数理解析研究所講究録
第 1756 巻 2011 年 106-111

Remarks on vertex operator algebras and Jacobi forms

Matt Krauel and Geoffrey Mason
Department of Mathematics
UC Santa Cruz

Abstract

We announce some results relating vertex operator algebras and Jacobi forms, and discuss some of the consequences. Proofs will be given elsewhere.

1 Background

We would like to thank the organizers, especially Masahiko Miyamoto, for giving us the opportunity to participate in the workshop in Kyoto.

For background concerning the Jacobi group and Jacobi forms that we use below, see the text of Eichler and Zagier [EZ]. Our notation is standard.

$$\begin{aligned}
\mathbb{C} &= \text{complex numbers, } \mathbb{H} = \text{complex upper half-plane,} \\
\mathbb{Q} &= \text{rational numbers, } \ \mathbb{Z} = \text{integers,} \\
q &= e^{2\pi i\tau} \ \ (\tau \in \mathbb{H}), \ \zeta = e^{2\pi iz} \ \ (z \in \mathbb{C}), \\
\Gamma &= SL_2(\mathbb{Z}), \ J = \Gamma \ltimes \mathbb{Z}^2, \\
\eta(\tau) &= q^{1/24}\prod_{i=1}^{\infty}(1-q^n) \ \ \text{(Dedekind eta-function).}
\end{aligned}$$

J is the *Jacobi group*, i.e., the semidirect product of Γ with $\mathbb{Z}^2$, where Γ acts naturally on $\mathbb{Z}^2$. Thus,

$$(\gamma_1, U)(\gamma_2, V) = (\gamma_1\gamma_2, U\gamma_2 + V)) \quad \text{for } \gamma_1, \gamma_2 \in \Gamma, \ U, V \in \mathbb{Z}^2.$$

There are left group actions

$$\Gamma \times \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau, z\right) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

and

$$\mathbb{Z}^2 \times \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$$
$$((u,v),\tau,z) \mapsto (\tau, z + u\tau + v)\,.$$

These jointly define an action of the Jacobi group

$$J \times \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}.$$

Consider the space

$$\mathfrak{F} = \{\text{holomorphic } F : \mathbb{H} \times \mathbb{C} \to \mathbb{C}\}.$$

For all integers k, m there are right group actions

$$\begin{array}{ll} \mathfrak{F} \times \Gamma \to \mathfrak{F}, & \qquad \mathfrak{F} \times \mathbb{Z}^2 \to \mathfrak{F}, \\ (F,\gamma) \mapsto F|_{k,m}\gamma, & \qquad (F,(u,v)) \mapsto F|_m(u,v), \end{array}$$

where

$$\begin{aligned} F|_{k,m}\gamma(\tau,z) &:= (c\tau+d)^{-k} e^{-2\pi i m c z^2/c\tau+d} F(\gamma.(\tau,z)) \\ &= (c\tau+d)^{-k} e^{-2\pi i m c z^2/c\tau+d} F\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \end{aligned}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; and

$$\begin{aligned} F|_m(u,v)(\tau,z) &:= e^{2\pi i m(u^2\tau+2uz)} F((u,v).(\tau,z)) \\ &= e^{2\pi i m(u^2\tau+2uz)} F(\tau, z+u\tau+v). \end{aligned}$$

Again these actions can be combined into an action of the Jacobi group

$$\mathfrak{F} \times J \to \mathfrak{F}$$
$$(F,(\gamma,(u,v))) \mapsto F|_{k,m}(\gamma,(u,v)).$$

2 Weak Jacobi forms

A *weak Jacobi form* of *weight* k and *index* m is an *invariant* of the J-action, i.e.

$$F|_{k,m}(\gamma,(u,v)) = F \quad \text{for all } (\gamma,(u,v)) \in J,$$

which is *holomorphic at the cusps.* Invariance is equivalent to

$$\begin{aligned} F(\gamma\tau, \frac{z}{c\tau+d}) &= (c\tau+d)^k e^{2\pi i m c z^2/c\tau+d} F(\tau,z) \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma), \\ F(\tau, z+u\tau+v) &= e^{-2\pi i m(u^2\tau+2uz)} F(\tau,z) \quad ((u,v) \in \mathbb{Z}^2). \end{aligned}$$

Holomorphy at the cusps means that there is a Fourier expansion

$$F(\tau, z) = \sum_{n\geq 0} \sum_{r} a(n, r) q^n \zeta^r.$$

The Fourier coefficients $a(n, r)$ of a weak Jacobi form necessarily satisfy

$$a(n, r) = 0 \text{ if } r^2 > m^2 + 4mn.$$

A *Jacobi form* of weight k and index m is a weak Jacobi form of weight k and index m such that the Fourier coefficients satisfy the stronger condition

$$a(n, r) = 0 \text{ if } r^2 > 4mn.$$

One may modify the definition of (weak) Jacobi form in various ways, e.g., by considering forms $F(\tau, z)$ invariant only under a subgroup of finite index in J, by taking $m \in \mathbb{Q}$, or by allowing poles at the cusps. (However, we shall never encounter forms with poles in $\mathbb{H}$.)

