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Higher Order Asymptotic Coupon Bond Option Valuation for Interest Rates with non-Gaussian Dependent Innovations

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Abstract

We consider the coupon bond option valuation when a short rate model has non-Gaussian dependent innovations. Higher-order asymptotic theory enables approximate coupon bond option price formula to be obtained. Some numerical examples are presented, where the process of innovation follows a particular model. The properties of these coupon bond options are very different from those implied by the continuous Vasicek model. This has important implications for hedging interest-rate risk with bond options.

Key Words: coupon bond option, Edgeworth expansion, short rates, Vasicek model.

1 Introduction

The interest rates dynamics has received much attention in the finance literature because the interest rate is a crucial input for asset pricing and a fundamental component of monetary policy. The stochastic process of the short-term interest rate (short rate) influences not only the valuation of bonds and interest rate derivatives, but also the pricing of equity options, which rely on the term structure of the interest rate. Therefore, an understanding of the dynamic properties of the short rate is of profound importance to policy makers and economic agents.

Many short rate models have been developed. Among these, Vasicek (1977) and Cox et al. (1985) consider the short rate as a diffusion process with mean reversion. Under both these models, bond yields can be expressed as an affine function of the short rate and thus belong to the class of affine term structure models. Duffie and Kan (1996), Duffie and Singleton (1997), and Dai and Singleton (2000) describe the characteristics of affine term structure models. Using these models, tremendous progress has been made in valuing interest rate derivative securities. For option pricing, the conditional characteristic function of an affine term structure model is known in closed form, and the prices of zero-coupon bond options are easily computed by Fourier inversion. Option formulas for zero-coupon bonds are discussed in Chen (1996), Nunes et al. (1999), and Duffie et al. (2000).

A number of studies have also examined the pricing of options on coupon bonds. Jamshidian (1989) argues that an option on a coupon bond can be decomposed into a portfolio of options on discount bonds. Wei (1997) develops an approximation for coupon bond option prices based on closed-form solutions for the corresponding discount bond options and the duration measure. Singlton and Umantsev (2002) propose an approximation of the prices of European options on coupon bonds, where the underlying short rate is an affine combination of the CIR-type processes.

Although the Vasicek model is useful and flexible, empirical research widely reports that it fails to appropriately capture the behavior of the short rate. Moreover, the normality assumption for residuals may not hold. Honda et al. (2010) extend the discretized Vasicek model to incorporate non-Gaussian dependent innovations and derive the closed-form formula of zero-coupon bonds and the term structure of interest rates. Shiohama and Tamaki (2011) give the evaluation of the price of European call options on zero-coupon bonds. These formulas are based on the Edgeworth expansion of the underlying discretized short rates and bond price densities. This paper discusses the extension to assess coupon bond options. Since the first four cumulants have intuitive meaning, the direct relation between the option price and the cumulants of the underlying distribution is appealing.

The Gram-Charlier and Edgeworth expansions have been used in many fields, including mathematics, statistics, and physics, and Sargan (1975, 1976) introduces these expansions to econometrics. In option pricing theory, several authors have proposed the use of a statistical series expansion method for pricing options when the risk-neutral density is asymmetric and leptokurtic: for example, see Rubinstein (1998) and the references therein. Jarrow and Rudd (1982) applied the log-normal Gram-Charlier series expansion to the density of stock prices and derived a formula for pricing options. Corrado and Su (1996) also considered the Gram-Charlier series expansion for the density of stock log returns rather than stock price itself. The Black-Scholes model is a special case of their models. Recently, Masuda and Yoshida (2005) considered the Edgeworth expansion for log returns of stock price in the stochastic volatility model of Barndorff-Nielsen and Shephard. Their results can simultaneously explain non-Gaussianity for short-time lags and approximated Gaussianity for long-time lags. Kunitomo and Takahashi (2001) developed an approach called small disturbance asymptotic expansion to derive various formulae for swaption and Asian options for interest rates. Perote and Del Brío (2003) investigated the effect of skewness and kurtosis on financial time series to improve Value at Risk (VaR) measures. Collin-Dufresne and Goldstein (2002) and Kawai (2003) propose approximations of the price of a swaption based on an Edgeworth expansion of the density of the coupon bond price.

All these reports reveal that densities, based on Hermite polynomials, are accurate and general specifications that capture the skewness and kurtosis of underlying stochastic models. However, these financial time series also have dependent structures. Tamaki and Taniguchi (2007) employs Edgeworth expansions to take into account the effects of non-Gaussianity and the dependence of stock log returns simultaneously. The methodology that we present in this paper is analogous to Tamaki and Taniguchi (2007). We extend the results of Honda et al. (2010) to evaluate zero-coupon bonds, the term structure of interest rates, and European call options on zero-coupon bond.

The remainder of this paper is organized as follows. In Section 2, a discretized method for zero-coupon bond option is introduced. Section 3 discusses the evaluation of the zero-coupon bond option based on the Edgeworth expansion. Section 4 derives a closed-form formula for the approximate coupon bond option prices. In Section 5, numerical examples are presented to demonstrate the effects of non-Gaussianity and dependence of short rates. Section 6 offers our conclusions.

