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ON THE REIDEMEISTER-TURAEV TORSION OF STANDARD SPIN^c STRUCTURES ON SEIFERT FIBERED 3-MANIFOLDS

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ABSTRACT. The Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin^c structures. Here, a Spin^c structure of a 3-manifold is a homology class of non-singular vector fields on it. Each Seifert fibered 3-manifold has a standard Spin^c structure, which is represented as a non-singular vector field the set of whose orbits gives a Seifert fibration. This short note provides an algorithm for computing the Reidemeister-Turaev torsion of the standard Spin^c structure on a Seifert fibered 3-manifold. The machinery used to compute the torsion is that of punctured Heegaard diagrams.

INTRODUCTION

Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin^c structures. This invariant is defined by Turaev [12] as a refinement of the Reidemeister torsion, which is one of the most well-known classical invariant of 3-manifolds. A Spin^c structure can be represented as a *homology class* of non-singular vector fields on the ambient 3-manifold. On the other hand, a branched standard spine of a 3-manifold carries a non-singular vector field. The computation of the Reidemeister-Turaev torsion using branched standard spines is first introduced in [3] for the case with non-empty boundary and then in [1] for the closed case. In [6], the author developed the method via Heegaard splittings compatible with the branched standard spines. In [7], the author introduced a Heegaard-type diagram, which we call a *punctured Heegaard diagram*, to present a branched spine and this diagram allows to compute the Reidemeister-Turaev torsion quite easily. In the case of closed 3-manifolds, a punctured Heegaard diagram is exactly a Heegaard diagram with a fixed complementary region of slopes satisfying a special condition, see Section 1.5.

In the present paper, we introduce the method for constructing punctured Heegaard diagrams of Seifert fibered 3-manifolds equipped with standard Spin^c structures as a parallel construction of [11] and then explain how to compute its Reidemeister-Turaev torsion. Each Seifert fibered 3-manifold has a standard Spin^c structure, which is represented as non-singular vector fields everywhere tangent to its Seifert fibration. Recall that most Seifert fibered 3-manifolds admits a unique Seifert fibration, see Section 1. For such Seifert fibered 3-manifolds, the Reidemeister-Turaev torsion of the standard Spin^c structure can be regarded as the *principal values* of the Reidemeister torsion of the manifold. Note that a general algorithm for computing Reidemeister-Turaev torsions of any 3-manifold equipped with any Spin^c structure has already been described by Turaev ([16, 17]) by means of surgery presentations on links in S^3.

In the final section, we observe that the Reidemeister-Turaev torsions of the standard Spin^c structures of a Seifert fibered 3-manifold have *standard* values among the set of the Reidemeister-Turaev torsions of all Spin^c structures on the manifold.

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Notation 0.1. Let X be a subset of a given topological space or a manifold Y. Throughout this paper, we will denote the interior of X by $\operatorname{Int} X$, the closure of X by $\overline{X}$ and the number of components of X by $\#X$. We will use $\eta(X;Y)$ to denote a regular neighborhood of X in Y. If the ambient space Y is clear from the context, we simply denote it by $\eta(X)$. By 3-manifold, we always mean a *connected, compact* and *oriented* one, with or without boundary, unless otherwise mentioned.

1. Preliminaries

1.1. **Spin^c structures.** Let M be a closed smooth 3-manifold. Two non-singular vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ on M are said to be *homologous* if there exists a closed 3-ball $B \subset M$ such that the restrictions of $\mathcal{V}_1$ and $\mathcal{V}_2$ to $M \setminus \operatorname{Int} B$ are homotopic as non-singular vector fields. A *Spin^c structure* is a homology class $[\mathcal{V}]$ of non-singular vector fields $\mathcal{V}$. We denote by $\operatorname{Spin}^c(M)$ the set of Spin^c structure on M. The action of $H_1(M)$ to $\operatorname{Spin}^c(M)$ is defined through Reeb surgery, see [17, 9] for details.

1.2. **Review of the Reidemeister-Turaev torsion.** Let F be a field and let E be an n-dimensional vector space over F. For two ordered bases $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ of E, we write $[b/c] = \det(a_{ij}) \in F^\times$, where $b_i = \sum_{j=1}^n a_{ij} c_j$. The bases b and c are said to be *equivalent* if $[b/c] = 1$.

Let $C = (0 \xrightarrow{\partial_m} C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} 0)$ be a finite dimensional chain complex over F. For each $0 \leqslant i \leqslant m$, set $B_i = \operatorname{Im} \partial_i$, $Z_i = \operatorname{Ker} \partial_{i-1}$ and $H_i = Z_i / B_i$. The chain complex is said to be *acyclic* if $H_i = 0$ for all i. Suppose that C is acyclic and C_i is endowed with a *distinguished* basis c_i for each i. Choose an ordered set of vectors b_i in C_i for each : $0 \leqslant i \leqslant m$ such that $\partial_{i-1}(b_i)$ forms a basis of B_{i-1}. By the above construction, $\partial_i(b_{i+1})$ and b_i are combined to be a new basis $\partial_i(b_{i+1})b_i$ of C_i. With this notation, the *torsion* of C is defined by

$$\tau(C) := \prod_{i=0}^{m} [\partial_i(b_{i+1})b_i/c_i]^{(-1)^{i+1}} \in F^\times.$$

Let M be a compact connected orientable smooth manifold of an arbitrary dimension. Let X be a CW-decomposition of M, $\hat{X} \to X$ be its maximal abelian covering and F be a field. We can equip $\hat{X}$ with the CW-structure naturally induced by that of X, and then we regard $C_*(\hat{X})$ as a left $\mathbb{Z}[\pi_1(X,*)]$-module via the monodromy. Let $\{e_i^k\}$ be the set of all oriented k-cells in X, and $\{\hat{e}_i^k\}$ be a family of their lifts to $\hat{X}$. Give an orientation with each of these cells and order the cells $\{\hat{e}_i^k\}$, for each k, in an arbitrary way. Then this family gives an ordered $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. In this way, we can regard $C_*(\hat{X})$ as an ordered, based chain complex.

Let $\varphi : \mathbb{Z}[H_1(X)] \to F$ be a ring homomorphism. If the based chain complex $C_*^\varphi(X) = F \otimes_\varphi C_*(\hat{X})$ over F is acyclic, the (φ*-twisted*) *Reidemeister torsion* of M is defined as

$$\tau^\varphi(M) := \tau(C_*^\varphi(X)) \in F^\times / \pm \varphi(H_1(M)).$$

Otherwise, set $\tau^\varphi(M) := 0 \in F$.

Let M be a smooth 3-manifold and let X be its CW-decomposition. A family of cells of $\hat{X}$ is said to be *fundamental* if over each cell of X exactly one cell of this family lies. When we choose a fundamental family $\{\hat{e}_i^k\}$ of cells of $\hat{X}$ and orient and order these cells in arbitrary way, this family becomes a free $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. (i.e. $C_k(\hat{X}) = \bigoplus_i \mathbb{Z}[H_1(X)]\hat{e}_i^k$). In this way, we can regard $C_*(\hat{X})$ as a chain complex with basis.

