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A class of preorders iterated under a type of RCS

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Abstract

We formulate a class of preorders which we call preproper. This class contains the class of proper preorders and is a modification to the subproper preorders by R. Jensen. We show that this class of preorders is iterable under a type of revised countable support iteration.

Introduction

Jensen formulates classes of notions of forcing in [1]. We take subproper among those. This class is wider than proper, may differ from semiproper of Shelah and is iterable under the revised countable support a la Donder according to [1]. We formulate a class of notion of forcing whose definition involves less parameters than Jensen's subproper and show that it iterates under a type of revised countable support of [2]. We say a preorder P is preproper, if there exist a set z and a regular cardinal θ with $z, P \in H_\theta$ such that for all (χ, A, N, p, s) such that

- χ is a regular cardinal with $H_\theta \subseteq L_\chi[A]$,
- $(N, \in, A \cap N)$ is a countable elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$ with $z, P \in N$,
- there exists a transitive set model M of ZFC$^-$ (i.e. no power set axiom) such that the transitive collapse of N appears as a $(H_\tau)^M$ with a regular τ in M,
- $p \in P \cap N$ and $s \in N$,

  there exists $(q, \dot{N})$ such that $q \leq p$ in P, $\dot{N}$ is a P-name and q forces the following three.
- $(\dot{N}, \in, A \cap \dot{N})$ is a countable elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$.
- There exists an isomorphism between the two structures $(N, \in, A \cap N)$ and $(\dot{N}, \in, A \cap \dot{N})$ which fixes s.
- For all dense subsets $D \in \dot{N}$ of P, we have $D \cap \dot{N} \cap \dot{G} \neq \emptyset$, where $\dot{G}$ denotes the canonical P-name of the P-generic filters over the ground model V.

Therefore we do not use H_χ's but $L_\chi[A]$ which are fat enough relative to P. A reason to use the structures $(L_\chi[A], \in, A \cap L_\chi[A])$ is to escape definabilities of H_χ's calculated in finitely many intermediate stages in iterated forcing. We do not expect genericity over N but over $\dot{N}$ which may exist in the generic extension. The condition on the collapse of N is very technical without which we may have no control over the isomorphic images of N. The roll of s is to fix finite-parts in an increasing manner to diagonally build a new isomorphic elementary substructures in the limit stages of iterated forcing.

§ 1. Preliminary

We collect what we think are basics in this subject.

Proposition. Let P be a preorder. Let θ and χ be regular cardinals such that $P \in H_\theta \subseteq L_\chi[A]$. Let $P \in N$ and $(N, \in, A \cap N)$ be an elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$, which we simply denote by

$$(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A]).$$

Let G be a P-generic filter over the ground model V. Let $L_\chi[A][G] = \{\sigma_G \mid \sigma \in L_\chi[A], \sigma \text{ is a } P\text{-name}\}$ and $N[G] = \{\sigma_G \mid \sigma \in N, \sigma \text{ is a } P\text{-name}\}$. Let $B = (\{0\} \times (A \cap L_\chi[A])) \cup (\{1\} \times G)$. Then we have

(1) $(L_\chi[A], \in, A \cap L_\chi[A])$ models ZFC$^-$ in the expanded language.

(2) $(L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$ models ZFC$^-$ in the expanded language.

(3) If $\pi \in N$ is a P-name and $\varphi(x, y)$ is a formula in the expanded language, then there exists a P-name $\tau \in N$ such that $\Vdash_P$"$(L_\chi[A][\dot{G}], \in, L_\chi[A], A \cap L_\chi[A]) \models$ "$\exists y\varphi(y, \pi) \longrightarrow \varphi(\tau, \pi)$"". And so we have

$$(N[G], \in, L_\chi[A] \cap N[G], (A \cap L_\chi[A]) \cap N[G]) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A]).$$

(4) The following five are all equivalent.

- $N[G] \cap L_\chi[A] = N$.
- $N[G] \cap \chi = N \cap \chi$.
- For all dense subsets $D \in N$, we have $D \cap N \cap G \neq \emptyset$.
- $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$.
- The transitive collapse of $(N, \in)$ gets extended to the transitive collapse of $(N[G], \in)$.

(5) $L_\chi[A][G] = L_\chi[B]$ holds. If $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$, then

$$(N[G], \in, B \cap N[G]) \prec (L_\chi[B], \in, B \cap L_\chi[B]).$$

(6) Let $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$ and two substructures $(N[G], \in, B \cap N[G])$ and $(M, \in, B \cap M) \prec (L_\chi[B], \in, B \cap L_\chi[B])$ be isomorphic fixing P. Let $X = L_\chi[A] \cap M$. Then $P \in X$, $(X, \in, A \cap X) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ and $M = X[G]$ hold. We also have

$$(X[G], \in, X, A \cap X) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A]).$$

The two substructures $(N[G], \in, N, A \cap N)$ and $(X[G], \in, X, A \cap X)$ are isomorphic by the given isomorphism. So are $(N, \in, A \cap N)$ and $(X, \in, A \cap X)$ by restricting the given isomorphism.

