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On Convexity of Cooperative Games

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1 Introduction

A cooperative game (transferable utility game, TU-game) is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a finite set of players, and $v : 2^N \to \mathbf{R}$ is a function satisfying $v(\emptyset) = 0$. Since N is generally fixed throughout this paper, we may simply regard v as a TU-game. Any set $S \subseteq N$ is called a coalition and the value $v(S)$ is the worth of S. We denote by Γ^N the set of all TU-games on N.

A main problem in a TU-game is to fix allocation rules such that the players may redistribute the utilities among themselves. Those rules are often referred to as solutions of the game. Among cooperative games (TU-games), convex games have several nice properties. For example the Shapley value is contained in the core. Therefore in this paper we review several interesting results concerning convexity of TU-games.

This paper is organized as follows: In Section 2 we show some characterization of convexity in TU-games. Section 3 is devoted to some variations of convexity. In section 4, we deal with convexity in restricted games. We discuss convexity of fuzzy cooperative games in Section 5.

2 Characterization of convexity in TU-games

First we provide some characterizations of convexity in TU-games.

Definition 1 *A game $v \in \Gamma^N$ is said to be superadditive if*

$$v(S \cup T) \geq v(S) + v(T), \quad \forall S, T \subseteq N, \ S \cap T = \emptyset.$$

It is said to be convex if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall S, T \subseteq N.$$

In other words, the game (N, v) is convex if and only if the set function v is supermodular (The terminology "supermodular" is used in discrete convex analysis).

Convexity has some equivalent characterization such as follows:

Proposition 1 *A game $v \in \Gamma^N$ is convex if and only if one of the following equivalent conditions is satisfied:*

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T), \ \forall S \subset T \subseteq N \setminus R$$
$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \ \forall S \subset T \subseteq N, i \notin T.$$

Since we can define the sum of two games $v, w \in \Gamma^N$ and the scalar multiplication of $v \in \Gamma^N$ by $\alpha \in \mathbf{R}$ in the usual manner, Γ^N is a $(2^n - 1)$-dimensional vector space. As a basis of this vector space, we generally take the *unanimity games* $u_T \in \Gamma^N$ for $T \subseteq N$, $T \neq \emptyset$ defined by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$$

Then a game $v \in \Gamma^N$ is represented as a linear combination of u_T as $v = \sum_{T \subseteq N} d_T(v) u_T$ ($d_\emptyset(v) = 0$ for convenience). Here the coefficient $d_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ is called the *Harsanyi dividend.* It can be also obtained by the recursive formula

$$d_T(v) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(T) - \sum_{S \subset T} d_S(v), & \text{if } T \neq \emptyset \end{cases}$$

Recently convexity has been characterized by the dividends as follows.

Theorem 1 *(Kuipers et al. [9]) A game $v \in \Gamma^N$ is convex if and only if $\sum_{T \subseteq S} d_{T \cup \{i,j\}}(v) \geq 0$ for all $i, j \in N$ $(i \neq j)$ and all $S \subseteq N \setminus \{i, j\}$.*

Theorem 2 *(Driessen [6]) The set of all convex games on N, $\mathcal{C}^N$, is a polyhedral cone in the linear space Γ^N and dim $\mathcal{C}^N = 2^n - 1$.*

Now we consider a permutation or ordering π on N, where player i is in the $\pi(i)$th position in this ordering. Let

$$P(\pi, i) = \{j \in N \mid \pi(j) < \pi(i)\}$$

The the marginal vector $m^\pi(v)$ for v with respect to π is defined by

$$m_i^\pi(v) = v(P(\pi, i) \cup \{i\}) - v(P(\pi, i)), \quad \forall i \in N$$

Definition 2 *The minimarg operator Mi assigns to each game $v \in \Gamma^N$ the game $Mi(v)$ given by*

$$Mi(v)(S) = \min_{\pi \in \Pi(N)} \sum_{i \in S} m_i^\pi(v), \quad \forall S \subseteq N$$

where $\pi(N)$ is the set of all permutations on N.

Characterization of convexity by minimarg operator is given as follows.