A Spin^c structure $[\mathcal{V}]$ on M instructs to obtain a fundamental family of cells of $\hat{X}$, and hence the Reidemeister torsion is refined to be an invariant $\tau^\varphi(M, [\mathcal{V}]) \in F/\pm 1$ of Spin^c structures on M, see [12, 13, 15, 17]. In [1, 3], this construction is described via the notion of *branched standard spine.*

Let M be a Seifert fibered 3-manifold. In this paper, all Seifert fibered 3-manifolds are assumed to be closed orientable ones having orientable base surfaces. Recall that a Seifert fibered 3-manifold is said to be *large* if its base surface is different from a sphere with less than four singular points.

We call a non-singular vector field (a Spin^c structure, respectively) on a Seifert fibered 3-manifold is *standard* if it is everywhere tangential to a Seifert fibration. In [11], Taniguchi, Tsuboi and Yamashita introduced an algorithm to obtain a *branched spine* of a standard vector field on an arbitrary closed Seifert fibered 3-manifold in term of the Seifert invariants $S(g; b; (p_1, q_1), (p_2, q_2), \dots, (p_r, q_r))$, where g is the genus of the base surface, b is its obstruction class, and (p_i, q_i), $i = 1, 2, \dots, r$, are the types of its singular fibers. It is well-known (see e.g. [5]) that a large Seifert fibered 3-manifold except $S(0; 4; (2,1), (2,1), (2,-1), (2,-1))$ has a unique (up to isotopy) Seifert fibration.

1.3. **Branched spines.** Let N be a compact orientable 3-manifold. A branched surface $P \subset N$ is a union of finitely many compact smooth surfaces glued together to form a compact subspace locally modeled on one of the three possibilities in Figure 1. Note that

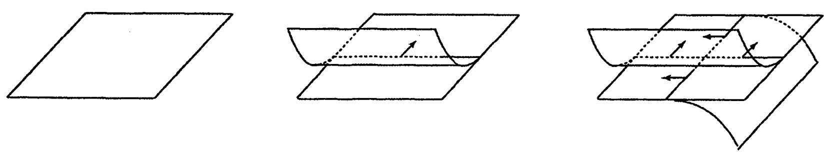

FIGURE 1. Local pictures of a branched surface.

the general definition of branched surface allows more sheets than just two on one side and one on the other side, but we only consider this situation (which is generic and stable, i.e. corresponds to an open dense set in the space of branched surfaces).

The *branch locus* $S(P)$ of P is the set of points none of whose neighborhoods (in P) is a disk. $S(P)$ is a collection of smooth immersed curves in P. Let $V(P)$ be the set of double points of $S(P)$. We associate with every component of $S(P) \setminus V(P)$ a vector (in P) pointing in the locally one-sheeted direction, as shown in Figure 1. We call a component of $P \setminus S(P)$ a *sector* of P. Let R be a sector of P. If all branch directions along $\partial \overline{R}$ point out from R, then $P \setminus R$ is still a branched surface, see Figure 2 (i). One can regard $\eta(P)$ as an interval bundle over P as drawn in Figure 2 (ii). The boundary $\partial\eta(P)$ decomposes into two parts: the endpoints of the fibers, $\partial_h\eta(P)$, and the rest, $\partial_v\eta(P)$. In this paper, all branched surfaces are assumed to be *transversely oriented*, that is, P is equipped with a global orientation on the 1-foliation of $\eta(P)$ whose leaves are fibers of $\eta(B)$. Refer to [4, 10] for more details about branched surfaces.

A branched surface $P \subset N$ is called a *branched spine* (of N) if N collapses onto P. A branched spine P is naturally stratified as $V(P) \subset S(P) \subset P$. A branched spine P is said

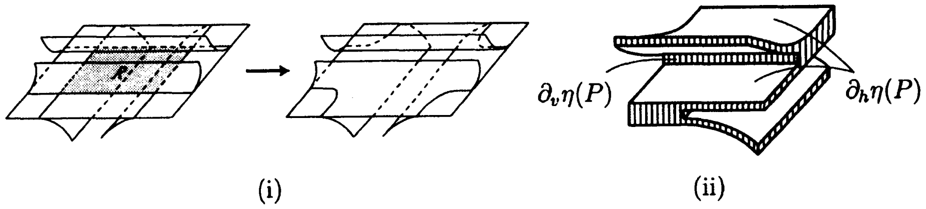

FIGURE 2. (i) Removable sector; (ii) A regular neighborhood of a branched surface.

to be *standard* if this stratification induces a CW decomposition of P, namely, there is no loop in $S(P)$ and sectors are disks. See [2] for a precise definition. If P is a branched spine of a compact 3-manifold N with $\partial N = S^2$, then P is also called a branched spine of the closed 3-manifold M obtained from N by attaching a 3-ball to the unique 2-sphere boundary. A branched spine of a closed 3-manifold is called a *flow-spine* if $\partial_v\eta(P)$ is an annulus.

In [2], Benedetti and Petronio proved that every orientable 3-manifold admits a branched (standard) spine and it naturally encodes a well-defined homotopy class of vector fields, which is called the *concave traversing field*, on the ambient manifold. We require that the flow intersects P in the same direction as the fixed transverse orientation. In the case where P is a flow-spine of a closed oriented 3-manifold M, one can extend the concave traversing field, whose orbits are the I-fibers of the regular neighborhood of the spine, to the whole of M.

1.4. **Oriented, based Heegaard diagrams.** Throughout the paper, we only consider closed orientable 3-manifolds.

By a *Heegaard diagram* we means a triple $(S_g; \alpha, \beta)$ where

(1) S_g is a closed, connected, orientable surface of genus $g \in \mathbb{N}$; and
(2) $\alpha = \bigcup_{i=1}^g \alpha_i$ and $\beta = \bigcup_{i=1}^g \beta_i$ are compact, mutually transverse 1-manifolds with g components on S_g.
(3) $\overline{S_g \setminus \eta(\bigcup_i^g \alpha_i; S_g)} \cong \overline{S_g \setminus \eta(\bigcup_i^g \beta_i; S_g)} \cong$ ($2g$-th punctured sphere)

A Heegaard diagram gives rise to a closed 3-manifold $M_{(S_g;\alpha,\beta)}$ by adding 2-handles $H_{\alpha_1}, \dots, H_{\alpha_g}$ and $H_{\beta_1}, \dots, H_{\beta_g}$ to $S_g \times [-1,1]$ along the curves $\alpha_1 \times \{-1\}, \dots, \alpha_g \times \{-1\}$ and $\beta_1 \times \{1\}, \dots, \beta_g \times \{1\}$, respectively, and then adding 3-handles along the resulting 2-sphere boundary components. We will denote the core disk of H_{α_i} (H_{β_i}, respectively) (fairly extended so that its boundary is on S_g) by D_{α_i} (D_{β_i}, respectively) for $1 \leqslant i \leqslant g$. When we consider (and draw in $\mathbb{R}^3$) a Heegaard diagram, we always equip the surface S_g with the positive normal $\boldsymbol{w}_p$ ($x \in S_g$) pointing toward the α side, and with the orientation $(\boldsymbol{u}_p, \boldsymbol{v}_p)$, $\boldsymbol{u}_p, \boldsymbol{v}_p \in T_pS_g$, such that $(\boldsymbol{u}_p, \boldsymbol{v}_p, \boldsymbol{w}_p)$ gives the right-hand orientation on $\mathbb{R}^3$.