(7) Let us further assume in (6) that we have a preorder $Q \in H_\theta^{V[G]} \cap M$, H is Q-generic over $V[G]$ and $(M[H], \in, M, B \cap M) \prec (L_\chi[B][H], \in, L_\chi[B], B \cap L_\chi[B])$. Then

$$(X[G][H], \in, X[G], X, A \cap X) \prec (L_\chi[A][G][H], \in, L_\chi[A][G], L_\chi[A], A \cap L_\chi[A]).$$

Proof. Mostly routine interpreting formulas in structures with unary predicates. Notice that we assume every dense subset $D \in V$ of P belongs to H_θ and so to $L_\chi[A]$, though $L_\chi[A]$ may not satisfy the power set axiom. Details are left.

□

§ 2. Iteration Lemma

We define the class of preorders under consideration and show that it iterates under the revised countable support iteration of [2].

Definition. A preorder P is *preproper*, if there exist θ and z such that

- θ is a regular cardinal and $z, P \in H_\theta$.
- Given any (A, χ, N, p, s) such that
  - $H_\theta \subseteq L_\chi[A]$, χ is a regular cardinal.
  - $z, P \in N$, N is countable and $(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A])$.
  - The transitive collapse $\overline{N}$ of N satisfies $\overline{N} = (H_\tau)^M$ for some M, where M is a transitive set model of ZFC$^-$ (i.e. ZFC minus the power set axiom) and τ is a regular cardinal in M.

• $p \in P \cap N$ and $s \in N$.

There exists $(q, \dot{N}, \dot{f})$ such that

• $q \leq p$ and $\dot{N}, \dot{f}$ are P-names.

• q forces the following.

(1) $\dot{f}$ is an isomorphism from $(N, \in, A \cap N)$ to $(\dot{N}, \in, A \cap \dot{N}) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ with $\dot{f}(s) = s$.

(2) For all dense subsets $D \in \dot{N}$ of P, $D \cap \dot{N} \cap \dot{G} \neq \emptyset$.

Remark. (1) The condition (2) in the definition is equivalent to

$$(\dot{N}[\dot{G}], \in, \dot{N}, A \cap \dot{N}) \prec (L_\chi[A][\dot{G}], \in, L_\chi[A], A \cap L_\chi[A]),$$

where $L_\chi[A][\dot{G}]$ denotes the generic extension of $L_\chi[A]$ via P.

(2) $(L_\chi[A][\dot{G}], \in, L_\chi[A], A \cap L_\chi[A])$ and $(L_\chi[B], \in, B \cap L_\chi[B])$ are suitably interpretable equivalent structures with $L_\chi[A][\dot{G}] = L_\chi[B]$, where $B = (\{0\} \times (A \cap L_\chi[A])) \cup (\{1\} \times \dot{G})$.

(3) If a preorder P is proper, then P is preproper.

(4) This formulation of preproper is tentative and is a modification to the subproper of [1]. There remains a chance to further reformulate this notion of forcing. A possible modification may include relativizing things in $L[A]$ so that if $j : V \longrightarrow M$ is an elementary embedding and P is preproper in V, then P is preproper in M whenever, say, $H_\theta \in M$.

However, we know little on the notion of preproper except the following.

Lemma. Let $\langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration such that for all $\alpha < \nu$, $\Vdash_{P_\alpha}$ "$P_{\alpha\alpha+1}$ is preproper with $\dot{\theta}_\alpha$ and $\dot{z}_\alpha$". Let θ and z be such that

• θ is a regular cardinal such that for all $\alpha < \nu$, $\Vdash_{P_\alpha}$ "$\dot{\theta}_\alpha < \theta$".

• $\langle P_\alpha \mid \alpha \leq \nu \rangle$, $\langle (\dot{\theta}_\alpha, \dot{z}_\alpha) \mid \alpha < \nu \rangle \in z \prec H_\theta$ and z is countable.

Then for all (α, β) with $\alpha < \beta \leq \nu$, we have $\Vdash_{P_\alpha}$ "$P_{\alpha\beta}$ is preproper with θ and $(\dot{G}_\alpha, z)$".

Proof. By induction on β (for all $\alpha < \beta$). We assume that for all (i, j) with $i < j < \beta$, $\Vdash_{P_i}$ "P_{ij} is preproper with θ and $(\dot{G}_i, z)$". Fix any $\alpha < \beta$. We want to show $\Vdash_{P_\alpha}$ "$P_{\alpha\beta}$ is preproper with θ and $(\dot{G}_\alpha, z)$".

Case. β is limit. Suppose $w \Vdash_{P_\alpha}$ "$(\dot{A}, \chi, \dot{N}, p\lceil[\alpha, \beta), \dot{s})$ as in the hypothesis". Namely, w forces

• $H_\theta^{V[\dot{G}_\alpha]} \subseteq L_\chi[\dot{A}]$, χ is regular.

• $\dot{G}_\alpha, z, P_{\alpha\beta} \in \dot{N}$ is countable and $(\dot{N}, \in, \dot{A} \cap \dot{N}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ and $\dot{N}$ satisfies the condition on the transitive collapse.

• $p\lceil[\alpha, \beta) \in P_{\alpha\beta} \cap \dot{N}$ and $\dot{s} \in \dot{N}$.

Then we have that

• $z, \beta \in \dot{N}$ and so $P_\beta \in \dot{N}$.

We may assume that

• $p \in P_\beta \cap \dot{N}$, $p\lceil\alpha \in \dot{G}_\alpha$ and p has stages $\langle \delta_k^p \mid k < \omega \rangle \in \dot{N}$.

We may also extend w so that

• $w \leq p\lceil\alpha$.

• $w^\frown(1_\beta\lceil[\alpha, \beta)) \Vdash_{P_\beta}$ "$\delta_0^p = \alpha$".

