数理解析研究所講究録
第 1756 巻 2011 年 101-105

Intertwining operator and C_2-cofiniteness of modules

Masahiko Miyamoto *
Institute of Mathematics,
University of Tsukuba,
Tsukuba, 305 Japan

Abstract

Let V be a vertex operator algebra and T a V-module. We show that if there are C_2-cofinite V-modules U and W and a surjective (logarithmic) intertwining operator $\mathcal{Y}$ of type $\binom{T}{U\ W}$, then T is also C_2-cofinite. So, when V is simple and $V'\cong V$, then if one of V-modules is C_2-cofinite, then so is V.

1 Introduction

A vertex algebra was introduced by axiomatizing the concept of a Chiral algebra in conformal field theory by Borcherds [1]. It is a triple $(V, Y, \mathbf{1})$ satisfying the several axioms, where V is a graded vector space $V=\oplus_{i\in\mathbb{Z}}V_i$ over the complex number field $\mathbb{C}$, $Y(v,z)=\sum_{m\in\mathbb{Z}}v_m z^{-m-1}\in \mathrm{End}(V)[[z,z^{-1}]]$ denotes a vertex operator of $v\in V$ on V, $\mathbf{1}\in V_0$ is a specified element called the vacuum. When V has another specified element $\omega\in V_2$ and V has a lower bound of weights and all homogeneous subspaces are of finite dimensional, then we call V a vertex operator algebra. We set $Y(\omega,z)=\sum_{n\in\mathbb{Z}}L(n)z^{-n-1}$.

For a VOA V-module W, we define $C_2(W)=\{v_{-2}u \mid v,u\in V, \mathrm{wt}(v)\geq 1\}$. When $C_2(W)$ has a finite co-dimension in W, W is called to be C_2-cofinite. A concept of C_2-cofiniteness is originally introduced by Zhu [8] as a technical assumption to prove a modular invariance property of the space of the trace functions on modules. However, we are now recognizing the real meaning and the importance of C_2-cofiniteness. For example, V is C_2-cofinite if and only if all V-modules are $\mathbb{N}$-gradable. (See [2] and [7] for the proof.) We will use this fact frequently in this paper.

Our main result in this paper is the following:

Theorem 1 *Let U be a vertex operator algebra of CFT-type. Let A, B, C be simple $\mathbb{N}$-graded U-modules and $\mathcal{I}$ a surjective (formal power series) intertwining operator of type $\binom{C}{A\ B}$. If both of A and B are C_h-cofinite as U-modules for $h=1,2$, then so is C.*

*e-mail: miyamoto@math.tsukuba.ac.jp Supported by the Grants-in-Aids for Scientific Research, No. 22340002, The Ministry of Education, Science and Culture, Japan.

2 Preliminary

From the axiom of VOAs, for $v \in V_r$ and $u \in V_n$, we have $v_m u \in V_{r-m-1+n}$. Hence there is an integer N such that $v_s u = 0$ for any $s > N$. This property is called a truncation property. In this paper, we will say that "v is truncated at u" to simplify the terminology,

Set $V^* = \mathrm{Hom}(V, \mathbb{C})$ and define a pairing $\langle \cdot, \cdot \rangle$ on $V^* \times V$ by $\langle \xi, v \rangle = \xi(v)$ for $\xi \in V^*$ and $v \in V$. For $T \subseteq V$, $\mathrm{Annh}(T)$ denotes an annihilator of T, that is, $\mathrm{Annh}(T) = \{\xi \in V^* \mid \langle \xi, t \rangle = 0 \text{ for all } t \in T\}$. For $v \in V$ and $m \in \mathbb{Z}$, an action v^*_m on V^* is defined by

$$\langle (\sum_{m \in \mathbb{Z}} v^*_m z^{-m-1})\xi, w \rangle = \langle \xi, Y(e^{L(1)z}(-z^{-2})^{L(0)}v, z^{-1})w \rangle$$