Examples

1. $E_{4,1}(\tau, z) = 1 + (\zeta^2 + 56\zeta + 126 + 56\zeta^{-1} + \zeta^{-2})q + (126\zeta^2 + 576\zeta + 756 + 576\zeta^{-1} + 126\zeta^{-2})q^2 + ...$ is the Jacobi Eisenstein series with $k = 4, m = 1$. It is a Jacobi form on the full group J.

2. Suppose that $F(\tau, z)$ is a (weak) Jacobi form of weight k and index m on J. If we take $z = z_0 \in \mathbb{Q}$ then $F(\tau, z_0)$ is a holomorphic modular form of weight k on a congruence subgroup of Γ. In particular, if $z = 0$ then $F(\tau, 0)$ is a holomorphic modular form of weight k on Γ. Eg., $E_{4,1}(\tau, 0) = 1 + 240q + \ldots$ is the weight 4 Eisenstein series on Γ.

3. Suppose that L is a positive-definite even lattice of even rank $2l$ with inner product $(\ ,)$. Let $\beta \in L$ with $m = (\beta, \beta)/2$. The *theta function of* L, defined by $\theta_{L,\beta}(\tau, z) := \sum_{\alpha\in L} q^{(\alpha,\alpha)/2}\zeta^{(\alpha,\beta)}$, is a Jacobi form of weight l and index m (on a subgroup of J). $\theta_{L,\beta}(\tau, 0) = \theta_L(\tau)$ is the usual theta function, a modular form of weight l on a congruence subgroup of Γ. E.g., $E_{4,1}(\tau, z)$ is the theta function of the E_8 root lattice with β taken to be a root of the lattice.

3 Statement of Main Results

We deal with simple vertex operator algebras V of central charge c which are *regular* (i.e. rational and C_2-cofinite) and of *strong CFT-type* (i.e. $V = \mathbb{C}1 \oplus V_1 \oplus \ldots$ and V is self-dual as V-module.) Let $M^1, \ldots, M^p$ be the distinct irreducible V-modules.

Theorem 3.1. *Suppose that $h \in V_1$ has the following properties:*

$$(a)\ h(0) \text{ is semisimple with eigenvalues in } \mathbb{Z}$$
$$(b)\ 1/2h(1)h = m\mathbf{1} \text{ and } m \in \mathbb{Z}.$$

For a V-module M, set

$$J_{M,h}(\tau, z) \quad = \quad Tr_M q^{L(0)-c/24} \zeta^{h(0)}.$$

Then the linear space spanned by the functions $J_{M^i,h}(\tau, z)$ $(1 \le i \le p)$ is a J-module with respect to the action $|_{0,m}$. In other words, $(J_{M^1,h}(\tau, z), \ldots), J_{M^p,h}(\tau, z))^t$ is a vector-valued weak Jacobi form of weight 0 and index m. Each $J_{M^i,h}(\tau, z)$ is holomorphic in $\mathbb{H} \times \mathbb{C}$, but generally has poles at the cusps.

As usual, if V is *holomorphic* (so that V is the unique irreducible V-module) we get a more precise result.

Theorem 3.2. *Suppose that V is holomorphic, and let h be as in Theorem 3.1. Then $Z_{V,h}(\tau, z)$ is a weak Jacobi form on J of weight 0 and index m. Alternatively, $\eta(\tau)^c Z_{V,h}(\tau, z)$ is a holomorphic weak Jacobi form of weight $c/2$ and index m. If $1 \le m \le 4$ then $\eta(\tau)^c Z_{V,h}(\tau, z)$ is a Jacobi form of weight $c/2$ and index m, (i.e. the adjective 'weak' may be dropped).*

4 Applications

Theorems 3.1 and 3.2 find a number of applications. We discuss some of them.

I). If we take $z = 0$ the trace function $J_{M,h}(\tau, z)$ becomes the usual partition function $\mathrm{Tr}_M q^{L(0)-c/24}$. Then Theorem 3.1 reduces to the modular-invariance of the space of partition functions of the irreducible V-modules, and Theorem 3.2 says that in the holomorphic case the partition function of V is a modular function of weight 0 on Γ (possibly with character). These results are due to Zhu [Z]. Generally, our results may be regarded as an extension of Zhu's theory (loc. cit.) from the case of modular forms to that of Jacobi forms. However, some of Zhu's results (concerning n-point functions) are no longer true in the more general setting.