This is because $z \in \dot{N} \prec L_\chi[\dot{A}]$ and so $\dot{N}$ contains various maps defined in V and so $\dot{N}$ is closed under those maps. For example, we have $\langle p \mapsto \langle \delta^p_k \mid k < \omega \rangle \rangle \in \dot{N}$.

Hence w decides P_α-names $\dot{A}$, $\dot{N}$, $\dot{s} \in V$ and the values of χ, p, $\langle \delta^p_k \mid k < \omega \rangle$ and δ^p_0.

It suffices to find $(q, \dot{X}_\omega, \dot{f}_\omega)$ such that

- $q \in P_\beta$, $q\lceil\alpha = w$, $q \leq p$.
- $\dot{X}_\omega$ and $\dot{f}_\omega$ are P_β-names.

q forces the following.

(1) $\dot{f}_\omega$ is an ismorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{X}_\omega, \in, \dot{A} \cap \dot{X}_\omega) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$, $\dot{f}_\omega(\dot{s}) = \dot{s}$ and $G_\alpha (= \dot{G}_\beta\lceil\alpha), P_\beta \in \dot{X}_\omega$.

(2) For all dense subsets $D \in \dot{X}_\omega$ of P_β, $D \cap \dot{X}_\omega \cap \dot{G}_\beta \neq \emptyset$.

This is because, given any dense subset $D' \in \dot{X}_\omega$ of $P_{\alpha\beta}$, let

$$D = \{y \in P_\beta \mid y\lceil\alpha \notin G_\alpha \mid\mid y\lceil[\alpha, \beta) \in D'\}.$$

Then $D \in \dot{X}_\omega$ is a dense subset of P_β with $D\lceil[\alpha, \beta) = D'$. Since $D \cap \dot{X}_\omega \cap \dot{G}_\beta \neq \emptyset$, we have $D' \cap \dot{X}_\omega \cap G_{\alpha\beta}(= \dot{G}_\beta\lceil[\alpha, \beta)) \neq \emptyset$.

In order to get $(q, \dot{X}_\omega, \dot{f}_\omega)$ as such, we construct a nested antichain T with associated structures. To do so, we present the following general construction.

Claim. Given any $(i, x, \langle \delta^x_k \mid k < \omega \rangle, a, \dot{X}, \dot{f}, \dot{D})$ such that

- $\alpha \leq i$, $a \in P_i$, $x \in P_\beta$, $a\lceil\alpha \leq w$, $a \leq x\lceil i$ and $x \leq p$.
- $\langle \delta^x_k \mid k < \omega \rangle$ are stages for x and $a^\frown 1 \Vdash_{P_\beta}$ "$\delta^x_0 = i$".
- $a \in P_i$ forces the following, where $\dot{X}, \dot{f}, \dot{D} \in V$ are P_i-names.

(1) $\dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{X}, \in, \dot{A} \cap \dot{X}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ with $\dot{f}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta)$.

(2) $P_i \in \dot{X}$ and $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X}) \prec (L_\chi[\dot{A}][G_{\alpha i}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$.

(3) $x \in P_\beta \cap \dot{X}$ and $\langle \delta^x_k \mid k < \omega \rangle \in \dot{X}$.

(4) $\dot{D} \in \dot{X}$ is a dense subset of P_β.

Get $(j, y, \langle \delta^y_k \mid k < \omega \rangle, b, \dot{Y}, \dot{g})$ such that

- $i < j$, $b \in P_j$, $y \in P_\beta$, $b\lceil i \leq a$, $b \leq y\lceil j$ and $y \leq x$.
- $\langle \delta^y_k \mid k < \omega \rangle$ are stages for y and are a step ahead of $\langle \delta^x_k \mid k < \omega \rangle$, namely, $\Vdash_{P_\beta}$ "$\delta^x_{k+1} \leq \delta^y_k$" for all $k < \omega$.
- $b^\frown 1 \Vdash_{P_\beta}$ "$\delta^y_0 = j$".
- $b \in P_j$ forces the following, where $\dot{Y}, \dot{g} \in V$ are P_j-names.

(1*) $\dot{g}$ is an isomorphism from $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X})$ to

$$(\dot{Y}[G_{\alpha i}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$$

with

$$\dot{g}(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta^y_k \mid k < \omega \rangle) = (G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta^y_k \mid k < \omega \rangle).$$

(2*) $j \in \dot{X} \cap \dot{Y}$ and so $P_{\alpha j} \in \dot{Y}$, $L_\chi[\dot{A}][G_{\alpha j}] = L_\chi[\dot{A}][G_{\alpha i}][G_{ij}]$ and $\dot{Y}[G_{\alpha j}] = \dot{Y}[G_{\alpha i}][G_{ij}]$. We have

$$(\dot{Y}[G_{\alpha i}][G_{ij}], \in, \dot{Y}[G_{\alpha i}], \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}][G_{ij}], \in, L_\chi[\dot{A}][G_{\alpha i}], L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

And so $\dot{Y}[G_{\alpha j}] \cap L_\chi[\dot{A}][G_{\alpha i}] = \dot{Y}[G_{\alpha i}]$ and $\dot{Y}[G_{\alpha i}] \cap L_\chi[\dot{A}] = \dot{Y}$.

(3*) $y \in \dot{D} \cap \dot{X} \cap \dot{Y}$ and $\langle \delta^y_k \mid k < \omega \rangle \in \dot{X} \cap \dot{Y}$.