2 Discretized Expression on Bond Options

The short rate, r_t, follows the SDE:

$$dr_t = \kappa(\mu - r_t)dt + \sigma dB_t, \tag{1}$$

where B_t is a standard Brownian motion. The parameter μ represents the long-term mean of the short rate, κ represents the rate at which the short rate reverts to its long-term mean, and σ represents the instantaneous volatility of the short rate. This process is well known as the Ornstein-Uhlenbeck process and the mean-reverting structure is its important feature. Let $C(r, 0, T, S, K)$ denote the price of a European call on a zero-coupon bond which matures at time S. The option matures at $T(T < S)$ with an exercise price K. For simplicity and without loss of generality, we assume that the current time is 0. Then, the price of a European call option is evaluated with risk-neutral measure Q such that

$$C(r, 0, T, S, K) = E_0^Q\left[e^{-\int_0^T r_t dt}(P(T, S) - K)_+\right], \tag{2}$$

where $P(T, S)$ denotes the time-T price of the zero-coupon bond maturing at time S. Based on Vasicek model (1), option price (2) is given by

$$C(r, 0, T, S, K) = P(0, S)\Phi(d_1) - KP(0, T)\Phi(d_2), \tag{3}$$

where $d_1 = \log\{P(0, S)/(K(0, T))\}/\sigma_p + \sigma_p/2$, $d_2 = d_1 - \sigma_p$,

$$\sigma_p = \frac{\sigma}{\kappa}(1 - e^{-\kappa(S-T)})\sqrt{\frac{1 - e^{-2\kappa T}}{2\kappa}},$$

and $\Phi(\cdot)$ is the standard normal distribution function. Here, the bond price is

$$P(0, T) = \exp\left[\{B(0, T) - T\}\left(\mu - \frac{\sigma^2}{2\kappa^2}\right) - \frac{\sigma^2 B(0, T)^2}{4\kappa} - B(0, T)r_0\right], \tag{4}$$

where $B(0, T) = (1 - e^{-\kappa T})/\kappa$.

In this paper, we assume that the short rate r_t is discretely sampled with interval Δ and the initial short rate r_0 is observable and fixed. Subsequently, the short rates are discretely sampled at time $0, \Delta, 2\Delta, \ldots, n_1\Delta(\equiv T), (n_1+1)\Delta, \ldots, n\Delta(\equiv S)$. Notice that $S - T = (n - n_1)\Delta \equiv n_2\Delta$.

By the Euler approximation, for $j = 1, 2, \ldots, n$, (1) is reduced to

$$r_j - r_{j-1} = \kappa(\mu - r_{j-1})\Delta + \sigma\Delta^{1/2}\varepsilon_j,$$

where $\{\varepsilon_j\}$ are i.i.d $N(0,1)$ random variables. Following Honda et al. (2010), we extend the discretized Vasicek model to possess the non-Gaussian and dependent innovations. Then the short rate model is expressed as follows:

$$r_j - r_{j-1} = \kappa(\mu - r_{j-1})\Delta + \Delta^{1/2}X_j. \tag{5}$$

It is clear that this expression is motivated by the Vasicek model. Obviously, if $\{X_j\}$ are i.i.d $N(0,\sigma^2)$ random variables, the expression (5) corresponds to the discretized Vasicek model. Empirical research shows that the processes of $\{X_j\}$ are highly non-Gaussian and serially correlated.

By using similar arguments of Honda et al. (2010) and Choi and Wirganto (2007), $P(0,T)$ can be expressed as follows:

$$\begin{aligned} P(0,T) &= E_0\left[e^{-\int_0^T r_t dt}\right] \approx E_{0,n_1-1}\left[\exp\left\{-\Delta\left(\frac{1}{2}r_0 + r_1 + \cdots + r_{n_1-1} + \frac{1}{2}r_{n_1}\right)\right\}\right] \\ &= e^{-A_{n_1} - B_{n_1}r_0}AF_{n_1,n_1} \equiv \tilde{P}_0(0,T), \end{aligned} \tag{6}$$

where $E_{j,k} = E_jE_{j+1}\cdots E_{k-1}$, $A_j = \mu(j\Delta - B_j)$, $B_j = (1+v)(1-v^j)/(2\kappa)$ and

$$AF_{j,k} = E_{k-j,k-1}\left[\exp\left(-\frac{\Delta^{3/2}}{2}\sum_{i=1}^{j} a_iX_{k-i+1}\right)\right].$$

Here, AF_{n_1,n_1} can be expressed as $AF_{n_1,n_1} = E_{0,n_1-1}\left[e^{-Y_{1,n_1}}\right]$ where

$$Y_{1,n_1} = \Delta^{1/2}\sum_{i=1}^{n_1} b_iX_{n_1-i+1} \quad \text{and} \quad b_i = \frac{1}{\kappa}\left\{1 - \frac{1}{2}v^{i-1}(1+v)\right\}. \tag{7}$$

Using recursive substitution in (5), r_j has another representation

$$r_j = (1-v^j)\mu + v^jr_0 + \Delta^{\frac{1}{2}}\sum_{i=1}^{j} v^{i-1}X_{j-i+1} \equiv (1-v^j)\mu + v^jr_0 + Y_{2,j}, \tag{8}$$

where $v = 1 - \kappa\Delta$ and $Y_{2,j} = \Delta^{1/2}\sum_{i=1}^{j} v^{i-1}X_{j-i+1}$.