for $w \in V$ and $\xi \in \mathrm{Hom}(V, \mathbb{C})$, where $Y^*(v, z) = \sum_{m \in \mathbb{Z}} v^*_m z^{-m-1}$ is called an adjoint operator of v. An important fact is that $(\oplus_{m \in \mathbb{Z}} \mathrm{Hom}(V_m, \mathbb{C}), Y^*)$ becomes a V-module as they proved in [3]. This module is called a restricted dual of V and denoted by V'. In particular, $Y^*(\cdot, z)$ satisfy the Borcherds identity:

$$\sum_{i=0}^{\infty} \binom{m}{i} (u^*_{r+i} v^*)_{m+n-i}\xi = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \{u^*_{r+m-i} v^*_{n+i}\xi - (-1)^r v^*_{r+n-i} u^*_{m+i}\xi\} \tag{2.1}$$

for any $m, n, r \in \mathbb{Z}$, $v, u \in V$, $\xi \in V'$. We note $V' = \oplus_{n \in \mathbb{Z}} V_n$ and $V^* = \prod_{n \in \mathbb{Z}} V_n$. Therefore we can express $\xi \in V^*$ by $\prod_n \xi_n$ with $\xi_n \in \mathrm{Hom}(V_n, \mathbb{C})$. We call that $\xi \in V^*$ is "$L(0)$-free" if $\dim \mathbb{C}[L(0)]\xi = \infty$, that is, $\xi_m \neq 0$ for infinitely many m. We note that any $\mathbb{N}$-gradable module does not contain any $L(0)$-free elements.

Let go back to (2.1). If $\xi \in \mathrm{Hom}(V_t, \mathbb{C})$, then all terms in (2.1) have the same weight $\mathrm{wt}(a) + \mathrm{wt}(b) - r - m - n - 2 + t$ and so the Borcherds' identity is also well-defined on V^*, as Li has pointed out in [5]. However, V^* is not a V-module because of failure of truncation properties. In order to find a V-module in V^*, we will start our arguments from one point ξ in V^*.

Lemma 2 *If u and v are truncated at ξ, then $v_m u$ is also truncated at ξ for any m. In particular, if all elements in Ω of V are truncated at ξ and $< \Omega >_{\mathrm{VA}} = V$, then all elements in V are truncated at ξ, where $\langle \Omega \rangle_{VA}$ denotes a vertex subalgebra generated by Ω.*

[Proof] By the assumption, there is an integer N such that $u_n\xi = v_n\xi = u_n v = 0$ for $n \geq N$. We assert that for $s \in \mathbb{N}$ and $n \geq 2N + s$, we have $(u_{N-s}v)_n\xi = 0$. Suppose false and let s be a minimal counterexample. Substituting $r = N - s$, $n = N + s + p$, $m = N + q$ in (2.1) with $p, q \geq 0$, we have

$$\begin{aligned}[\text{LeftSide}] &= \textstyle\sum_{i=0}^{\infty} \binom{N+q}{i} (u_{N-s+i}v)_{2N+q+s+p-i}\xi = \sum_{i=0}^{s} \binom{N+q}{i} (u_{N-(s-i)}v)_{2N+s-i+p+q}\xi \\ &= (u_{N-s}v)_{2N+s+p+q}\xi\end{aligned}$$

by the minimality of s. On the other hand, we have:

$$[\text{RightSide}] = \textstyle\sum_{i=0}^{\infty} (-1)^i \binom{N-s}{i} (u_{2N-s+q-i} v_{N+s+p+i}\xi - (-1)^{N-s} v_{2N-s+p-i} u_{N+q+i}\xi = 0,$$

which contradicts the choice of s. ∎

Since $v_n u_m \xi = u_m v_n \xi + \sum_{i=0}^{\infty} \binom{n}{i} (v_i u)_{n+m-i} \xi$, the above lemma also implies:

Lemma 3 *If v and u are truncated at ξ, then v is truncated at $u_m \xi$ for any m. In particular, if all elements of V are truncated at ξ, then $< u^1_{m_1} \cdots u^k_{m_k} \xi \mid u^i \in V, m_i \in \mathbb{Z} >_{\mathbb{C}}$ is a V-module.*

As Buhl has shown in [2], if V is C_2-cofinite, then all V-modules are $\mathbb{N}$-gradable and so there are no $L(0)$-free elements at which all elements in V are truncated. Namely, we have proved the following, which we will frequently use.