Hence, we have

- $b \in P_j$ forces the following.

(1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ with

$$\dot{g} \circ \dot{f}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta).$$

(2) $P_j \in \dot{Y}$ and $(\dot{Y}[G_{\alpha j}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha j}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$.

(3) $y \in P_\beta \cap \dot{Y}$ and $\langle \delta^y_k \mid k < \omega \rangle \in \dot{Y}$.

Proof. Let G_i be P_i-generic over V with $a \in G_i$. In $V[G_i]$, let $G_\alpha = G_i \lceil \alpha, G_{\alpha i} = G_i \lceil [\alpha, i), A = \dot{A}_{G_\alpha}$, $X = \dot{X}_{G_i}$, $f = \dot{f}_{G_i}$ and $D = \dot{D}_{G_i}$. Then we have

- $x \in X \cap P_\beta$, $P_{\alpha i} \in X$ and

$$(X[G_{\alpha i}], \in, X, A \cap X) \prec (L_\chi[A][G_{\alpha i}], \in, L_\chi[A], A \cap L_\chi[A]).$$

- $D \in X$ is a dense subset of P_β.

Get $y \in D \cap X[G_{\alpha i}] = D \cap X$, $y \lceil i \in G_i$, $y \le x$ and stages $\langle \delta^y_k \mid k < \omega \rangle \in X$ for y which is a step ahead of $\langle \delta^x_k \mid k < \omega \rangle \in X$.

Then decide the value of δ^y_0 as $u^\frown 1 \Vdash_{P_\beta}$ "$\delta^y_0 = j$". We may assume $j \in \beta \cap X[G_{\alpha i}] = \beta \cap X$, $u \in P_j \cap X[G_{\alpha i}] = P_j \cap X$, $u \le y \lceil j$ and $u \lceil i \in G_i$.

Since P_{ij} is preproper with θ, G_i, z and in $V[G_i]$, we have

- $H^{V[G_i]}_\theta = H^{V[G_\alpha]}_\theta[G_{\alpha i}] \subseteq L_\chi[A][G_{\alpha i}]$, χ is regular.
- $G_i, z, P_{ij} \in X[G_{\alpha i}]$, $(X[G_{\alpha i}], \in, X, A \cap X) \prec (L_\chi[A][G_{\alpha i}], \in, L_\chi[A], A \cap L_\chi[A])$.
- And so the transitive collapse $\overline{X[G_{\alpha i}]}$ of $X[G_{\alpha i}]$ satisfies

$$\overline{X[G_{\alpha i}]} = \overline{X}[\overline{G_{\alpha i}}] = (H_\tau)^M[\overline{G_{\alpha i}}] = (H_\tau)^{M[\overline{G_{\alpha i}}]} \text{ with } \overline{P_{\alpha i}} \in (H_\tau)^M.$$

- $u \in P_j \cap X[G_{\alpha i}], u \lceil i \in G_i$ and $G_\alpha, z, G_{\alpha i} \in X[G_{\alpha i}]$.

Hence we may fix $(b, \dot{Y}, \dot{g})$ in V such that

- $b \le u$ and $b \lceil i \le a$.
- $b \in P_j$ forces the following.

(1)* $\dot{g}$ is an isomorphism from $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X})$ to

$$(\dot{Y}[G_{\alpha i}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$$

with

$$\dot{g}(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta^y_k \mid k < \omega \rangle) = (G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta^y_k \mid k < \omega \rangle).$$

(2)*

$$(\dot{Y}[G_{\alpha i}][G_{ij}], \in, \dot{Y}[G_{\alpha i}], \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}][G_{ij}], \in, L_\chi[\dot{A}][G_{\alpha i}], L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

(3)* $y \in \dot{D} \cap \dot{X} \cap \dot{Y}$ and $\langle \delta^y_k \mid k < \omega \rangle \in \dot{X} \cap \dot{Y}$.

And so, we have

- $b \in P_j$ forces the following.

(1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$.

(2) $P_j \in \dot{Y}[G_{\alpha j}] \cap L_\chi[\dot{A}] = \dot{Y}$ and $(\dot{Y}[G_{\alpha j}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha j}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$.

(3) $y \in \dot{D} \cap \dot{Y}$ and $\langle \delta^y_k \mid k < \omega \rangle \in \dot{Y}$.

□

In V, we construct a nested antichain $\langle (a,n) \mapsto p^{(a,n)} \mid a \in T_n, n < \omega \rangle$ together with an associated structures $(\dot{X}^{(a,n)}, p^{(a,n)}, \langle \delta^{(a,n)}_k \mid k < \omega \rangle, \dot{D}^{(a,n)}, \dot{f}^{(a,n)}, S^{(a,n)}, \dot{f}^{(a,n)(b,n+1)})$ for $a \in T_n$, $b \in \mathrm{suc}^n_T(a)$, $n < \omega$ such that

- $T_0 = \{w\}$, $\dot{X}^{(w,0)} = \dot{N}$, $p^{(w,0)} = p$, $\langle \delta^{(w,0)}_k \mid k < \omega \rangle = \langle \delta^p_k \mid k < \omega \rangle$, $\dot{D}^{(w,0)} = \check{P}_\beta$, $\dot{f}^{(w,0)} = (id \text{ on } \dot{N})$, $S^{(w,0)} = \{p\}$.