Similar arguments to (6) yield that $P(T,S)$ can be expressed as follows

$$\begin{aligned} P(T,S) &= E_T\left[e^{-\int_T^S r_u du}\right] \approx E_{n_1,n-1}\left[\exp\left\{-\Delta\left(\frac{1}{2}r_{n_1} + r_{n_1+1} + \cdots + r_{n-1} + \frac{1}{2}r_n\right)\right\}\right] \\ &= e^{-A_{n_2} - B_{n_2}r_{n_1}}AF_{n_2,n} \equiv \tilde{P}_T(T,S). \end{aligned} \tag{9}$$

In order to evaluate (9) at the current time $t = 0$, we substitute (8) into (9), then we have

$$\tilde{P}_0(T,S) = \exp\left[-A_{n_2} - B_{n_2}\left\{(1-v^{n_1})\mu + v^{n_1}r_0 + Y_{2,n_1}\right\}\right] AF_{n_2,n} \equiv C_{n_2}e^{-B_{n_2}Y_{2,n_1}}, \quad (10)$$

where $AF_{n_2,n}$ can be expressed as $AF_{n_2,n} = E_{n_1,n-1}\left[e^{-Z_{n_2}}\right]$ with $Z_{n_2} = \Delta^{1/2}\sum_{i=1}^{n_2} b_i X_{n-i+1}$.

By the similar calculation to (9), we observe that

$$\begin{aligned} C(r_0,0,T,S,K) &= E_0^Q\left[e^{-\int_0^T r_u du}(P(T,S)-K)_+\right] \\ &\approx E_{0,n_1-1}\left[\exp\left\{-\Delta\left(\frac{1}{2}r_0 + r_1 + \cdots + r_{n_1-1} + \frac{1}{2}r_{n_1}\right)\right\}(\tilde{P}_0(T,S)-K)_+\right] \\ &= e^{-A_{n_1}-B_{n_1}r_0}E_{0,n_1-1}\left[e^{-Y_{1,n_1}}(\tilde{P}_0(T,S)-K)_+\right] \equiv \tilde{C}(r_0,0,T,S,K). \end{aligned} \quad (11)$$

From (11) and (10), we can evaluate the European call options on zero-coupon bonds as follows

$$\tilde{C}(r_0,0,T,S,K) = e^{-A_{n_1}-B_{n_1}r_0}E_{0,n_1-1}\left[e^{-Y_{1,n_1}}\left(C_{n_2}e^{-B_{n_2}Y_{2,n_1}} - K\right)_+\right]. \quad (12)$$

3 Assumptions and Main Results

In this section, we evaluate the European call options on zero-coupon bond based on the Edgeworth expansion of the joint density function of $\boldsymbol{Y}_{n_1} = (Y_{1,n_1}, Y_{2,n_1})'$ where Y_{1,n_1} and Y_{2,n_1} are defined by (7) and (8), respectively. The proofs of the lemmas and theorems stated in this section are given in Shiohama and Tamaki (2011).

First, we make some assumptions on $\{X_j\}$.

Assumption 1. The processes of $\{X_j\}$ are fourth-order stationary in the sense that

1. $E[X_j] = 0$,
2. $\mathrm{cum}(X_j, X_{j+u}) = c_{X,2}(u)$,
3. $\mathrm{cum}(X_j, X_{j+u_1}, X_{j+u_2}) = c_{X,3}(u_1,u_2)$,
4. $\mathrm{cum}(X_j, X_{j+u_1}, X_{j+u_2}, X_{j+u_3}) = c_{X,4}(u_1,u_2,u_3)$.

Assumption 2. The cumulants $c_{X,k}(u_1,\ldots,u_{k-1})$ for $k = 2,3,4$ satisfy

$$\sum_{u_1,\cdots,u_{k-1}=-\infty}^{\infty} |c_{X,k}(u_1,\ldots,u_{k-1})| < \infty.$$

Assumptions 1 and 2 are satisfied by a wide class of time series models which contain the usual ARMA and GARCH processes. The following lemma gives the evaluation of the cumulants of $\boldsymbol{Y}_{n_1}$.

Lemma 1. *Under Assumptions 1 and 2, the cumulants of* $\boldsymbol{Y}_{n_1}$ *are evaluated as follows:*

1. $E[\boldsymbol{Y}_{n_1}] = 0$,

2. $\mathrm{Cov}[\boldsymbol{Y}_{n_1}] = \Sigma_{n_1} + o(n_1^{-1})$, *where*

$$\Sigma_{n_1} = \begin{pmatrix} \sigma_{1,n_1}^2 & \sigma_{12}^{(n_1)} \\ \sigma_{21}^{(n_1)} & \sigma_{2,n_1}^2 \end{pmatrix}.$$

3. $\mathrm{cum}(Y_{i,n_1}, Y_{j,n_1}, Y_{k,n_1}) = n_1^{-1/2} C_{ijk}^{(n_1)} + o(n_1^{-1})$,

4. $\mathrm{cum}(Y_{i,n_1}, Y_{j,n_1}, Y_{k,n_1}, Y_{\ell,n_1}) = n_1^{-1} C_{ijk\ell}^{(n_1)} + o(n_1^{-1})$,

for $i, j, k, \ell \in \{1, 2\}$.

