) = ||\psi||_T$ for a nontrivial knot satisfying that $\deg \Delta_K = 2g(K)$.

The following theorem shows non-triviality of the fiberedness obstruction in Theorem 3.6.

Theorem 3.8. *There are infinitely many knots satisfying the Neuwirth condition and that* $c(\psi) \neq 1$ *for both orientations.*

Proof. Let $K \subset S^3$ be an oriented fibered knot and let $J \subset S^3$ be an oriented knot with nontrivial Δ_J. We take an oriented knot η in the exterior of a fiber surface Σ of K which is unknot in S^3 and which represents a nontrivial element $[\eta] \in A$. We consider the result $K_0 \subset S^3$ of *infecting* K by J along η. (See [1, Section 8].) Namely, E_{K_0} is homeomorphic to the result of attaching $-E_J$ to $E_{K \sqcup \eta}$ along the boundaries so that a longitude and a meridian of η correspond to a meridian and a longitude of J.

Regarding E_K as $E_{K \sqcup \eta} \cup (D^2 \times S^1)$ and extending a degree 1 map $(E_J, \partial E_J) \to (D^2 \times S^1, \partial D^2 \times S^1)$ by the identity map on $E_{K \sqcup \eta}$, we have $f\colon E_{K_0} \to E_K$. Comparing the Meyer-Vietoris homology long exact sequences for the decompositions of E_{K_0} and E_K, we can see that the Alexander modules of them are isomorphic by f_*. Hence $f_*\colon \pi_1 E_{K_0}/(\pi_1 E_{K_0})'' \to \pi_1 E_K/(\pi_1 E_K)''$ is also isomorphic. Moreover, since $f^{-1}(\Sigma)$ is a Seifert surface of K_0 and has the minimal genus $g(K)$, we can see that K_0 also satisfies the Neuwirth condition.

Since $H_*^{\rho \circ f_*}(E_{K_0}; \mathbb{Q}(A)(t)) = H_*^{\rho \circ f_*}(E_J; \mathbb{Q}(A)(t)) = H_*^{\rho \circ f_*}(\partial E_J; \mathbb{Q}(A)(t)) = 0$, it follows again from the Meyer-Vietoris homology long exact sequence that $H_*^{\rho}(E_{K \sqcup \eta}; \mathbb{Q}(A)(t)) = 0$. We have the following short exact sequences of acyclic chain complexes:

$$0 \to C_*(\widetilde{\partial E_J}) \to C_*(\widetilde{E_{K \sqcup \eta}}) \oplus C_*(\widetilde{E_J}) \to C_*(\widetilde{E_{K_0}}) \to 0,$$

$$0 \to C_*(\widetilde{\partial D^2 \times S^1}) \to C_*(\widetilde{E_{K \sqcup \eta}}) \oplus C_*(\widetilde{D^2 \times S^1}) \to C_*(\widetilde{E_K}) \to 0,$$

where we implicitly tensor all the chain complexes with $\mathbb{Q}(A)(t)$. By Lemma 2.2 we obtain

$$\tau_{\rho \circ f_*}(\partial E_J) \cdot \tau_\rho(E_{K \sqcup \eta}) = \tau_{\rho \circ f_*}(E_J) \cdot \tau_{\rho \circ f_*}(E_{K_0}),$$
$$\tau_\rho(\partial D^2 \times S^1) \cdot \tau_\rho(E_{K \sqcup \eta}) = \tau_\rho(D^2 \times S^1) \cdot \tau_\rho(E_K).$$

Here

$$\tau_{\rho \circ f_*}(E_J) = [\Delta_K([\eta])([\eta] - 1)^{-1}],$$
$$\tau_\rho(D^2 \times S^1) = [[\eta] - 1]^{-1},$$
$$\tau_{\rho \circ f_*}(\partial E_J) = \tau_\rho(\partial D^2 \times S^1) = 1,$$

which are easy to check. Combining them, we obtain

$$\tau_{\rho \circ f_*}(E_{K_0}) = [\Delta_K([\eta])] \cdot \tau_\rho(E_K).$$

Now it follows from Theorem 3.6 that

$$c(\tau_{\rho \circ f_*}(E_{K_0})) = [\Delta_K([\eta])] \neq 1.$$

Since we can choose K, J and $[\eta]$ arbitrarily, the knot type of K_0 can be changed into infinitely many types, which proves the theorem. □

Remark 3.9. We have actually given how to construct knots satisfying the desired conditions. By a similar technique we can show that there are also infinitely many non-fibered knots satisfying the Neuwirth condition and that $c(\psi) = 1$ for both orientations. See [12] for a proof.

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