Theorem 3 *(Curiel and Tijs [4]) A game $v \in \Gamma^N$ is convex if and only if $Mi(v) = v$, i.e., v is a fixed point of the minimarg operator.*

Definition 3 *For a game $v \in \Gamma^N$, the upper vector M^v of v is defined by*

$$M_i^v = v(N) - v(N \setminus \{i\}), \quad \forall i \in N$$

and the gap function $g^v : 2^N \to \mathbf{R}$ of v is defined by

$$g^v(S) = \sum_{i \in S} M_i^v - v(S), \quad \forall S \subseteq N.$$

Theorem 4 *A game $v \in \Gamma^N$ is convex if and only if one of the following equivalent conditions is satisfied:*

$$g^v(S \cup \{i\}) - g^v(S) \geq g^v(T \cup \{i\}) - g^v(T),\ \forall i \in N, \forall S, T \subseteq N : S \subset T \subseteq N \setminus \{i\}.$$
$$g^v(S \cup R) - g^v(S) \geq g^v(T \cup R) - g^v(T),\ \forall R, S, T \subseteq N : S \subset T \subseteq N \setminus R.$$

Proposition 2 *If a game $v \in \Gamma^N$ is convex, then $g^v(S) \leq g^v(S \cup \{i\})$ for all $i \in N$ and all $S \subseteq N \setminus \{i\}$.*

Now we consider solution concepts for TU-games and provide characterization of convexity by discussing some relationships between those solution concepts.

The most fundamental set-valued solution concept *core* is defined by

$$C(v) = \{x \in \mathbf{R}^n \mid \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S)\ \forall S \subset N\}$$

Another important point-valued solution concept, the *Shapley value* is defined by

$$\varphi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v) = \sum_{S \subseteq N, S \ni i} \frac{1}{|S|} d_S(v)$$

Moreover, the *Weber set* is defined by

$$W(v) = \mathrm{conv}\{m^\pi(v) \mid \pi \in \Pi(N)\}$$

Each element of the Weber set $W(v)$ is called a *random order value* for v.

The following theorem is very famous.

Theorem 5 *(Shapley [14], Ichiishi [7]) A game $v \in \Gamma^N$ is convex if and only if $m^\pi(v) \in C(v)$ for each permutation $\pi \in \Pi(N)$.*

Corollary 1 *If a game $v \in \Gamma^N$ is convex, then $\varphi_i(v) \in C(v)$.*

Theorem 6 *(Kuipers [9]) A game $v \in \Gamma^N$ satisfies*

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S)$$

for all $i, j \in N$ $(i \neq j)$ and all $S \subseteq N \setminus \{i, j\}$ if and only if all marginal vectors are elements of $C(v)$.

Theorem 7 *(Derks [5]) For each game $v \in \Gamma^N$ and a convex game $v' \in \mathcal{C}^n$ with $v' \leq v$, the intersection $C(v') \cap I(v)$ is either empty or externally stable.*

Definition 4 *Let $v \in \Gamma^N$. A scheme $a = (a_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ of real numbers is a population monotonic allocation scheme (pmas) of v if*

1. *$\sum_{i \in S} a_{iS} = v(S)$ for all $S \subseteq N$, $S \neq \emptyset$,*

2. *$a_{iS} \leq a_{iT}$ for all $S, T \subseteq N$, $\emptyset \neq S \subset T$ and $i \in S$.*

Definition 5 *Let $v \in \Gamma^N$ and $\pi \in \Pi(N)$. The extended vector of marginal contributions associated with π is the vector $a^\pi = (a_{iS}^\pi)_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ defined component-wise by*

$$a_{iS}^\pi = v((P(\pi, i) \cap S) \cup \{i\}) - v(P(\pi, i) \cap S).$$

Theorem 8 *(Sprumont 1990) If v is a convex game, then every extend vector of marginal contributions is a pmas for v.*

3 Variations of convex games

In this section we explain some variations of convex games briefly.