In order to derive the Edgeworth expansion of $\boldsymbol{Y}_{n_1}$ and Z_{n_2}, we need the following assumption.

Assumption 3. J-th order ($J \geq 5$) cumulants of $\boldsymbol{Y}_{n_1}$ and Z_{n_2} are of order $O(n_1^{-J/2+1})$ and $O(n_2^{-J/2+1})$, respectively.

For simplicity, let $\boldsymbol{Y} = (Y_1, Y_2)' = (Y_{1,n_1}/\sigma_{1,n_1}, Y_{2,n_1}/\sigma_{2,n_1})'$. From Lemma 1, the covariance matrix Σ of $\boldsymbol{Y}$ is reduced to

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where ρ is the correlation coefficient of Y_1 and Y_2. Then, we obtain the following theorem.

Theorem 1. *Under Assumptions 1-3, the third-order Edgeworth expansion of the joint density function of* $\boldsymbol{Y}$ *is given by*

$$g(y) = \phi_\Sigma(y)\left\{1 + \frac{1}{6\sqrt{n_1}}\tilde{C}_{ijk}^{(n_1)} H_\Sigma^{ijk}(y) + \frac{1}{24 n_1}\tilde{C}_{ijk\ell}^{(n_1)} H_\Sigma^{ijk\ell}(y) + \frac{1}{72 n_1}\tilde{C}_{ijk}^{(n_1)}\tilde{C}_{i'j'k'}^{(n_1)} H_\Sigma^{ijki'j'k'}(y)\right\} + o(n_1^{-1}), \tag{13}$$

where $\tilde{C}_{i_1\cdots i_j}^{(n_1)} = C_{i_1\cdots i_j}^{(n_1)}/(\sigma_{i_1,n_1}\cdots\sigma_{i_j,n_1})$, $\phi_\Sigma(y)$ *is the bivariate normal density function with mean zero and covariance matrix* Σ *and* $H_\Sigma^{i_1\cdots i_j}(y)$ *is the Hermite polynomials with* $\phi_\Sigma(y)$,

$$H_\Sigma^{i_1\cdots i_j}(y) = \frac{(-1)^j}{\phi_\Sigma(y)}\frac{\partial^j}{\partial y_{i_1}\cdots\partial y_{i_j}}\phi_\Sigma(y).$$

Remark 1. Throughout the paper, we use the Einstein notation. Indices appearing twice in a single term, once as a superscript and once as a subscript, are interpreted as summing over all of its possible values, as in, for example, $C_{ijk}^{(n_1)} H_\Sigma^{ijk}(y) = \sum_{i,j,k=1}^2 C_{ijk}^{(n_1)} H_\Sigma^{ijk}(y)$.

The following lemma is useful for calculating the expectation of (12) based on the density function (13).

Lemma 2. *For any constants c_1, c_2, and d, and $i_1, \ldots, i_j \in \{1, 2\}$,*

1.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{d} e^{-c_1 y_1 - c_2 y_2} \phi_\Sigma(y) dy = \exp\left\{\frac{1}{2}\left(c_1^2 + 2\rho c_1 c_2 + c_2^2\right)\right\} \Phi(d + \rho c_1 + c_2).$$

2.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{d} e^{-c_1 y_1 - c_2 y_2} H_\Sigma^{i_1 \cdots i_j}(y) \phi_\Sigma(y) dy$$
$$= \exp\left\{\frac{1}{2}\left(c_1^2 + 2\rho c_1 c_2 + c_2^2\right)\right\} (-1)^j c_1^{j-j(2)} \left(c_2 + D\right)^{j(2)} \Phi(d + \rho c_1 + c_2),$$

where $j(2)$ is the number of i_k's equal to 2 among $i_1, \ldots, i_j$ and $D = d/dx$.

We then obtain the following theorem.

Theorem 2. *Under Assumptions 1-3, the European call options on the zero-coupon bond is expressed as follows:*

$$\tilde{C}(r_0, 0, T, S, K) = \tilde{P}_G(0, S) N D_0 D_1 \Phi(\tilde{d}_1) - K \tilde{P}_G(0, T) D_2 \Phi(\tilde{d}_2) + o(\min(n_1, n_2)^{-1}), \quad (14)$$

where $\tilde{P}_G(0, T) = \exp\left(-A_{n_1} - B_{n_1} r_0 + \sigma_{1,n_1}^2/2\right)$, $\tilde{P}_G(0, S) = \exp\left(-A_n - B_n r_0 + \sigma_{1,n}^2/2\right)$,