II). In a similar vein, the representation of the Jacobi group J on the trace functions $J_{M^i,h}(\tau, z)$ restricts to a representation of Γ that is the same as the representation ρ of Γ on the space of partition functions $\mathrm{Tr}_{M^i} q^{L(0)-c/24}$. Thus the conjecture that each $J_{M^i,h}(\tau, z)$ is a (weak) Jacobi form on a congruence subgroup is *equivalent* to the well-known conjecture that each $\mathrm{Tr}_{M^i} q^{L(0)-c/24}$ is a modular form on a congruence subgroup of Γ.

On the other hand, if it is *known* that ρ factors through a congruence subgroup for a given V, then it follows that each $J_{M^i,h}(\tau, z)$ is indeed a (weak)

Jacobi form on a congruence subgroup. This is the case, for example, for lattice theories and for VOAs based on an affine Lie algebra. Theorem 3.1 in this stronger form was proved for lattice theories by Dong-Liu-Ma [DLMa] and for affine algebras it is a consequence of the Kac-Peterson theory [KP], [K].

III). One can generally find a large supply of states h satisfying (a) and (b) of Theorem 3.1 as follows. There is a decomposition $V_1 = \mathfrak{A} \oplus \mathfrak{G}_{1,m_1} \oplus \ldots \oplus \mathfrak{G}_{r,m_r}$ where $\mathfrak{A}$ is abelian and $\mathfrak{G}_{i,m_i}$ is a simple Lie algebra of *positive integral level* m_i $(1 \leq i \leq r)$ ([DM1], [DM2]). Let $h := h_\alpha \in \mathfrak{G}_i$ where α is long root element normalized so that $\kappa_i(h,h) = 2$ ($\kappa_i :=$ Killing form of $\mathfrak{G}_i$). By Lie theory one knows that $h(0)$ has integer eigenvalues on any finite-dimensional $\mathfrak{G}_i$-module, in particular $h(0)$ has integer eigenvalues on each homogeneous space V_n. Moreover the definition of level implies that $h(1)h = 2m_i$.

IV). Assume now that V is holomorphic, and let $h = h_\alpha$ be chosen as in III). We already pointed out in Section 2, Example 2 that $J_{V,h}(\tau, z_0)$ is a modular form on a congruence subgroup of Γ whenever $z_0 \in \mathbb{Q}$. Note that $J_{V,h}(\tau, z_0) = \mathrm{Tr}_V q^{L(0)-c/24} e^{2\pi i z_0 h(0)}$ is the trace function of the *finite order automorphism* $e^{2\pi i z_0 h(0)}$ of V. Conversely, if L is the group of *linear automorphisms* of V, i.e. the Lie group obtained by exponentiating elements of $\mathfrak{L} := \mathfrak{G}_{1,m_1} \oplus \ldots \oplus \mathfrak{G}_{r,m_r}$ in the usual way, and if $g \in L$ is an element of finite order, then we may choose a Chevalley basis of $\mathfrak{L}$ so that $g = e^{2\pi i z_0 h(0)}$ as above for suitable h. In this way we obtain

Theorem 4.1. *Suppose that V is holomorphic and that $g \in L$ is a linear automorphism of finite order. Then the trace function $\mathrm{Tr}_V q^{L(0)-c/24} g$ is a modular function of weight zero on a congruence subgroup of Γ.*

For many holomorphic VOAs (e.g. the lattice theory based on the E_8 root lattice), the full automorphism group Aut$V$ coincides with L. In these cases, Theorem 4.1 establishes the conjectured modular-invariance of trace functions of all finite order automorphisms. See [DLM] for more on this subject.

V). It is a simple arithmetic result from the theory of Jacobi forms that any weak (holomorphic) Jacobi form of index $m \leq 4$ with Fourier series expansion of the form $1 + O(q)$ is, in fact, a Jacobi form. This is why the 'weak' condition is not needed in Theorem 3.2 when $m \leq 4$. It seems likely that (as for affine Lie algebra theories) the adjective 'weak' can always be dropped from the assumptions of Theorem 3.1 and 3.2. On the other hand, for general index m there are plenty of weak Jacobi forms which are not true Jacobi forms.

References

[DLM] C. Dong, H. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, Comm. Math. Phys. **214** (2000), 1-56.

[DLMa] C. Dong, K. Liu, and X. Ma, Elliptic genus and vertex operator algebras, Pure Appl. Math. Quart. **1** (2005), 791-815.

[DM1] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, Int. Math. Res. Not. **56** (2004), 2989-3008.

[DM2] C. Dong and G. Mason, Integrability of C_2-cofinite Vertex Operator Algebras, Int. Math. Res. Not. Vol. 2006, Art. ID **80468**, 1-15.

[EZ] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics Vol. 55, Birkhäuser, Boston, 1985.

[K] V. Kac, *Infinite-dimensional Lie algebras, 3rd. ed.*, C.U.P, 1990.

[KP] V. Kac and D. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. in Math. **53** (1984), 125-264.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, J.A.M.S. **9** No. 1 (1996), 237-302.