Definition 6 *A game $v \in \Gamma^N$ is called semiconvex if it is superadditive and $g^v(\{i\}) \leq g^v(S)$ for all $i \in N$ and $S \subseteq N$ with $i \in S$.*

Proposition 3 *Every convex game is semiconvex.*

Definition 7 *A game $v \in \Gamma^N$ is called 1-convex if*

$$0 \leq g^v(N) \leq g^v(S) \ \ \forall S \subset N, \ S \neq \emptyset$$

Definition 8 *A game $v \in \Gamma^N$ is called positive if $d_T(v) \geq 0$ for all $T \subseteq N$.*

Proposition 4 *If a game $v \in \Gamma^N$ is positive, then it is convex.*

Definition 9 *(Izawa and Takahashi [8]) A game $v \in \Gamma^N$ is said to be totally convex if for any $T \subseteq N$,*

$$\sum_{S \subseteq N} \sum_{i \in S \cap T} \frac{(s-1)!(n-s)!}{n!} [v^i(S) - v^i(S \cap T)] \geq 0$$

where $v^i(S) = v(S) - v(S \setminus \{i\})$.

Theorem 9 *([8]) Let $v \in \Gamma^N$. The Shapley value $\varphi(v)$ lies in the core $C(v)$ if and only if v is totally convex.*

4 Convexity in restricted games

In practical situations of TU-games, some coalitions may not be formed because of several reasons. Thus we consider restrictions on feasibility of coalitions. It is realized by introducing the concept of set systems over N,which is a pair $(N, \mathcal{F})$ with $\mathcal{F} \subseteq 2^N$. We often impose appropriate combinatorial structures on $\mathcal{F}$.

Definition 10 *A partition system is a set system $(N, \mathcal{F})$ with the following properties*
(P1) $\emptyset \in \mathcal{F}$ and $\{i\} \in \mathcal{F}$ for all $i \in N$,
(P2) $S \cup T \in \mathcal{F}$ for any $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$.

Definition 11 *Let $(N, \mathcal{F})$ be a set system and $S \subseteq N$. The maximal nonempty feasible subsets of S are called components of S. We denote by $\mathcal{C}_{\mathcal{F}}(S)$ the set of all componets of S.*

Proposition 5 *(Bilbao [2]) Let $(N, \mathcal{F})$ be a partition system. Then for each $S \subseteq N$, $\mathcal{C}_{\mathcal{F}}(S)$ is a partition of S.*

Definition 12 *Let $v \in \Gamma^N$ and $(N, \mathcal{F})$ be a partition system. The $\mathcal{F}$-restricted game $v^{\mathcal{F}}$ is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in \mathcal{C}_{\mathcal{F}}(S)} v(T).$$

Definition 13 *A intersecting system is a set system $(N, \mathcal{F})$ with the following properties*
(P1) $\emptyset \in \mathcal{F}$ and $\{i\} \in \mathcal{F}$ for all $i \in N$,
(P2) $S \cup T \in \mathcal{F}$ for any $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$.
(P3) $S \cap T \in \mathcal{F}$ for any $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$.

Theorem 10 *Let $v \in \Gamma^N$ be convex and $(N, \mathcal{F})$ be an intersecting system. Then the $\mathcal{F}$-restricted game $v^{\mathcal{F}}$ is also convex.*

5 Convexity in fuzzy games

An ordinary TU-game is a function $v : 2^N \to \mathbf{R}$. This function is defined for each coalition $S \subseteq N$, which can be identified with the vector $e^S \in \{0,1\}^n$ through the following correspondence

$$e_i^S = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Thus we may extend the domain $\{0,1\}^n$ of cooperative games to $[0,1]^n$, and consider a new game $\xi : [0,1]^n \to \mathbf{R}$, which is called a cooperative fuzzy game.

We simply denote $e^{\{i\}}$ by e^i. For $s, t \in [0,1]^n$

$$(s \vee t)_i = \max\{s_i, t_i\}, \ (s \wedge t)_i = \min\{s_i, t_i\}, \ i = 1, 2, \ldots, n,$$

and the support of a vector $s \in [0,1]^n$ si given by supp $s = \{i \in N \mid s_i > 0\}$. We denote by Δ^N the set of cooperative fuzzy games on N.