$$N = \exp\left\{\frac{1}{2}\left(\sigma_{1,n_1}^2 + \sigma_{1,n_2}^2 - \sigma_{1,n}^2 + 2B_{n_2}\sigma_{12}^{(n_1)} + B_{n_2}^2 \sigma_{2,n_1}^2\right)\right\},$$

$$D_0 = 1 - \frac{C_{111}^{(n_2)}}{6\sqrt{n_2}} + \frac{C_{1111}^{(n_2)}}{24 n_2} + \frac{(C_{111}^{(n_2)})^2}{72 n_2},$$

$$D_\alpha = 1 - \frac{1}{6\sqrt{n_1}} C_{ijk}^{(n_1)} D_\alpha^{ijk} + \frac{1}{24 n_1} C_{ijk\ell}^{(n_1)} D_\alpha^{ijk\ell} + \frac{1}{72 n_1} C_{ijk}^{(n_1)} C_{i'j'k'}^{(n_1)} D_\alpha^{ijki'j'k'}$$

for $\alpha = 1, 2$, *are the polynomials of* $D_1^{i_1 \cdots i_j} = (B_{n_2} + D/\sigma_{2,n_1})^{j(2)}$ *and* $D_2^{i_1 \cdots i_j} = (D/\sigma_{2,n_1})^{j(2)}$,

$$\tilde{d}_1 = \frac{1}{B_{n_2}\sigma_{2,n_1}} \log \frac{\tilde{P}_G(0, S) N D_0}{K \tilde{P}_G(0, T)} + \frac{1}{2} B_{n_2} \sigma_{2,n_1} \quad \textit{and} \quad \tilde{d}_2 = \tilde{d}_1 - B_{n_2}\sigma_{2,n_1}.$$

The following theorem shows that if $\{X_j\}$ are i.i.d. $N(0, \sigma^2)$, then as $\Delta \to 0$, approximate bond option formula (14) converges to the price of that of Vasicek (1977).

Theorem 3. *Suppose that $\{X_j\}$ are independent and identically distributed normal random variables with mean 0 and variance σ^2. Then, (14) corresponds to Vasicek (1977).*

$$\tilde{C}(r_0, 0, T, S, K) \to P(0, S)\Phi(d_1) - K P(0, T) \Phi(d_2), \quad \textit{as} \quad \Delta \to 0,$$

where d_1, d_2 and $P(0, T)$ are defined in (3).

4 Asymptotic Valuation for Coupon Bond Option Prices

The difficulty of pricing coupon bond options is that the exercise region is defined implicitly and its probability is often difficult to compute. Because a coupon bond is just a portfolio of discount bonds of different maturities, the value of any riskless coupon bond can be expressed as a weighted sum of discount bond prices. Let S, M and δ be the bond maturity, the number of coupon payments, and the cashflow frequency, respectively. Then $M = S/\delta$. The price of a coupon bond with coupon rate α is given by

$$\sum_{i=1}^{M} \alpha_i \tilde{P}_0(0, S_i)$$

where $\tilde{P}_0(0, S_i)$ is given by (6). Most coupon bonds have semi-annual coupons, and swaps have either annual or semi-annual cashflows. As an example, consider a 10-year 10% bond with a face amount of 100 and coupons are paid annually. In this case, $M = 10$ since the bond makes annual coupon payments of 10 as well as a final payment of 110, that is $\alpha_1 = \cdots = \alpha_9 = 10$ and $\alpha_{10} = 110$. In addition, $S_1 = 1, S_2 = 2, \ldots, S_{10} = 10$.

Let $n_{1,i} = (S_i - T)/\Delta + n_1$ and $n_{2,i} = n_{1,i} - n_1$. The payoff function for a T-maturity European call on a bond with strike price K is

$$\max\left(0, \sum_{i=1}^{(S-T)/\delta} \alpha_i \tilde{P}_0(T, S_i) - K\right), \tag{15}$$

where $\tilde{P}_0(T, S_i)$ is given by (10). Since $\tilde{P}_0(T, S_i)$ is a monotonic function of r_0 for all S_i, there is a critical interest rate r^* such that the call is exercised if $0 \leq r_0 < r^*$ of its expiration date. This critical interest rate is easily found by solving the following expression for r^*.

$$\sum_{i=1}^{(S-T)/\delta} \alpha_i \tilde{P}_0(r^*, T, S_i) = K, \tag{16}$$

where

$$\tilde{P}_0(r^*, T, S_i) = \exp\left[-A_{n_{2,i}} - B_{n_{2,i}}\{(1 - v^{n_{1,i}})\mu + v^{n_{1,i}} r^* Y_{2,n_{1,i}}\}\right] AF_{n_{2,i},n}.$$