A Heegaard diagram is said to be *oriented* if the 1-manifolds α and β are oriented. A Heegaard diagram $(S_g; \alpha, \beta)$ with a fixed point $b_i \in \beta_i \setminus \alpha$ for each β_i is said to be *based.* A Heegaard diagram $(S_g; \alpha, \beta)$ is said to be *standard* if every connected component of $S_g \setminus (\alpha \cup \beta)$ is an open ball. It is clear that we can make any Heegaard diagram standard up to isotopy of β. We often denote an oriented, based Heegaard diagram by $(S_g; \vec{\alpha}, \vec{\beta}, \{b_k\}_{k=1}^g)$. A system of pairwise disjoint, simple, closed, oriented curves $\gamma = \bigcup_{i=1}^g \gamma_i$ on S_g is called a *dual system* of β if each γ_i intersects β_i transversely once at the point b_i in the positive direction shown in Figure 3, where (u_x, v_x) is compatible with the fixed orientation of S_g, and $\gamma_i \cap \beta_j = \emptyset$ when $i \neq j$.

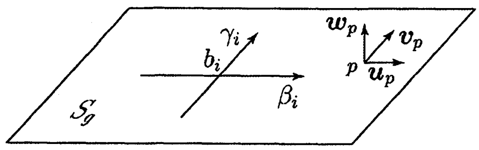

FIGURE 3. The positive intersection with a dual loop.

1.5. **Punctured Heegaard diagrams.** Given a genus g Heegaard diagram $(S_g; \alpha, \beta)$, let D be a disk component of $S_g \setminus (\alpha \cup \beta)$. Then D is said to be *joining* if it satisfies the following: i) $\partial\overline{D}$ is a simple loop, where the closure is taken in the surface S_g; and ii) $\partial\overline{D} \cap \alpha_i$ ($\partial\overline{D} \cap \beta_i$, respectively) is a single connected arc for all $1 \leqslant j \leqslant g$. See Figure 4. We call a Heegaard diagram $(S_g; \alpha, \beta)$ with joining disk D a *punctured Heegaard diagram*

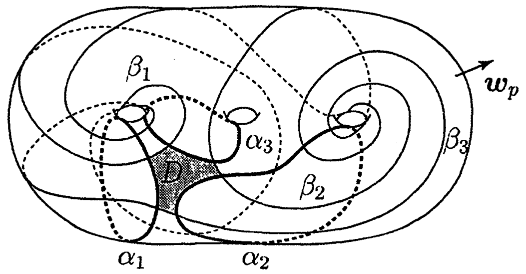

FIGURE 4. A punctured Heegaard diagram of genus 3.

and denote it by $(S_g; \alpha, \beta; D)$. Given a punctured Heegaard diagram $(S_g; \alpha, \beta; D)$, we may equip the polyhedron

$$P_{(S_g;\alpha,\beta;D)} := \left(S_g \cup \left(\bigcup_{i=1}^{g} D_{\alpha_i} \right) \cup \left(\bigcup_{i=1}^{g} D_{\beta_i} \right) \right) \setminus \operatorname{Int} D \subset M_{(S_g;\alpha,\beta)}$$

with a structure of an transversely-oriented flow-spine. We denote by $\mathcal{V}_{P_{(S_g;\alpha,\beta;D)}}$ a vector field on $M_{(S_g;\alpha,\beta;D)}$ obtained by extending the concave traversing field on a regular neighborhood of $P_{(S_g;\alpha,\beta;D)}$, see Section 1.3. Note that such a vector field $\mathcal{V}_{P_{(S_g;\alpha,\beta;D)}}$ is uniquely defined up to homotopy.

Each punctured Heegaard diagram $(S_g; \alpha, \beta)$ defines an oriented, based Heegaard diagram as in the following way:

- Since each of the slopes α and β appears on $\partial\overline{D}$ exactly as a single arc, the orientation of $\partial\overline{D}$ determines orientations of all of these slopes. Here, we consider that D inherits the orientation from S_g and we use "outernomal first" convention.
- For each $1 \leqslant i \leqslant g$, take a base point b_i on the interior of the arc $\beta_i \cap \partial\overline{D}$.

Let $(S_g; \vec{\alpha}, \vec{\beta}; \{b_k\}_{k=1}^{g})$ be an oriented, based Heegaard diagram and set $M := M_{(S_g;\alpha,\beta)}$. Let p be a point on α_i. Then we define the normal vector $\boldsymbol{n}_p \in T_pS_g$ of α_i at p in such a way that $(\boldsymbol{n}_p, \boldsymbol{a}_p)$ is coherent to the fixed orientation of S_g, where $\boldsymbol{a}_p \in T_p\alpha_i$ is coherent to the orientation of α_i. Then α_i determines an element $x_i \in \pi_1(M, *)$ and β_j determines

$r_j = r_j(x_1, \dots, x_g) \in \pi_1(M, *)$ starting at the point b_j and following the oriented loop β_j, for each $i, j = 1, \dots, g$. Namely, we use the convention such that at each point $p \in \alpha_i \cap \beta_j$ we read x_i (${x_i}^{-1}$, respectively) when the normal vector $\boldsymbol{n}_p \in T_pS_g$ of α_i at p is coherent (not coherent, respectively) to the orientation of β_j at p.

Moreover, if we choose a dual system $\gamma = \bigcup_{i=1}^{g} \gamma_i$ of β, γ_i determines $y_j \in \pi_1(M, *)$ in the same manner. Let $p : \mathbb{Z}[\pi_1(M, *)] \to \mathbb{Z}[H_1(M)]$ be the canonical projection and denote $[z] = p(z)$ for $z \in \pi_1(M, *)$. The following is immediate from the above setting and definition of the Reidemeister-Turaev torsion.

Corollary 1.1. *Let (S_g, α, β) be a punctured Heegaard diagram and set $M = M((S_g, \alpha, \beta))$. Let $(S_g; \vec{\alpha}, \vec{\beta}; \{b_j\})$ be an oriented, based Heegaard diagram defined by (S_g, α, β). Let the twisted chain complex $C_*^{\varphi}(M)$ be acyclic. Then there exist two integers $k, l \in \{1, \dots, n\}$ such that*

$$\tau^{\varphi}(M, [\mathcal{V}_{(S_g;\alpha,\beta;D)}]) = \frac{\det B_{k,l}}{(\varphi([x_k]) - 1)(\varphi([y_l]) - 1)} \in F^{\times}/\pm 1,$$

where $B_{k,l}$ is the (k,l)-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1 \le i,j \le g}$, namely the matrix obtained by removing k-th row and l-th column from the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1 \le i,j \le g}$. Here, $\frac{\partial}{\partial x_j}$ denotes the Fox's free differential calculus, and if $B_{k,l} = \emptyset$, we set $\det B_{k,l} = 1$.

1.6. **BW-decompositions and DS-diagrams.** Let P be a flow-spine of a closed 3-manifold M. Let N be a regular neighborhood of P. Recall that $\partial N \cong S^2$. Then the collapsing $N \searrow P$ induced a retraction π such that N is the mapping cylinder of $\pi|_{\partial N} : \partial N \to P$. This map satisfies the following:

(1) $\pi^{-1}(S(P)) \cap \partial N$ is a trivalent graph;
(2) For $x \in P$, $\phi^{-1}(x)$ consists of 2, 3 or 4 points according as $x \in P \setminus S(P)$, $x \in S(P) \setminus V(P)$ or $x \in V(P)$; and
(3) There exists a circle e in $\pi^{-1}(S(P)) \cap \partial N$ such that
  (a) $\partial N \setminus e$ is the disjoint union of B and W (this is called a *Black and White* (or simply *B-W*) *decomposition*);
  (b) Every component of e has B on one side and W on the other side;
  (c) π maps $e \setminus \pi^{-1}(V(P))$ bijectively onto $S(P) \setminus V(P)$; and
  (d) π maps B (W, respectively) bijectively onto P.