Definition 14 *A cooperative fuzzy game $\xi \in \Delta^N$ is said to be weakly superadditive if*

$$\xi(s) + \xi(t) \leq \xi(s \vee t), \ \ \forall s, t \in [0,1]^n, \ s \wedge t = 0.$$

It is said to be strongly superadditive if

$$\xi(s) + \xi(t) \leq \xi(s + t), \ \ \forall s, t \in [0,1]^n, \ s + t \in [0,1]^n.$$

It is said to be convex if

$$\xi(s) + \xi(t) \leq \xi(s \vee t) + \xi(s \wedge t), \ \ \forall s, t \in [0,1]^n.$$

It is clear that a strongly superadditive fuzzy game is weakly superadditive, and a convex game is weakly superadditive.

Another slightly stronger definition of convexity was given by Branzei et al.

Definition 15 *(Branzei et al. [3]) A cooperative fuzzy game $\xi \in \Delta^N$ is said to be B-convex if*

$$\xi(s) + \xi(t) \leq \xi(s \vee t) + \xi(s \wedge t), \ \ \forall s, t \in [0,1]^n.$$

and if for each $i \in N$ and each s_{-i}, the function $g_{-i} : [0,1] \to \mathbf{R}$ with $g_{-i}(t) = \xi(s_{-i}, t)$ for $t \in [0,1]$ is a convex function.

In practical situations, we may consider a cooperative fuzzy game $\xi_v \in \Delta^N$ by extending a crisp cooperative game $v \in \Gamma^N$. It should satisfy the relation $\xi_v(e^S) = v(S)$ for all $S \subseteq N$.

Some extensions of cooperative games have been already considered, though they have not been necessarily regarded as cooperative fuzzy games. Typical examples are the multilinear extension, and the Lovász extension.

Let $a_T \in \Delta^N$ be an extension of the unanimity game $u_T \in \Gamma^N$, i.e.,

$$a_T(e^S) = \begin{cases} 1, & \text{if } S \supseteq T \text{ (i.e., if } e^S \geq e^T) \\ 0, & \text{otherwise} \end{cases}$$

If the way (operation) of extension $\xi_v^a \in \Delta^N$ of $v \in \Gamma^N$ depending on a is assumed to be linear with respect to v, we have

$$\xi_v^a(s) = \sum_{T \subseteq N} d_T(v) a_T(s), \quad \forall s \in [0,1]^n.$$

Now we explain two representative examples in the above class.

Definition 16 *(Owen [13]) The multilinear extension $m_v \in \Delta^N$ of $v \in \Gamma^N$ is defined by*

$$m_v(s) = \sum_{T \subseteq N} d_T(v) \prod_{i \in T} s_i.$$

Definition 17 *(Lovász [10]) The Lovász extension $m_v \in \Delta^N$ of $v \in \Gamma^N$ is defined by*

$$l_v(s) = \sum_{T \subseteq N} d_T(v) \min_{i \in T} s_i.$$

Theorem 11 *(Tanino [15]) If a game v is positive, then the multilinear extension m_v is a convex cooperative fuzzy game.*

Theorem 12 *(Tanino [15]) If a cooperative game $v \in \Gamma^N$ is superadditive, then its Lovász extension $l_v \in \Delta^N$ is weakly superadditive.*

Theorem 13 *(Tanino [15]) If a cooperative game $v \in \Gamma^N$ is convex, then its Lovász extension $l_s \in \Delta^N$ is a convex cooperative fuzzy game.*

Proposition 6 *(Bilbao [2]) Let $v \in \Gamma^N$ and $l_v \in \Delta^N$ be its Lovász extension. The game v is convex if and only if the function l_v is concave (on $\mathbf{R}^n_+$).*

Theorem 14 *(Tanino [15]) A game $v \in \Gamma^N$ is convex if and only if its Lovász extension l_s is a strongly superadditive cooperative fuzzy game.*

Now we consider solution concepts in fuzzy games. For a cooperative fuzzy game $\xi \in \Delta^N$, in Aubin [1], the *core* of ξ is defined by

$$C(\xi) = \{x \in \mathbf{R}^n \mid \sum_{i \in N} x_i = \xi(e^N), \ \sum_{i \in N} s_i x_i \geq \xi(s), \forall s \in [0,1]^n\}$$