The critical interest rate can be determined by solving (16) numerically. Having specified critical interest rate r^*, the strike price K is decomposed as follows

$$K = \sum_{i=1}^{(S-T)/\delta} \tilde{P}_0(r^*, 0, S_i) \equiv \sum_{i=1}^{(S-T)/\delta} K_i,$$

see, for example, Jamshidian (1989). Then the coupon bond option price is expressed as follows

$$\tilde{C}_\alpha = \sum_{i=1}^{(S-T)/\delta} \tilde{C}(r_0, 0, T, S_i, K_i),$$

where the function $\tilde{C}(r_0, 0, T, S_i, K_i)$ is defined by (12). Using the results of Theorem 2, The price of European option on this coupon with the strike K and the maturity T is given by

$$\begin{aligned}\tilde{C}_\alpha &= \sum_{i=1}^{(S-T)/\delta} \alpha_i \tilde{P}_G(0, S_i) N_i D_{0,i} D_{1,i} \Phi(\tilde{d}_{1,i}) - K\tilde{P}_G(0,T) D_2 \Phi(\tilde{d}_2) \\ &\quad + o(\{\min_i \min(n_{1,i}, n_{2,i})\}^{-1})\end{aligned}$$

where

$$\tilde{d}_{1,i} = \frac{1}{B_{n_{2,i}}\sigma_{2,n_{1,i}}} \log \frac{\tilde{P}_G(0,S_i)N_i D_{0,i}}{K_i \tilde{P}_G(0,T)} + \frac{1}{2} B_{n_{2,i}} \sigma_{2,n_{1,i}},$$

$$N_i = \exp\left\{\frac{1}{2}\left(\sigma_{1,n_{1,i}}^2 + \sigma_{1,n_{2,i}}^2 - \sigma_{1,n}^2 + 2B_{n_{2,i}}\sigma_{12}^{(n_{1,i})} + B_{n_{2,i}}^2 \sigma_{2,n_{1,i}}^2\right)\right\},$$

$$D_{0,i} = 1 - \frac{C_{111}^{(n_{2,i})}}{6\sqrt{n_{2,i}}} + \frac{C_{1111}^{(n_{2,i})}}{24 n_{2,i}} + \frac{(C_{111}^{(n_{2,i})})^2}{72 n_{2,i}},$$

and

$$D_{1,i} = 1 - \frac{1}{6\sqrt{n_{1,i}}} C_{jk\ell}^{(n_{1,i})} D_{1,i}^{jk\ell} + \frac{1}{24 n_{1,i}} C_{jk\ell m}^{(n_{1,i})} D_{1,i}^{jk\ell m} + \frac{1}{72 n_{1,i}} C_{jk\ell}^{(n_{1,i})} C_{j'k'\ell'}^{(n_{1,i})} D_{1,i}^{jk\ell j'k'\ell'}.$$

Here $D_{1,i}^{i_1 \cdots i_j} = (B_{n_{2,i}} + D/\sigma_{2,n_{1,i}})^{j(2)}$.

5 Numerical Examples

The closed-form expressions for coupon bond option prices allow us to examine the comparative statics properties of these contingent claims directly. First, we consider the relation between coupon bond option prices and the riskless interest rate with different values of Δ. The innovation density is assumed to be normal distribution. The parameter values of interest rate models are as follows:

$$\mu = 0.085, \quad \sigma = 0.02, \quad \kappa = 0.20.$$

The option is a call with a maturity of five years, written on a 15-year coupon bond with a face value of 100 and a coupon rate of 10%. Coupons are paid annually. These coupon bond option valuation settings are the same as those investigated by (Wei (1997), p.139, Table I). The results are summarized in Table 1 and Figure 1.

Table 1: Continuous Vasicek versus Discretely Observed Vasicek Option Prices

r_0	$\Delta = 1/12$	$\Delta = 1/52$	$\Delta = 1/252$	$\Delta = 1/1000$	Vasicek
0.04	12.4634	12.5060	12.5161	12.5181	12.5188
0.06	9.8300	9.8465	9.8505	9.8512	9.8515
0.08	7.5978	7.5943	7.5935	7.5933	7.5933
0.10	5.7386	5.7209	5.7166	5.7158	5.7155
0.12	4.2231	4.1965	4.1901	4.1889	4.1885
0.14	3.0193	2.9884	2.9811	2.9797	2.9792
0.16	2.0910	2.0599	2.0526	2.0512	2.0507
0.18	1.3989	1.3705	1.3638	1.3625	1.3620
0.20	0.9018	0.8777	0.8721	0.8710	0.8706
0.22	0.5588	0.5398	0.5353	0.5345	0.5342
0.24	0.3321	0.3181	0.3148	0.3142	0.3139
0.26	0.1889	0.1792	0.1769	0.1765	0.1764
0.28	0.1027	0.0963	0.0949	0.0946	0.0945
0.30	0.0532	0.0494	0.0485	0.0483	0.0483

From Table 1 and Figure 1, it is seen that the discretely sampled Vasicek option prices converge to the continuous Vasicek option prices for all levels of interest rates as Δ tends to zero.