The left-hand side of Figure 5 depicts the B-W decomposition of ∂N. In the figure, the arrows show the concave traversing field on N defined by the branched spine P. Remark that the curve e consists of the concave points on the boundary. The right-hand side shows the trivalent graph $\pi^{-1}(S(P)) \cap \partial N$. In the figure, the arrows shows the retraction π induced by the collapsing, see [2, Section 3.3] for more details on B-W decomposition.

The above description provides a way to present the flow-spine P by a 3-regular graph $G := \pi^{-1}(S(P)) \cap \partial N \subset \partial N \cong S^2$ and the pairing on S^2 given by π. This presentation is called a *DS-diagram.*

2. The Reidemeister-Tureav torsions of the standard Spin^c structures

In this section, we introduce an algorithmic method for constructing punctured Heegaard diagrams of Seifert fibered 3-manifolds in terms of the Seifert invariants.

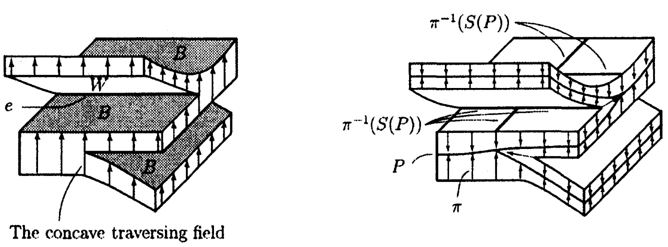

FIGURE 5. The B-W decomposition of ∂N.

2.1. **Construction of punctured Heegaard diagrams of the standard Spin^c structures.** It is easy to see that each Seifert fibered 3-manifold decomposes into finite copies of the pieces (trice-punctured sphere) $\times$ S^1, (once-punctured torus) $\times$ S^1 and a fibered torus, where D_1, D_2 and D_3 are mutually disjoint closed disks in S^2 and D' is a closed disk in $S^1 \times S^1$, by cutting along tori on which the fibers are tangential. Our construction of a punctured Heegaard diagram of a Standard Spin^c structure of a Seifert fibered 3-manifold is based on this decomposition.

Let H_R, H_L, $H_{\overline{R}}$, $H_{\overline{L}}$ and H_C be the pieces of a punctured Heegaard diagram shown in Figure 6. In the figure, the curves α are bold and the curves β are thin. For H_R or H_L, the disks D^- and D^+ are identified to be a meridian disk D of genus 1 compact orientable surface with two boundary components.

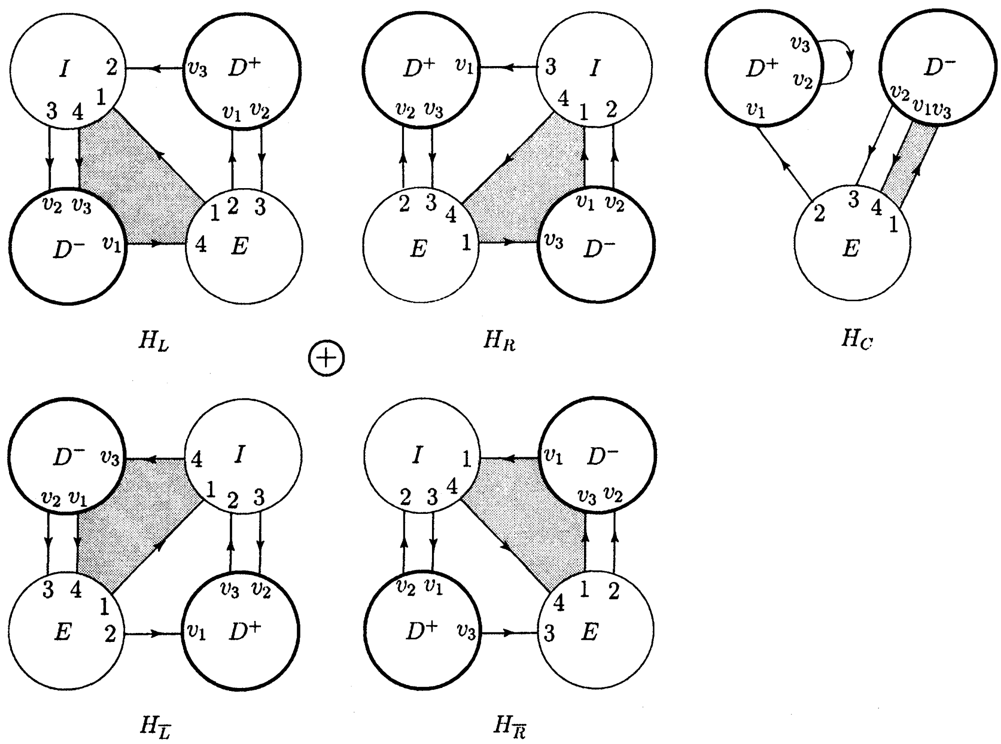

FIGURE 6. The pieces H_L, H_R, $H_{\overline{L}}$, $H_{\overline{R}}$ and H_C.

We use the following notation for a continued fraction:

$$[a_1, a_2, \ldots, a_n] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

For a pair of mutually coprime natural numbers p, q such that $p > q$, we define a word $w(p, q)$ of the letters L and R as follows:

$$w(p,q) := \begin{cases} L^{a_1}R^{a_2}L^{a_3}\cdots L^{a_{n-2}}R^{a_{n-1}}L^{a_n} & \text{(if } n \text{ is odd)} \\ L^{a_1}R^{a_2}L^{a_3}\cdots R^{a_{n-2}}L^{a_{n-1}}R^{a_n} & \text{(if } n \text{ is even),} \end{cases}$$

where $a_1, a_2, \ldots, a_n$ are natural numbers with $q/p = [a_1, a_2, \ldots, a_n, 1]$.

Given a word $w(p, q)$, where $q/p = [a_1, a_2, \ldots, a_n, 1]$, we construct a piece of punctured Heegaard diagram $H_{(p,q)}$, which corresponds to a fibered solid torus of type (p, q), in the following way. Take a_1 copies of the diagram H_L. Then attach the boundary ∂E of the i-th diagram H_L and the disk ∂I of the $(i + 1)$-th one along their boundaries following the numbers $1, 2, 3, 4$, for each $i = 1, 2, \ldots, a_1 - 1$. For the disk I of the first diagram H_L, attach the disk E of the diagram H_C. Next, take a_2 copies of the diagrams H_R. Then attach the boundary ∂E of the j-th diagram H_R and the boundary ∂I of the $j+1$-th one along their boundaries so that the numbers $1, 2, 3, 4$ on the both boundary circles match, for each $j = 1, 2, \ldots, a_2 - 1$. For the disk I of the first diagram H_R, attach the boundary ∂E of the a_1-th diagram H_L. Continuing this process, we finally get a diagram by gluing $1 + \sum_{i=1}^{n} a_i$ pieces of H_L, H_R and H_C, , see Figure 7. We denote the resulting piece of a