Lemma 4 *Let V be a C_2-cofinite vertex operator algebra and $\xi \in V^*$. If $\Omega \subseteq V$ generates V as a vertex subalgebra and all elements of Ω are truncated on ξ, then ξ is not $L(0)$-free.*

For $A, B \subseteq V$, we will often use the notation $A_{(m)}B$ to denote a subspace spanned by $\{a_m b \mid a \in A, b \in B\}$. We note that if A is a $\mathbb{C}[L(-1)]$-module, then $A_{(-2-m)}B \subseteq A_{(-2)}B$ for $m \in \mathbb{N}$ since $(L(-1)a)_{-m} b = m a_{-m-1} b$ for $a \in A$ and $b \in B$. Not only V, we use this notation for a pair (U, W) of a VOA U and its module W. For example, we set $C_2(W) = U^+_{(-2)} W$, where $U^+ = \oplus_{k=1}^{\infty} U_k$. We also set $C_1(W) = U^+_{(-1)} W$. We say that W is C_h-cofinite as a U-module if $\dim W/C_h(W) < \infty$ for $h = 1, 2$. We note any VOA U is C_1-cofinite as a U-module and so this definition is not equal to the ordinary C_1-cofiniteness.

We start the proof of Theorem 1. Namely, we will prove:

Theorem 1 *Let U be a vertex operator algebra of CFT-type. Let A, B, C be simple $\mathbb{N}$-graded U-modules and $\mathcal{I}$ a surjective (formal power series) intertwining operator of type $\binom{\ \ C\ \ }{A\ \ B}$. If both of A and B are C_h-cofinite as U-modules for $h = 1, 2$, then so is C.*

We note that if U is of CFT-type and an $\mathbb{N}$-graded U-module $A = \oplus_{k=0}^{\infty} A_{r+k}$ is C_1-cofinite, then $\dim A_{r+k} < \infty$ for any k since $A_{r+k} \cap C_1(A) = \sum_{s=1}^{k-1} (U_s)_{-1} A_{r+k-s}$ has a finite codimension in A_{r+k}.

In the remainder part of this section, we assume the hypotheses of Theorem 1. Since A and B are C_h-cofinite, there are finite dimensional subspaces $F^1 \subsetneq A$ and $F^2 \subsetneq B$ such that $A = U^+{}_{(-h)} A + F^1$ and $B = U^+{}_{(-h)} B + F^2$. Let c_A and c_B be conformal weights of A and B, respectively. We may assume that there is an integer N such that $F^1 = \oplus_{k=0}^{N} A_{c_A+k}$ and $F^2 = \oplus_{k=0}^{N} B_{c_B+k}$. Fix bases $\{p^i \mid i \in I\}$ of F^1 and $\{q^j \mid j \in J\}$ of F^2. In order to prove Theorem 1, we prove the following lemma by applying an idea in [4] to $(C/U^+_{(-h)} C)^*$.

Lemma 5 *For $p \in A$, $q \in B$ and $\theta \in \mathrm{Annh}(U^+_{(-h)} C) \cap C'$,*

$$F(\theta, p, q; z) := \langle \theta, \mathcal{I}(p, z) q \rangle$$

is a linear combination of $\{F(\theta, p^i, q^j; z) \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z, z^{-1}]$ and we may choose these coefficients independentlly of the choice of θ.

[Proof] We will prove the assertion by the induction on the total weight $\mathrm{wt}(p)+\mathrm{wt}(q)$. If $\mathrm{wt}(p) > N + c_B$, then $p = \sum_k u^k_{-h}a^k$ for some $u^k \in U$ and $a^k \in A$. We note this expression does not depend on the choice of θ. So we may assume $p = u_{-h}a$ with $u \in U$ and $a \in A$. Then for $\theta \in \mathrm{Annh}(U^+_{(-h)}C)$, we have:

$$\begin{aligned}\langle\theta, \mathcal{I}(p,z)q\rangle &= \langle\theta, \mathcal{I}(u_{-h}a,z)q\rangle \\ &= \langle\theta, Y^-(L(-1)^{h-1}u,z)\mathcal{I}(a,z)q + \mathcal{I}(a,z)Y^+(L(-1)^{h-1}u,z)q\rangle \\ &= \langle\theta, \mathcal{I}(a,z)Y^+(L(-1)^{h-1}u,z)q\rangle,\end{aligned}$$

where $Y^-(v,z) = \sum_{m<0} v_m z^{-m-1}$ and $Y^+(v,z) = \sum_{m\geq 0} v_m z^{-m-1}$. This is a reduction on the sum of weights because $Y^+(L(-1)^{h-1}u,z)q$ is a sum of finite terms and all weights of the coefficients are less than $\mathrm{wt}(u) + \mathrm{wt}(q)$.

Similarly, if $\mathrm{wt}(q) > N + c_B$, then we may assume $q = u_{-h}b$ with $u \in U$ and $b \in B$ and

$$\begin{aligned}\langle\theta, \mathcal{I}(p,z)q\rangle &= \langle\theta, \mathcal{I}(p,z)u_{-h}b \\ &= \langle\theta, u_{-h}\mathcal{I}(p,z)b\rangle + \textstyle\sum_{i=0}^{\infty}\binom{-h}{i}z^{-h-i}\mathcal{I}(u_ip,z)b\rangle \\ &= \textstyle\sum_{i=0}^{\infty}\binom{-h}{i}z^{-h-i}\langle\theta, \mathcal{I}(u_ip,z)b\rangle.\end{aligned}$$

Again, these process do not depend on the choice of θ and this is also a reduction on the weights because $\mathrm{wt}(u_ip) + \mathrm{wt}(b) < \mathrm{wt}(u_{-h}b) + \mathrm{wt}(p)$ for $i \geq 0$. Therefore, $\langle\theta, \mathcal{I}(p,z)q\rangle$ is a linear combination of $\{\langle\theta, \mathcal{I}(p^i,z)q^j\rangle \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z,z^{-1}]$. We note the coefficients do not depend on the choice of θ. ∎

Now we are able to prove Theorem 1. By the proof of the above lemma,

$$\frac{d}{dz}F(\theta, p^s, q^t; z) = F(\theta, L(-1)p^s, q^t; z)$$

is a linear combination of $\{F(\theta, p^i, q^j; z) \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z,z^{-1}]$ for any $s \in I, t \in J$ and all coefficients do not depend on the choice of θ. Therefore, there is a differential linear equation such that $F(\theta, p^s, q^t)$ are all its solutions for any $s \in I$, $t \in J$ and θ. Furthermore, since $\{\mathcal{I}(p,z)q \mid p \in A, q \in B, z \in \mathbb{Z}\}$ spans C modulo $U^+_{(-h)}C$ and $\langle\theta, \mathcal{Y}(p,z)q\rangle$ are a linear sum of $\langle\theta, \mathcal{I}(p^i,z)q^j\rangle$, $\theta \in C' \cap \mathrm{Annh}(U^+_{(-2)}C) \to \prod_{i\in I, j\in J}\langle\theta, \mathcal{I}(p^i,z)q^j\rangle$ is injective. Therefore, we have $\dim C/U_{(-h)}C < \infty$.
This completes the proof of Theorem 1.

References

[1] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA 83 (1986), 3068-3071.

[2] G. Buhl, *A spanning set for VOA modules*, J. Algebra. 254 (2002), no. 1,125-151.

[3] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104** (1993).

[4] Y.-Z. Huang, *Differential equations, duality and modular invariance*, Commun. Contemp. Math. 7 (2005), no. 5, 649-706.

[5] H. Li, *Some finiteness properties of regular vertex operator algebras*, J. Algebra 212 (1999), 495-514.

[6] M. Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra. 179, (1996) 523-548.

[7] M. Miyamoto, *Modular invariance of vertex operator algebra satisfying C_2-cofiniteness*, Duke Math. J. 122 (2004), no. 1, 51-91.

[8] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc., **9** (1996), 237-302.