Then for all $a \in T_n$, we have (with $n = 0$)

- $\alpha \leq l(a)$, $a \in P_{l(a)}$, $p^{(a,n)} \in P_\beta$, $a \lceil \alpha \leq w$, $a \leq p^{(a,n)} \lceil l(a)$ and $p^{(a,n)} \leq p$.
- $\langle \delta^{(a,n)}_k \mid k < \omega \rangle$ are stages for $p^{(a,n)}$ and $a^\frown 1 \Vdash_{P_\beta}$ "$\delta^{(a,n)}_0 = l(a)$".
- $S^{(a,n)}$ is a finite subset of P_β with $p^{(a,n)} \in S^{(a,n)}$.
- $a \in P_{l(a)}$ forces the following, where $\dot{X}^{(a,n)}$, $\dot{D}^{(a,n)}$, $\dot{f}^{(a,n)}$ are $P_{l(a)}$-names.

(1) $\dot{f}^{(a,n)}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{X}^{(a,n)}, \in, \dot{A} \cap \dot{X}^{(a,n)}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$$

with

$$\dot{f}^{(a,n)}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta).$$

(2) $S^{(a,n)} \cup \{G_\alpha, z, \dot{s}, P_{l(a)}\} \subset \dot{X}^{(a,n)}$ and so $G_{l(a)} \in \dot{X}^{(a,n)}[G_{\alpha l(a)}]$ and

$$(\dot{X}^{(a,n)}[G_{\alpha l(a)}], \in, \dot{X}^{(a,n)}, \dot{A} \cap \dot{X}^{(a,n)}) \prec (L_\chi[\dot{A}][G_{\alpha l(a)}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

(3) $p^{(a,n)} \in P_\beta \cap \dot{X}^{(a,n)}$ and $\langle \delta^{(a,n)}_k \mid k < \omega \rangle \in \dot{X}^{(a,n)}$.

(4) $\dot{D}^{(a,n)} \in \dot{X}^{(a,n)}$ is a dense subset of P_β. We demand the following for a bookkeeping.

$$\dot{D}^{(a,n)} = \begin{cases} \dot{f}^{(a,n)}(\dot{x}_n), & \text{if it is dense in } P_\beta. \\ P_\beta, & \text{otherwise,} \end{cases}$$

where $\dot{N} = \{\dot{x}_n \mid n < \omega\}$ with $\dot{x}_0 = \check{P}_\beta$.

For $b \in \mathrm{suc}^n_T(a)$,

- $p^{(b,n+1)} \leq p^{(a,n)}$ and $\langle \delta^{(b,n+1)}_k \mid k < \omega \rangle$ is a step ahead of $\langle \delta^{(a,n)}_k \mid k < \omega \rangle$.
- $S^{(b,n+1)} = S^{(a,n)} \cup \{p^{(b,n+1)}\}$.
- $b \in P_{l(b)}$ forces the following, where $\dot{f}^{(a,n)(b,n+1)}$ is a $P_{l(b)}$-name.

(1)* $\dot{f}^{(a,n)(b,n+1)}$ is an isomorphism from $(\dot{X}^{(a,n)}, \in, \dot{A} \cap \dot{X}^{(a,n)})$ to

$$(\dot{X}^{(b,n+1)}, \in, \dot{A} \cap \dot{X}^{(b,n+1)}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$$

with

$$\dot{f}^{(a,n)(b,n+1)}\big(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha l(a)}, G_{\alpha l(a)}, l(b), S^{(b,n+1)}\big) = \big(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha l(a)}, G_{\alpha l(a)}, l(b), S^{(b,n+1)}\big)$$

and

$$\dot{f}^{(a,n)(b,n+1)}\big(\langle \delta_k^{(b,n+1)} \mid k < \omega\rangle\big) = \langle \delta_k^{(b,n+1)} \mid k < \omega\rangle$$

and furthermore

$$\dot{f}^{(a,n)(b,n+1)}\big(\dot{f}^{(a,n)}(\dot{x}_0), \cdots, \dot{f}^{(a,n)}(\dot{x}_n)\big) = \big(\dot{f}^{(a,n)}(\dot{x}_0), \cdots, \dot{f}^{(a,n)}(\dot{x}_n)\big).$$

$\dot{f}^{(b,n+1)} = \dot{f}^{(a,n)(b,n+1)} \circ \dot{f}^{(a,n)}$ and so

$\dot{f}^{(b,n+1)}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{X}^{(b,n+1)}, \in, \dot{A} \cap \dot{X}^{(b,n+1)})$.

(2)* $p^{(b,n+1)} \in S^{(b,n+1)} \subset \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)}$ and

$$(\dot{X}^{(b,n+1)}[G_{\alpha l(b)}], \in, \dot{X}^{(b,n+1)}, \dot{A} \cap \dot{X}^{(b,n+1)}) \prec (L_\chi[\dot{A}][G_{\alpha l(b)}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

(3)* $p^{(b,n+1)} \in \dot{D}^{(a,n)} \cap \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)}$ and $\langle \delta_k^{(b,n+1)} \mid k < \omega\rangle \in \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)}$.