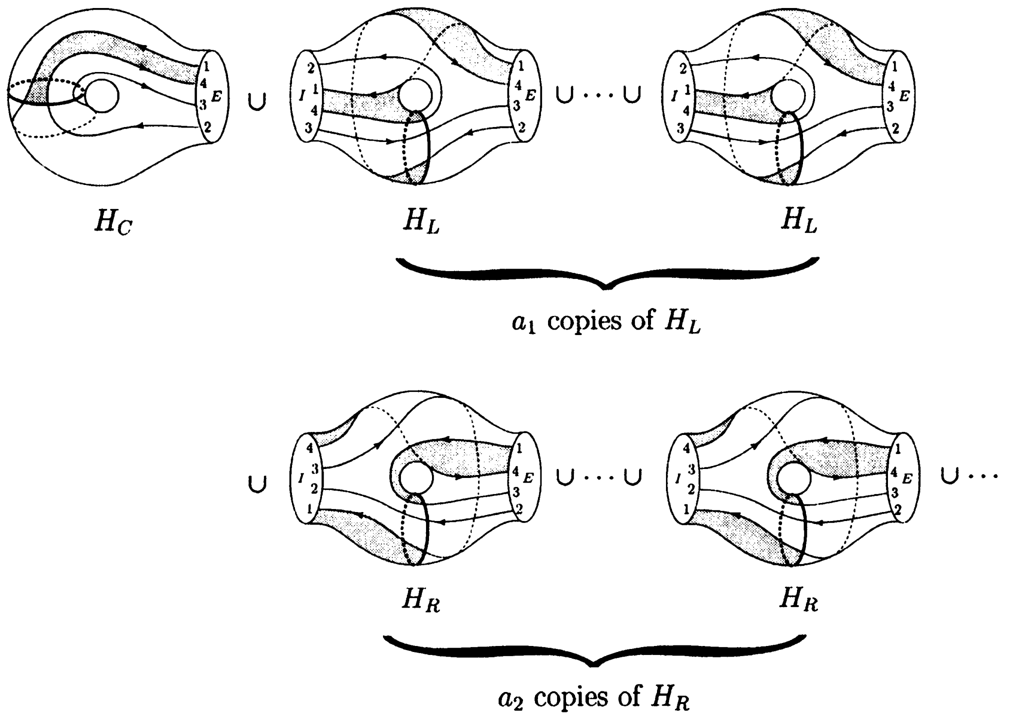

FIGURE 7. Gluing H_C and a_1 copies of H_L makes a larger piece of a punctured Heegaard diagram.

punctured Heegaard diagram by $H_{(p,q)}$.

We define H_b ($b \in \mathbb{Z}$) to be another piece of a punctured Heegaard diagram constructed following the same argument using the word $LR^b\overline{L}$ when b is non-negative and $L\overline{R}^{-b}\overline{L}$ otherwise.

Let H_S and H_T be the pieces of a punctured Heegaard diagram shown in Figure 8 and 9, respectively. These pieces correspond to either (trice-punctured sphere) $\times$ S^1 and (once-punctured torus) $\times$ S^1, respectively. Again, we consider that the curves α are bold and the curves β are thin in the figure.

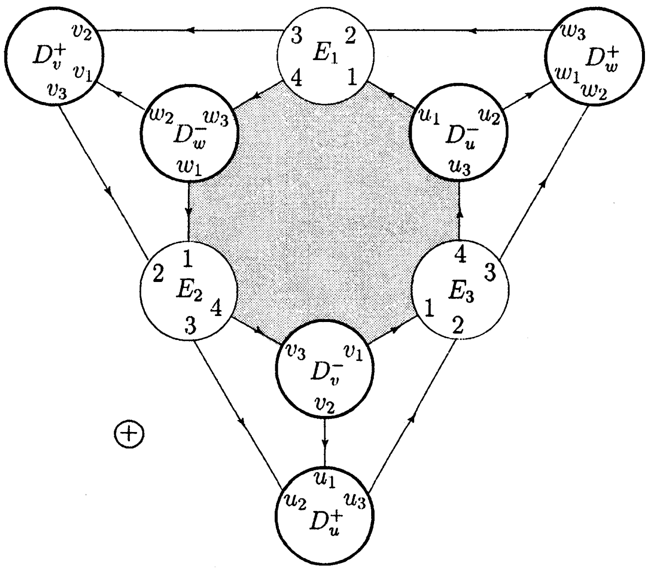

FIGURE 8. The piece H_S.

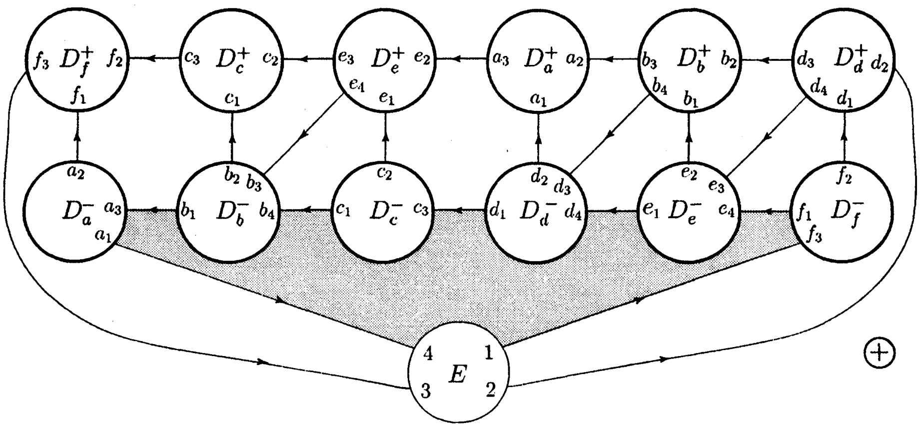

FIGURE 9. The piece H_T.

Let g be a non-negative integer and b be an integer. Let $(p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)$ be pairs of mutually coprime integers such that $1 < p_i$ and $0 < q_i < p_i$ ($i = 1, 2, \ldots, r$).

Assume that $g+r \geqslant 2$. Prepare $g+r-1$ copies $H_S^1, H_S^2, \ldots, H_S^{g+r-2}$ of the piece H_S and g copies $H_T^1, H_T^2, \ldots, H_T^g$ of the piece H_T. First, attach the boundary E of the piece H_b of punctured Heegaard diagram to the boundary ∂E_1 of the piece H_S^1 so that the numbers $1, 2, 3, 4$ on the both boundary circles match. For odd k with $1 \leqslant k \leqslant r$, attach the boundary I of the piece $H_{(p_k,q_k)}$ of piece to the boundary ∂E_2 of the piece H_S^k in the same manner as above. For even k with $1 \leqslant k \leqslant r$, attach the boundary E of the piece $H_{(p_k,p_k-q_k)}$ of a punctured Heegaard diagram to the boundary ∂E_2 of the piece H_S^k in the same manner as above. For $1 \leqslant k \leqslant g-1$, attach the boundary E of the piece H_T^k to the boundary ∂E_2 of the piece H_S^{r+k} in the same manner as above. Attach the boundary E of the piece H_T^g to the boundary ∂E_3 of the piece H_S^{g+r-1} in the same manner as above. Note that now we have $g+r-1$ components of pieces $W_1, W_2, \ldots, W_{g+r-1}$ of a punctured Heegaard diagram such that

- W_1 contains both H_b and H_S^1;
- W_k contains H_S^k for $2 \leqslant k \leqslant r$;
- W_k contains H_T^k for $r < k \leqslant g+r-2$; and
- W_{g+r-1} contains both H_T^{g+r-1} and H_T^{g+r}.