Next, we consider the cases when the underlying innovation processes are non-Gaussian and dependent. For this, we consider the following three models:

1. Skewed student t-distribution,
2. Gaussian GARCH(1,1) processes: $X_j = h_j^{1/2}\varepsilon_j$, with

$$h_j = \alpha_0 + \alpha_1 X_{j-1}^2 + \beta_1 h_{j-1},$$

3. Gaussian AR(1) processes: $X_j = \phi X_{j-1} + \varepsilon_j$.

For a skewed student t-distribution of (Hansen, 1994), the innovation density is given by

$$g(x|\eta,\lambda) = \begin{cases} bc\left(1+\frac{1}{\eta-2}\left(\frac{bx+a}{1-\lambda}\right)^2\right)^{-(\eta+1)/2}, & \text{if } x < -a/b, \\ bc\left(1+\frac{1}{\eta-2}\left(\frac{bx+a}{1+\lambda}\right)^2\right)^{-(\eta+1)/2}, & \text{if } x \geq -a/b, \end{cases}$$

where $-1 < \lambda < 1$, $\eta > 2$ and

$$a = 4\lambda c\frac{\eta-2}{\eta-1}, \quad b^2 = 1 + 3\lambda^2 - a^2, \quad c = \frac{\Gamma((\eta+1)/2)}{\sqrt{\pi(\eta-2)}\Gamma(\eta/2)},$$

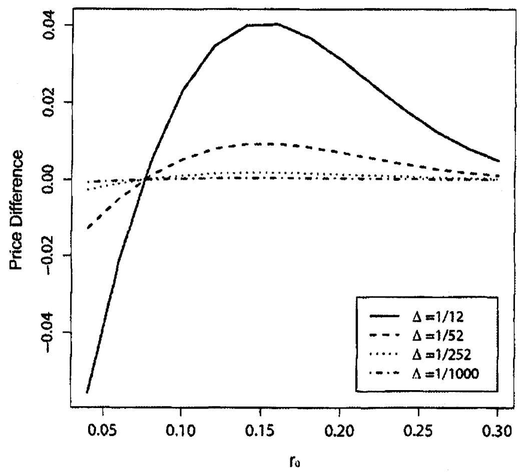

Figure 1: Relationship between r_0 and the option prices differences between the discretized and continuous Vasicek models, with $\Delta = 1/12, 1/52, 1/252$, and $1/1000$. Option values are computed for 5 year call on a 15 year 10% coupon bond with a face value of 100. Selected parameter values are $\kappa = 0.20$, $\mu = 0.085$, and $\sigma = 0.02$.

with zero mean and unit variance, and where $\Gamma(\cdot)$ is a gamma function. Suppose $\eta > 4$, Jondeau and Rockinger (2003) show that skewness and kurtosis of X_j are

$$\gamma_{1,\mathrm{ST}} = \frac{m_3 - 3am_2 + 2a^3}{b^3} \quad \text{and} \quad \gamma_{2,\mathrm{ST}} = \frac{m_4 - 4am_3 + 6a^3 m_2 - 3a^4}{b^4},$$

respectively, where $m_2 = 1 + 3\lambda^2, m_3 = 16c\lambda(1+\lambda^2)(\eta-2)^2/\left[(\eta-1)(\eta-3)\right], m_4 = 3(\eta-2)(1+10\lambda^2+5\lambda^4)/(\eta-4)$. Hereafter, we consider the process $\{\sigma_{\mathrm{ST}} X_j\}$. The covariances $\sigma_{i_1}\sigma_{i_2}$ are given by

$$\sigma_{i_1}\sigma_{i_2} = \frac{T}{n_1}\sum_{k=1}^{n_1} b_k^{p_2} v^{(k-1)(2-p_2)} \sigma_{\mathrm{ST}}^2. \tag{17}$$

The third- and fourth-order joint cumulants of $\boldsymbol{Y}_{n_1}$ are evaluated as

$$\mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}\right) = \frac{(T\sigma_{\mathrm{ST}}^2)^{3/2}}{\sqrt{n_1}}\left(\frac{1}{n_1}\sum_{k=1}^{n_1} b_k^{p_3} v^{(k-1)(3-p_3)}\right)\gamma_{1,\mathrm{ST}}, \tag{18}$$

and

$$\mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}, Y_{i_4,n_1}\right) = \frac{(T\sigma_{\mathrm{ST}}^2)^2}{n_1}\left(\frac{1}{n_1}\sum_{k=1}^{n_1} b_k^{p_4} v^{(k-1)(4-p_4)}\right)(\gamma_{2,\mathrm{ST}} - 3), \tag{19}$$

respectively.

Next, we consider the evaluation of third- and fourth-order cumulants of GARCH(1,1) process. Let $\{X_j\}$ be a GARCH(1,1) process (Bollerslev, 1986), given by

$$X_j = h_j^{1/2}\varepsilon_j, \quad h_j = \alpha_0 + \alpha_1 X_{j-1}^2 + \beta_1 h_{j-1},$$

where $\{\varepsilon_j\}$ is a sequence of i.i.d. normal random variables, each with mean zero and variance one. The parameter values must satisfy $\alpha_0 > 0, \alpha_1, \beta_1 \geq 0, \alpha_1 + \beta_1 < 1$ and $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$. Then, we obtain

$$\sigma_{i_1}\sigma_{i_2} = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}\frac{1}{n_1}\sum_{k=1}^{n_1} b_k^{p_2} v^{(k-1)(2-p_2)}, \quad \text{and} \quad \mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}\right) = 0. \tag{20}$$