This completes the construction. Let $q \in P_\beta$ be a fusion of $\langle T_n \mid n < \omega\rangle$. Then $q \leq p$ and $q \lceil \alpha = w$ holds. Let G_β be P_β-generic over V with $q \in G_\beta$. We want to construct X_ω and f_ω. In $V[G_\beta]$, let $\langle a_n \mid n < \omega\rangle$ be the generic cofinal path through T. Let $\alpha_n = l(a_n)$, $X_n = \dot{X}^{(a_n,n)}_{G_{\alpha_n}}$, $p_n = p^{(a_n,n)}$, $D_n = \dot{D}^{(a_n,n)}_{G_{\alpha_n}}$, $f_n = \dot{f}^{(a_n,n)}_{G_{\alpha_n}}$, $S_n = S^{(a_n,n)}$ and $f_{nn+1} = \dot{f}^{(a_n,n)(a_{n+1},n+1)}_{G_{\alpha_{n+1}}}$. We also let $A = \dot{A}_{G_\alpha}$, $s = \dot{s}_{G_\alpha}$ and $N = \dot{N}_{G_\alpha} = \{x_n \mid n < \omega\}$. Then in $V[G_\beta]$ we have

- $a_n \in G_{\alpha_n} (= G_\beta \lceil \alpha_n)$ and $p_n \in G_\beta$.
- $(X_n, \in, A \cap X_n) \prec (L_\chi[A], \in, A \cap L_\chi[A])$.
- $S_0 = \{p\}$, $S_{n+1} = S_n \cup \{p_{n+1}\}$ and so $S_n = \{p_0, \cdots, p_n\}$.
- f_n is an isomorphism from $(N, \in, A \cap N)$ to $(X_n, \in, A \cap X_n)$ with

$$f_n(G_\alpha, z, s, P_\beta) = (G_\alpha, z, s, P_\beta)$$

- $f_{n+1} = f_{nn+1} \circ f_n$ and

$$f_{nn+1}(p_0, \cdots, p_n, p_{n+1}) = (p_0, \cdots, p_n, p_{n+1})$$

$$f_{nn+1}\big(f_n(x_0), \cdots, f_n(x_n)\big) = \big(f_n(x_0), \cdots, f_n(x_n)\big).$$

$$D_n = \begin{cases} f_n(x_n), & \text{if it is dense in } P_\beta. \\ P_\beta, & \text{otherwise.} \end{cases}$$

- $p_0 = p \in N$, $D_n \in X_n$ and $p_{n+1} \in D_n \cap X_n \cap X_{n+1}$.

Let $X_\omega = \{f_n(x_n) \mid n < \omega\}$ and f_ω be a map from N to X_ω defined by $f_\omega(x_n) = f_n(x_n)$.

Claim. (1) f_ω is a well-defined isomorphism from $(N, \in, A \cap N)$ to

$$(X_\omega, \in, A \cap X_\omega) \prec (L_\chi[A], \in, A \cap L_\chi[A])$$

such that $f_\omega(s, G_\alpha, P_\beta) = (s, G_\alpha, P_\beta)$ and so $G_\alpha, P_\beta \in X_\omega$.

(2) For all dense subsets $D \in X_\omega$ of P_β, we have $D \cap X_\omega \cap G_\beta \neq \emptyset$.

Proof. (well-defined): Suppose $x_n = x_m$. Then $(N, \in, A \cap N) \models$ "$x_n = x_m$". Hence $(X_l, \in, A \cap X_l) \models$ "$f_l(x_n) = f_l(x_m)$", where $l = \max\{n, m\}$. Hence $f_n(x_n) = f_m(x_m)$.

(One-to-one): Suppose $x_n \neq x_m$: Then $(N, \in, A \cap N) \models$ "$x_n \neq x_m$". Hence $(X_l, \in, A \cap X_l) \models$ "$f_l(x_n) \neq f_l(x_m)$", where $l = \max\{n, m\}$. Hence $f_n(x_n) \neq f_m(x_m)$.

($\in$-homo): Suppose $x_n \in x_m$. Then $(N, \in, A \cap N) \models$ "$x_n \in x_m$". Hence $(X_l, \in, A \cap X_l) \models$ "$f_l(x_n) \in f_l(x_m)$", where $l = \max\{n, m\}$. Hence $f_n(x_n) \in f_m(x_m)$ and so $(X_\omega, \in, A \cap X_\omega) \models$ "$f_\omega(x_n) \in f_\omega(x_m)$".

($\notin$-homo): Suppose $x_n \notin x_m$. Then $(N, \in, A \cap N) \models$ "$x_n \notin x_m$". Hence $(X_l, \in, A \cap X_l) \models$ "$f_l(x_n) \notin f_l(x_m)$", where $l = \max\{n, m\}$. Hence $f_n(x_n) \notin f_m(x_m)$ and so $(X_\omega, \in, A \cap X_\omega) \models$ "$f_\omega(x_n) \notin f_\omega(x_m)$".

(A-homo): Suppose $x_n \in A$. Then $(N, \in, A \cap N) \models$ "$\dot{A}(x_n)$". Hence $(X_n, \in, A \cap X_n) \models$ "$\dot{A}(f_n(x_n))$". Hence $f_n(x_n) \in A$ and so $(X_\omega, \in, A \cap X_\omega) \models$ "$\dot{A}(f_\omega(x_n))$".

($\neg A$-homo): Suppose $x_n \notin A$. Then $(N, \in, A \cap N) \models$ "$\neg\dot{A}(x_n)$". Hence $(X_n, \in, A \cap X_n) \models$ "$\neg\dot{A}(f_n(x_n))$". Hence $f_n(x_n) \notin A$ and so $(X_\omega, \in, A \cap X_\omega) \models$ "$\neg\dot{A}(f_\omega(x_n))$".