For each even k with $1 \leqslant k \leqslant g+r-2$, change the fixed normal direction of the diagram W_k and

Now we get a punctured Heegaard diagram by attaching the boundary ∂E_3 of the diagram W_k to the boundary ∂E_1 of the diagram W_{k+1} for $1 \leqslant k \leqslant g+r-2$. We denote it by $H_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))}$.

If $g+r \leqslant 2$, attach the piece H_b of a punctured diagram to the boundary ∂E_1 of the piece H_S^1. Moreover, attach the rest of the pieces $H_{(p_i,q_i)}$ and copies of H_T, if any, to the boundaries E_2 and E_3. In particular, if $g+r<2$, attach the copies of H_C to all the remaining boundary components of H_S^1.

Theorem 2.1. *The punctured Heegaard diagram $H_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))}$ corresponds to the Seifert fibered 3-manifold $S(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))$ with a standard Spin^c structure.*

Proof. The idea of the proof is to construct the pieces of the punctured Heegaard diagram corresponding to the pieces of the DS-diagram constructed in [11] following the proof of Theorem 5.5.

Let π, B, W and e be as described in Section 1.6. Set $A := \eta(e; \partial\eta(P))$. Recall that e has the B part on one side and the W one on the other side. The key idea is to draw a simple closed curve C in A such that

(1) C is isotopic to e in A;
(2) $C \cap e \neq \emptyset$ and C intersects e transversely; and
(3) $C \cap \pi^{-1}(S(P)) \subset e \setminus \pi^{-1}(V(P))$.

Let $\mathcal{H}_L$ be a piece of DS-diagram (on the annulus) shown in Figure 10 (i). This diagram was constructed in [11]. The curve e lies horizontally in the middle part of the diagram and it separates the diagram into B-part, on the upper side, and W-part, on the lower side. Then the intersection $C \cap \mathcal{H}_R$ is depicted by the bold lines in Figure 10 (ii). The two curves $C \cap \mathcal{H}_R$ cut the annulus into two disks, the under piece of which corresponds to the joining disk. Note that the disk D^- shown in the figure is identified via the projection π with D^+. Now we get a piece H_L of a punctured Heegaard diagram. See Figure 11.

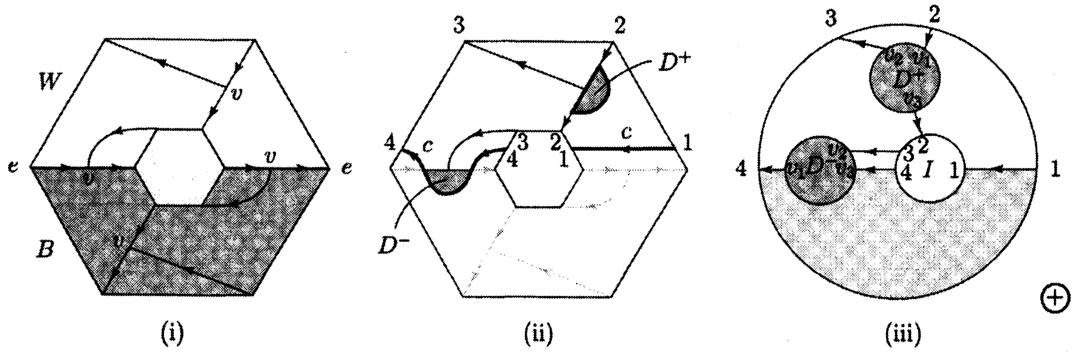

FIGURE 10. From $\mathcal{H}_L$ to H_L.

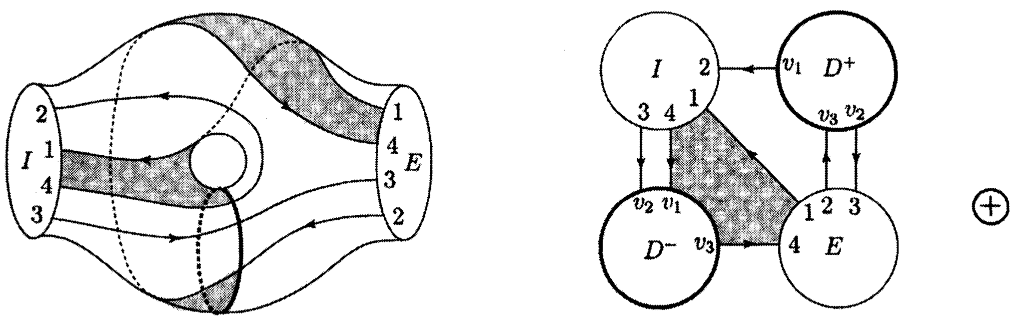

FIGURE 11. The piece H_L of a punctured Heegaard diagram.

For the other pieces shown in [11], we can apply the same argument. Cnsequently, we get the assertion. □

Remark 2.2. Forgetting the joining disk of the diagram $H_{(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))}$, one has a Heegaard diagram of the Seifert fibered manifold $S(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))$. For each piece of the Heegaard diagram corresponding to a singular fiber obtained in the above construction, the diagram can be destabilized so that it is a diagram on a once-punctured torus.

2.2. **Algorithm.** Let M be a Seifert fibered 3-manifold $S(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))$. Let $H_{(S(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))} = (S_g;\alpha,\beta,D)$ be the punctured Heegaard diagram constructed as above. Recall that once given a punctured Heegaard diagram, the Heegaard surface S_g assumed to be naturally oriented as explained in Section 1. Let F be a field and $\varphi:\mathbb{Z}[H_1(M_{(S;\alpha,\beta;D)})]\to F$ be a ring homomorphism. We can calculate the Reidemeister-Turaev torsion of the standard Spin^c structure of M, i.e. the principal Reidemeister torsion $T^\varphi(M)$, in the following algorithmic way (cf. [7]):

- **Step 1:** Orient α and β, and take base points of β following the rule prescribed in Section 1.
- **Step 2:** Get a presentation $\langle x_1,\dots,x_g \mid r_1,\dots,r_g\rangle$ of $\pi_1(M,*)$ using the punctured Heegaard diagram $(S;\alpha,\beta;D)$ as in the rule of Section 1.5.
- **Step 3:** Find an arbitrary dual system γ of β in the diagram $(S;\alpha,\beta;D)$ and relate a word y_i of $x_1,\dots,x_g$ to each loop γ_i in γ in the same rule as in Section 1.5.

Step 4: If there exist two integers $k, l \in \{1, \dots, g\}$ such that all of $\det B_{k,l}$, $\varphi([y_l])-1$ and $\varphi([y_l])-1$ are nonzero, then we have

$$\tau^{\varphi}(M, \mathcal{V}_{st}) = \pm \frac{\det B_{k,l}}{(\varphi([x_k])-1)(\varphi([y_l])-1)} \in F^{\times}/\pm 1,$$

where $B_{k,l}$ is the (k,l)-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\le i,j\le g}$. If there are not such integers k and l, then it turns out that the twisted chain complex $C^{\varphi}(M)$ is not acyclic, hence we have $\tau^{\varphi}(M, \mathcal{V}_{st}) = 0$ by definition.

Remark that due to [8] and [14], the above also gives an purely combinatorial algorithm to compute the Seiberg-Witten invariant of standard Spin^c structure when the given Seifert fibered 3-manifold has the first homology group of infinite order.