We induce a crisp game v^ξ from a fuzzy game $\xi \in \Delta^N$ by $v^\xi(S) = \xi(e^S)$ for all $S \subseteq N$. Then the *Shapley value* of ξ can be defined by

$$\varphi(\xi) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m^\pi(v^\xi).$$

Moreover the *Weber set* of ξ is defined by

$$W(\xi) = conv\{m^\pi(v^\xi) \mid \pi \in \Pi(N)\}.$$

Theorem 15 *(Branzei et al. [3]) Let ξ be a B-convex fuzzy game. Then $m^\pi(v^\xi) \in C(\xi)$ for each $\pi \in \Pi(N)$. Moreover, $C(\xi) = W(\xi)$, and $C(\xi) = C(v^\xi)$.*

Definition 18 *Let $\xi \in \Delta^N$. A scheme $(a_{it})_{i\in N, t\in[0,1]^n, t\neq 0}$ is called a participation monotonic allocation scheme (pmas) if*

1. *$(a_{it})_{i\in N} \in C(\xi_t)$ for each $t \in [0,1]^n$, $t \neq 0$,*
2. *$t_i^{-1} a_{it} \geq s_i^{-1} a_{is}$ for each $s, t \in [0,1]^n$ and each $i \in$ supp s.*

Theorem 16 *(Branzei et al. [3]) Let $\xi \in \Delta^N$ be B-convex. Then for each $x \in C(\xi)$, there exists a pmas (a_{it}) such that $a_{ie^N} = x_i$ for each $i \in N$.*

Now we deal with some restrictions on feasible coalitions in cooperative fuzzy games.

Definition 19 *A set $F \subseteq [0,1]^n$ is said to be a feasible coalition set (FCS) if it satisfies the following two conditions: F is a closed set in $[0,1]^n$ and $\alpha e^i \in F$, for all $\alpha \in [0,1]$.*

For a fuzzy coalition$s \in [0,1]^n$ and $l \in \mathbf{R}$, $\{s^1, \ldots, s^l\}$ such that $\sum_{j=1}^{l} s^j = s$ is said to be a partition of s. Especially for an FCS F, a partition of s, $\{s^1, \ldots, s^l\}$ such that $s^j \in F$ for all $j = 1, \ldots, l$ is said to be an F-partition of s. $P^F(s)$ denotes the set of all F-partiions of s.

For a fuzzy coalition $s \in [0,1]^n$ and an FCS F, a fuzzy coalition $t \in [0,1]^n$ is said to be and F-vector of s if it satisfies $t \leq s$, $t \in F$, and $t' = t$ for $t \leq t' \leq s$ such that $t' \in F$. $C^F(s)$ denote the set of all F-vectors of s. If $s \in F$, $C^F(s) = \{s\}$.

Definition 20 *An FCS F is said to be a partirion fuzzy coalition system (PFCS) if $C^F(s)$ is a partition of s for any $s \in [0,1]^n$.*

Proposition 7 *An FCS F is a PFCS if and only if one of the following conditions is satisfied:*

1. *For any $s \in [0,1]^n$, there exists a partition $\{I_1, \ldots, I_l\}$ of N such that $C^F(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}$,*
2. *$s, t \in F$, $s \wedge t \neq 0$, $\Longrightarrow$ $s \vee t \in F$.*

Definition 21 *Let $\xi \in \Delta^N$ be a strongly superadditive fuzzy cooperative game and F be a PFCS. Then the F-restricted game of ξ is defined by $\xi^F(s) = \sum_{t\in C^F(s)} \xi(t)$.*

Definition 22 *An FCS F is said to be an intersecting set if it satisfies*

$$s, t \in F, s \wedge t \neq 0 \implies s \vee t \in F,\ s \wedge t \in F.$$

Theorem 17 *(Moritani et al. [11]) Let ξ be a strongly superadditive and convex cooperative fuzzy game ana F be an intersecting system. Then the F-restricted game ξ^F is also convex.*

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