Since f_ω is onto, it is an isomorphism.

(Elementarity): Suppose

$$(L_\chi[A], \in, A \cap L_\chi[A]) \models \text{"}\exists y \varphi(y, f_0(x_0), \cdots, f_n(x_n))\text{"}.$$

Since $(X_n, \in, A \cap X_n) \prec (L_\chi[A], \in, A \cap L_\chi[A])$, we have

$$(X_n, \in, A \cap X_n) \models \text{"}\exists y \varphi(y, f_0(x_0), \cdots, f_n(x_n))\text{"}.$$

Take $f_n(x_m) \in X_n$ such that

$$(X_n, \in, A \cap X_n) \models \text{"}\varphi(f_n(x_m), f_0(x_0), \cdots, f_n(x_n))\text{"}.$$

Then

$$(X_l, \in, A \cap X_l) \models \text{"}\varphi(f_m(x_m), f_0(x_0), \cdots, f_n(x_n))\text{"},$$

where $l = \max\{n, m\}$. Hence

$$(L_\chi[A], \in, A \cap L_\chi[A]) \models \text{"}\varphi(f_m(x_m), f_0(x_0), \cdots, f_n(x_n))\text{"}.$$

(Generic): Let $D \in X_\omega$ be a dense subset of P_β. Let $D = f_n(x_n)$. By definition, $D_n = f_n(x_n) = D$ holds. Hence $p_{n+1} \in D_n \cap G_\beta$. But $p_{n+1} \in X_n$ and so $p_{n+1} = f_n(x_m) = f_m(x_m) \in X_\omega$. Therefore, $D \cap X_\omega \cap G_\beta \neq \emptyset$.

Case. β is successor. We write $\beta + 1$ for β for convenience. Since $\alpha < \beta + 1$, we have two subcases.

Subcase. $\alpha = \beta$. We want $\Vdash_{P_\beta}$ "$P_{\beta\beta+1}$ is preproper with θ and $(\dot{G}_\beta, z)$". Let G_β be any P_β-generic filter over V. Argue in $V[G_\beta]$. Let (A, χ, N, p, s) satisfy

- $G_\beta, z, P_{\beta\beta+1} \in H_\theta^{V[G_\beta]} \subseteq L_\chi[A]$.
- $G_\beta, z, P_{\beta\beta+1} \in N$ is countable and $(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ and N satisfies the condition on the transitive collapse.
- $p\lceil[\beta, \beta+1) \in P_{\beta\beta+1} \cap N$ and $s \in N$.

  We may assume
- $p \in P_{\beta+1} \cap N$ and $p\lceil\beta \in G_\beta$.

Then $z_\beta = (\dot z_\beta)_{G_\beta} \in N$ and $z_\beta, P_{\beta\beta+1} \in H_{\theta_\beta}^{V[G_\beta]} \subset H_\theta^{V[G_\beta]} \subseteq L_\chi[A]$. By assumption, $P_{\beta\beta+1}$ is preproper with θ_β and z_β. Hence there exists $(q, \dot N, \dot f)$ such that

- $q \in P_{\beta+1}$, $q\lceil\beta \in G_\beta$ and $q \le p$.
- $q\lceil[\beta, \beta+1)$ forces the following, where $\dot N$ and $\dot f$ are $P_{\beta\beta+1}$-names.

(1) $\dot f$ is an isomorphism from $(N, \in, A \cap N)$ to

$$(\dot N, \in, A \cap \dot N) \prec (L_\chi[A], \in, A \cap L_\chi[A])$$

with $\dot f(s) = s$.

(2) $(\dot N[\dot G_{\beta\beta+1}], \in, \dot N, A \cap \dot N) \prec (L_\chi[A][\dot G_{\beta\beta+1}], \in, L_\chi[A], A \cap L_\chi[A])$.

Subcase. $\alpha < \beta < \beta+1$. We want $\Vdash_{P_\alpha}$ "$P_{\alpha\beta+1}$ is preproper with θ and $(\dot G_\alpha, z)$".
Let $w \Vdash_{P_\alpha}$ "$(\dot A, \chi, \dot N, p, \dot s)$ be as in the hypothesis". Namely, w forces the following.

- $\dot G_\alpha, z, P_{\alpha\beta+1} \in H_\theta^{V[\dot G_\alpha]} \subseteq L_\chi[\dot A]$.
- $\dot G_\alpha, z, P_{\alpha\beta+1} \in \dot N$ is countable and $(\dot N, \in, \dot A \cap \dot N) \prec (L_\chi[\dot A], \in, \dot A \cap L_\chi[\dot A])$ and $\dot N$ satisfies the condition on the transitive collapse.
- $p\lceil[\alpha, \beta+1) \in P_{\alpha\beta+1} \cap \dot N$ and $\dot s \in \dot N$.

We may assume

- $w \le p\lceil\alpha$.
- $w \Vdash_{P_\alpha}$ "$p \in \dot N \cap P_{\beta+1}$".

We want $(q, \dot Y, \dot h)$ such that

- $q \in P_{\beta+1}$, $q\lceil\alpha \le w$ and $q \le p$.
- q forces the following, where $\dot Y$ and $\dot h$ are $P_{\beta+1}$-names.

(1) $\dot h$ is an isomorphism from $(\dot N, \in, \dot A \cap \dot N)$ to

$$(\dot Y, \in, \dot A \cap \dot Y) \prec (L_\chi[\dot A], \in, \dot A \cap L_\chi[\dot A])$$

with $\dot h(\dot s) = \dot s$.