3. Examples and observations

3.1. **Lens spaces.** Using the algorithm in Section 2.2 for a lens space $L(p,q)$, we get a Spin^c structure on $L(p,q)$ and a presentation of $\pi_1(L(p,q))$ corresponding to the Spin^c structure can be written as $\pi_1(L(p,q)) = \langle x \mid x^p \rangle$ after simplifying the generators and relators. Then for a representation $\varphi : H_1(L(p,q)) \to F^{\times}$, we have a well-known result $\tau^{\varphi}(L(p,q), [\mathcal{V}_{st}]) = \pm 1/(\zeta-1)(\zeta^r-1)$, where $\zeta = \varphi([x])$.

Let us focus on the lens space $L(11,1)$. The set of the values of the Reidemeister-Turaev torsions of the Spin^c structures of $L(11,1)$ is:

$$\{\tau^{\varphi}(L(11,1),[\mathcal{V}]) \mid [\mathcal{V}] \in \mathrm{Spin}^c(L(11,1))\} = \left\{ \pm\frac{\zeta^i}{(\zeta-1)^2} \in F^{\times}/\pm 1 \;\middle|\; 0 \leqslant i < 11 \right\}.$$

In this set, only the two values $\pm 1/(\zeta-1)^2$ and $\pm\zeta^2/(\zeta-1)^2$ can be modified so that the numerator is ± 1 and the denominator are the form of $(\zeta^a-1)(\zeta^b-1)$ for some $a, b \in \mathbb{Z}$. In fact, we have $\pm\zeta^2/(\zeta-1)^2 = \pm 1/(\zeta^{10}-1)^2$. Note that the value $\pm 1/(\zeta-1)^2$ is the torsion of the Spin^c structure derived from the standard Seifert fibration of $(L(11,1))$ and $\pm\zeta^2/(\zeta-1)^2$ is that of the Spin^c structure derived from the standard Seifert fibration of $(L(11,10))$.

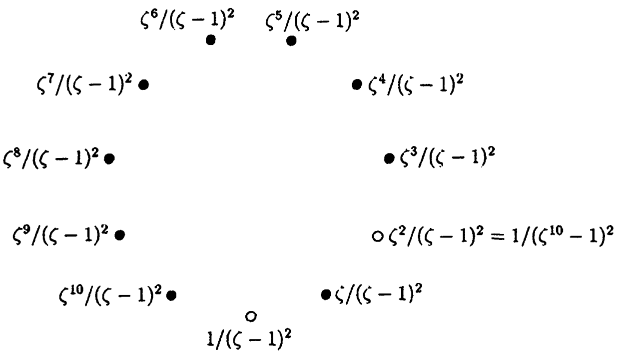

FIGURE 12. The set of Spin^c structures on $L(11,1)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Next, consider the lens space $L(11,2)$. For this manifold, the set of the values of the Reidemeister-Turaev torsions of the Spin^c structures is:

$$\{\tau^{\varphi}(L(11,2),[\mathcal{V}]) \mid [\mathcal{V}] \in \mathrm{Spin}^c(L(11,2))\} = \left\{ \pm\frac{\zeta^i}{(\zeta-1)(\zeta^6-1)} \in F^{\times}/\pm 1 \;\middle|\; 0 \leqslant i < 11 \right\}$$

In this set, exactly the four values $\pm 1/(\zeta-1)(\zeta^6-1)$, $\pm\zeta/(\zeta-1)(\zeta^6-1)$, $\pm\zeta^6/(\zeta-1)(\zeta^6-1)$ and $\pm\zeta^7/(\zeta-1)(\zeta^6-1)$ can be modified so that the numerator is ± 1 and the denominator are the form of $(\zeta^a-1)(\zeta^b-1)$ for some $a, b \in \mathbb{Z}$. In fact, we have $\pm\zeta/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta^6-1)(\zeta^{10}-1)$, $\pm\zeta^6/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta-1)(\zeta^5-1)$ and $\pm\zeta^7/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta^5-1)(\zeta^{10}-1)$.

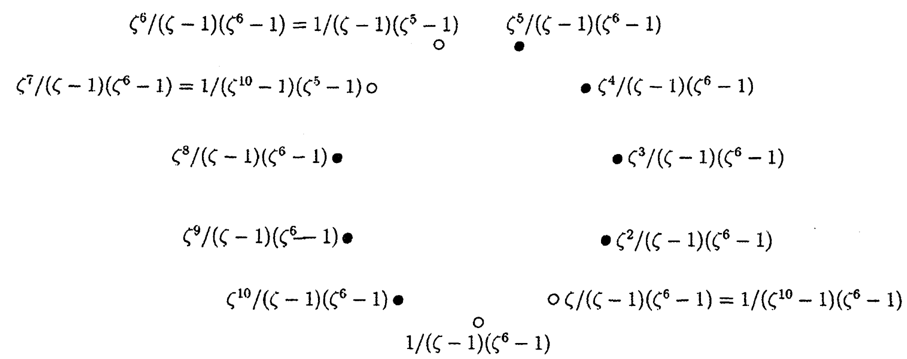

FIGURE 13. The set of Spin^c structures on $L(11,2)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Observation 3.1. *The Reidemeister-Turaev torsion of a Spin^c structure of a lens space is of the form $\pm 1/(\zeta^a-1)(\zeta^b-1)$ for some $a, b \in \mathbb{Z}$ if and only if the Spin^c structure is standard.*

3.2. $S_g \times S^1$. Let S_g be a closed orientable surface of genus $g > 1$ and consider the Seifert fibered 3-manifold $S_g \times S^1$. Using the algorithm in Section 2.2 for $S_g \times S^1$, we get a Spin^c structure $\mathcal{V}_{st}$ on $S_g \times S^1$ and a presentation of $\pi_1(S_g \times S^1)$ corresponding to the Spin^c structure can be written as

$$\pi_1(S_g \times S^1) = \langle x_1, x_2, \ldots, x_{2g}, y \mid x_i y {x_i}^{-1} y^{-1}, i = 1, 2, \ldots, 2g, \prod_{i=1}^{g} (x_{2i-1} x_{2i} {x_{2i-1}}^{-1} {x_{2i}}^{-1}) \rangle,$$

and its abelianization is:

$$H_1(S_g \times S^1) := \left(\bigoplus_{i=1}^{2g} \mathbb{Z}\langle [x_i] \rangle \right) \oplus \mathbb{Z}\langle [y] \rangle.$$

Let $\varphi : \mathbb{Z}[H_1(S_g \times S^1; \mathbb{Z})] \to F$ be a ring homomorphism to a field F such that each of $\zeta_i = \varphi([x_i])$ and $\zeta = \varphi([y])$ has an infinite order. Then we have

$$\tau^{\varphi}(S_g \times S^1, [\mathcal{V}_{st}]) = \pm(\zeta-1)^{2g-2}.$$

The set of the values of the Reidemeister-Turaev torsions of the Spin^c structures of $S_g \times S^1$ is:

$$\begin{aligned} &\{\tau^{\varphi}(S_g \times S^1, [\mathcal{V}]) \mid [\mathcal{V}] \in \mathrm{Spin}^c\,(S_g \times S^1)\} \\ = &\left\{ \pm\zeta_1^{i_1} \cdots \zeta_{2g}^{i_{2g}} \zeta^i (\zeta - 1)^{2g-2} \in F^{\times}/\pm 1 \;\middle|\; i_1, \ldots, i_{2g}, i \in \mathbb{Z} \right\} \end{aligned}$$

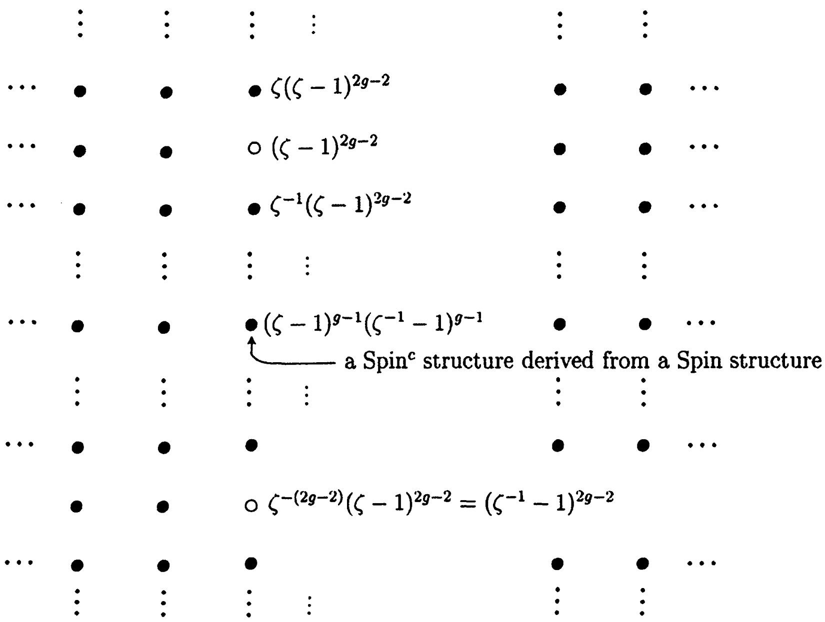

FIGURE 14. The set of Spin^c structures on $S_g \times S^1$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Observation 3.2. *The Reidemeister-Turaev torsion of a Spin^c structure of $S_g \times S^1$ is of the form $\pm(\zeta^a - 1)^{2g-2}$ for some $a \in \mathbb{Z}$ if and only if the Spin^c structure is standard.*

3.3. **Brieskorn 3-manifolds.** The Brieskorn manifold $\Sigma(p,q,r)$ of type (p,q,r) is a closed 3-manifold defined by:

$$\Sigma(p,q,r) := \{(x,y,z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 = 1, x^p + y^q + z^r = 0\},$$

where p, q and r are integers greater than 1.

$\Sigma(p,q,r)$ is the r-fold branched covering of the 3-sphere S^3 branched along a torus knot or link of type (p,q). The first integral homology groups of the Brieskorn manifolds is

$$H_1(\Sigma(p,q,r);\mathbb{Z}) = \begin{cases} 1 & n = \pm 1 \pmod 6 \\ \mathbb{Z}/3\mathbb{Z} & n = \pm 2 \pmod 6 \\ \mathbb{Z}\ 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 3 \pmod 6 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 0 \pmod 6 \end{cases}.$$

Using the algorithm in Section 2.2 for $\Sigma(2,3,6n)$, we get a Spin^c structure $\mathcal{V}_{st}$ on $\Sigma(2,3,6n)$ and a presentation of $\pi_1(\Sigma(2,3,6n))$ corresponding to the Spin^c structure can be written as

$$\pi_1(\Sigma(2,3,6n)) = \langle x_1, x_2, \dots, x_{6n} \mid x_i {x_{i+6n-1}}^{-1} {x_{i+1}}^{-1}, 1 \leqslant i \leqslant 6n \rangle$$

and its abelianization is:

$$H_1(\Sigma(2,3,6n);\mathbb{Z}) := \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [x_2] \rangle.$$

Let $\varphi : \mathbb{Z}[H_1(\Sigma(2,3,6n);\mathbb{Z})] \to F$ be a ring homomorphism to a field F such that each of $\zeta_1 = \varphi([x_1])$ and $\zeta_2 = \varphi([x_2])$ has an infinite order. Then we have

$$\tau^{\varphi}(\Sigma(2,3,6n),[\mathcal{V}_{st}]) = \pm \frac{\det\left(\varphi([\frac{\partial x_i {x_{i+6n-1}}^{-1} {x_{i+1}}^{-1}}{\partial x_j}])\right)_{1,1}}{({\zeta_1}^{-1}-1)(\zeta_1 - 1)} = \pm n.$$

The set of the values of the Reidemeister-Turaev torsions of the Spin^c structures of $S_g \times S^1$ is:

$$\{\tau^{\varphi}(\Sigma(2,3,6n),[\mathcal{V}]) \mid [\mathcal{V}] \in \mathrm{Spin}^c(\Sigma(2,3,6n))\} = \left\{ \pm n \zeta_1^{i_1} \zeta_2^{i_2} \in F^{\times} / \pm 1 \;\middle|\; i_1, i_2 \in \mathbb{Z} \right\}$$

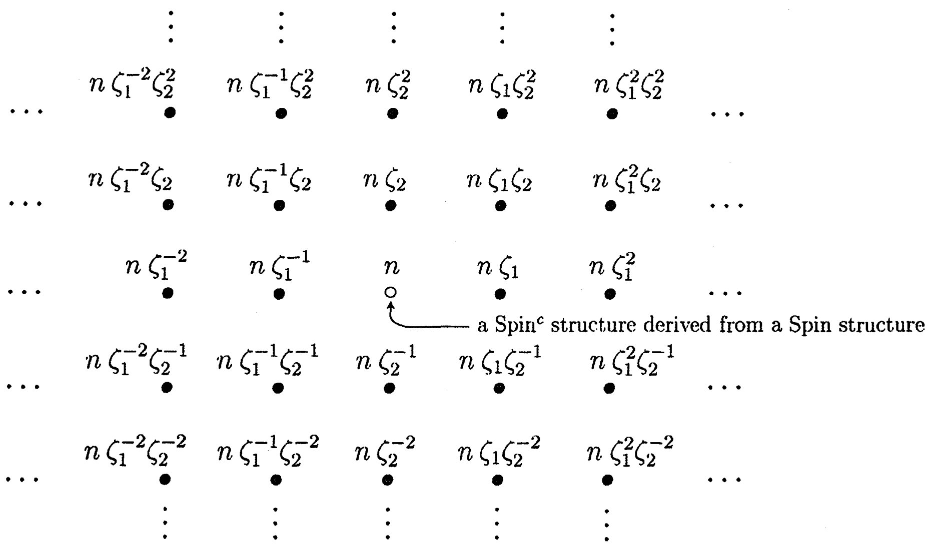

FIGURE 15. The set of Spin^c structures on $\Sigma(2,3,6n)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dot is the standard Spin^c structure.

Observation 3.3. *The Reidemeister-Turaev torsion of a Spin^c structure of the Brieskorn 3-manifolds $\Sigma(2,3,6n)$ $(n \in \mathbb{N})$ is of the form $\pm a$ for some $a \in \mathbb{Z}$ if and only if the Spin^c structure is standard.*

From the above observations, we may roughly say that the Reidemeister-Turaev torsions of the standard Spin^c structures of a Seifert fibered 3-manifold have *standard* values among the set of the Reidemeister-Turaev torsions of all Spin^c structures on the manifold.

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