(2) $(\dot Y[G_{\alpha\beta+1}], \in, \dot Y, \dot A \cap \dot Y) \prec (L_\chi[\dot A][G_{\alpha\beta+1}], \in, L_\chi[\dot A], \dot A \cap L_\chi[\dot A])$.

But w forces

- $\dot G_\alpha, z, P_{\alpha\beta} \in H_\theta^{V[\dot G_\alpha]} \subseteq L_\chi[\dot A]$.
- $\dot G_\alpha, z, P_{\alpha\beta} \in \dot N$ is countable and $(\dot N, \in, \dot A \cap \dot N) \prec (L_\chi[\dot A], \in, \dot A \cap L_\chi[\dot A])$ and $\dot N$ satisfies the condition on the transitive collapse.
- $p\lceil[\alpha, \beta) \in P_{\alpha\beta} \cap \dot N$ and $\dot s \in N$.

Since $\Vdash_{P_\alpha}$ "$P_{\alpha\beta}$ is preproper with θ and $(\dot G_\alpha, z)$" by induction, we have $(q\lceil\beta, \dot X, \dot f) \in V$ such that

- $q\lceil\beta \in P_\beta$, $q\lceil\alpha \le w$ and $q\lceil\beta \le p\lceil\beta$.
- $q\lceil\beta$ forces the following, where $\dot X$ and $\dot f$ are P_β-names.

(1) $\dot f$ is an isomorphism from $(\dot N, \in, \dot A \cap \dot N)$ to

$$(\dot X, \in, \dot A \cap \dot X) \prec (L_\chi[\dot A], \in, \dot A \cap L_\chi[\dot A])$$

with $\dot{f}(G_\alpha, z, \dot{s}, P_\beta, p) = (G_\alpha, z, \dot{s}, P_\beta, p)$.

(2) $(\dot{X}[G_{\alpha\beta}], \in, \dot{X}, \dot{A} \cap \dot{X}) \prec (L_X[\dot{A}][G_{\alpha\beta}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}])$.

Now we want $q\lceil[\beta, \beta+1)$. To do so let G_β be any P_β-generic filter over V with $q\lceil\beta \in G_\beta$. Then argue in $V[G_\beta]$ to specify the interpretation of $q\lceil[\beta, \beta+1)$. Let $G_\alpha = G_\beta\lceil\alpha$, $G_{\alpha\beta} = G_\beta\lceil[\alpha, \beta)$, $A = \dot{A}_{G_\alpha}$, $N = \dot{N}_{G_\alpha}$, $s = \dot{s}_{G_\alpha}$, $X = \dot{X}_{G_\beta}$. Then we have

- $G_\alpha, P_{\alpha\beta} \in X$ and so $G_\beta \in X[G_{\alpha\beta}]$.
- $(\dot{z}_\beta)_{G_\beta}, P_{\beta\beta+1} \in H_{\theta_\beta}^{V[G_\beta]} \subset H_\theta^{V[G_\beta]} = H_\theta^{V[G_\alpha]}[G_{\alpha\beta}] \subseteq L_X[A][G_{\alpha\beta}]$, as $P_{\alpha\beta} \in H_\theta^{V[G_\alpha]}$.
- $(\dot{z}_\beta)_{G_\beta}, P_{\beta\beta+1} \in X[G_{\alpha\beta}]$ is countable and $(X[G_{\alpha\beta}], \in, X, A \cap X) \prec (L_X[A][G_{\alpha\beta}], \in, L_X[A], A \cap L_X[A])$ and $X[G_{\alpha\beta}]$ satisfies the condition on the transitive collapse.
- $p\lceil[\beta, \beta+1) \in P_{\beta\beta+1} \cap X[G_{\alpha\beta}]$ and $s \in X[G_{\alpha\beta}]$.

Since $P_{\beta\beta+1}$ is preproper with θ_β and z_β, we have $(q\lceil[\beta, \beta+1), \dot{Y}, \dot{g})$ such that

- $q\lceil[\beta, \beta+1) \leq p\lceil[\beta, \beta+1)$.
- $q\lceil[\beta, \beta+1)$ forces the following, where $\dot{Y}$ and $\dot{g}$ are $P_{\beta\beta+1}$-names.

(1) $\dot{g}$ is an isomorphism from $(X[G_{\alpha\beta}], \in, X, A \cap X)$ to

$$(\dot{Y}[G_{\alpha\beta}], \in, \dot{Y}, A \cap \dot{Y}) \prec (L_X[A][G_{\alpha\beta}], \in, L_X[A], A \cap L_X[A])$$

with $\dot{g}(s) = s$.

(2) $(\dot{Y}[G_{\alpha\beta}][\dot{G}_{\beta\beta+1}], \in, \dot{Y}[G_{\alpha\beta}], \dot{Y}, A \cap \dot{Y}) \prec (L_X[A][G_{\alpha\beta}][\dot{G}_{\beta\beta+1}], \in, L_X[A][G_{\alpha\beta}], L_X[A], A \cap L_X[A])$.

Hence we may assume $q \in P_{\beta+1}$, $q \leq p$ and q forces the following.

(1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$$

with $\dot{g} \circ \dot{f}(\dot{s}) = \dot{s}$.

(2) $(\dot{Y}[G_{\alpha\beta+1}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_X[\dot{A}][G_{\alpha\beta+1}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}])$.

□

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