The fourth-order cumulants are evaluated as follows:

$$\begin{aligned}
&\mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}, Y_{i_4,n_1}\right) \\
&= \frac{3}{n_1}\left[\int_{-\pi}^{\pi} \frac{1}{n_1}\sum_{k=1}^{n_1}\sum_{\ell=1}^{n_1} b_k^{p_2} v^{(k-1)(2-p_2)} b_\ell^{p_2'} v^{(\ell-1)(2-p_2')} e^{i(k-\ell)\lambda} f_{X^2}(\lambda)d\lambda \right. \\
&\left. - \left(\frac{2\alpha_0^2(1 + \alpha_1 + \beta_1)}{(1 - (\alpha_1 + \beta_1))(1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2)}\right)\frac{2}{n_1}\sum_{k=1}^{n_1} h^{(k-1)p_4} b_k^{4-p_4}\right],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
f_{X^2}(\lambda) &= \frac{\sigma_{x^2}^2}{2\pi}\frac{1 + \beta_1^2 - 2\beta_1\cos(\lambda)}{1 + (\alpha_1 + \beta_1)^2 - 2(\alpha_1 + \beta_1)\cos(\lambda)}, \\
\sigma_{x^2}^2 &= \frac{2\alpha_0^2(1 + \alpha_1 + \beta_1)}{(1 - (\alpha_1 + \beta_1))(1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2)}.
\end{aligned}$$

Finally, we evaluate the second-order cumulant of Gaussian AR(1) process. Let $\{X_j\}$ be the Gaussian AR(1) process,

$$X_j = \phi X_{j-1} + \varepsilon_j,$$

where $|\phi| < 1$ and $\{\varepsilon_j\}$ is a sequence of i.i.d. $N(0, \sigma_{\mathrm{AR}}^2)$ random variables. Since ε_j is a Gaussian random variable, we observe that for $i_1, i_2, i_3, i_4 \in \{1, 2\}$,

$$\mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}\right) = \mathrm{cum}\left(Y_{i_1,n_1}, Y_{i_2,n_1}, Y_{i_3,n_1}, Y_{i_4,n_1}\right) = 0. \tag{22}$$

The covariance of $\boldsymbol{Y}_{n_1}$ can be evaluated as follows:

$$\sigma_{i_1}\sigma_{i_2} = \frac{T}{n_1}\sum_{k,\ell=1}^{n_1} b_k^{p_2} v^{(\ell-1)(2-p_2)} R(k - \ell), \tag{23}$$

where $R(j)$ is the autocovariance function of X_t at lag j; that is, $R(j) = \sigma^2_{\mathrm{AR}}\phi^j(1-\phi^2)^{-1}$. After some extensive calculations, (23) becomes

$$\sigma^2_{1,n_1} = \frac{\sigma^2_{\mathrm{AR}}\beta(\phi v)(1-v^{2n_1})}{\kappa(1-\phi^2)(1+v)} + o(n_1^{-1}), \tag{24}$$

$$\sigma^2_{2,n_1} = \frac{\sigma^2_{\mathrm{AR}}}{\kappa^2(1-\phi^2)}\left[\Delta n_1\beta(\phi) - \frac{2\Delta\phi}{(1-\phi)^2}\right.$$
$$\left. -\frac{1+v}{4\kappa}\left\{2\left(\beta(\phi)+\beta(\phi v)\right)(1-v^{n_1}) - \beta(\phi v)(1-v^{2n_1})\right\}\right] + o(n_1^{-1}), \tag{25}$$

and

$$\sigma^{(n_1)}_{12} = \frac{\sigma^2_{\mathrm{AR}}}{2\kappa^2(1-\phi^2)}\left[\{\beta(\phi)+\beta(\phi v)\}(1-v^{n_1}) - \beta(\phi v)(1-v^{2n_1})\right] + o(n_1^{-1}), \tag{26}$$

where $\beta(x) = (1+x)/(1-x)$.

Table 2 reports the prices of the coupon bond options with three different innovation models for 5-year call on a 15-year 10% coupon bond with face value 100. The time interval Δ is fixed at $\Delta = 1/12$ and the parameters of the short rates are the same as those in Table 1. The skewed t-distributed option prices are calculated with $\eta = 4.5$ and $\lambda = -0.8, 0, 0.8$. When $\lambda = 0$, which corresponds to the standardized t-distribution, the option prices become higher than those obtained by using Gaussian distribution, (see, second column of Table 1). When λ is negative, and r_0 is less than 0.18, the option prices get larger than those with $\lambda = 0$, and when r_0 is greater than 0.20, the option prices become smaller than those with $\lambda = 0$, due to the asymmetry of the underlying innovation distribution. When λ is positive, the results are vice versa.

As for the AR(1) distributed call options, when the autoregressive coefficient is 0 ($\phi = 0$) and which corresponds to the i.i.d. Gaussian case, the option prices obtained are the same as those in Table 1 with $\Delta = 1/12$. For negative ϕ, the option prices become lower, and for positive ϕ, the option prices become higher. The option prices are much more sensitive to the positive autocorrelation than the negative autocorrelation.

As for the GARCH(1,1) distributed call options, the option prices obtained with the parameters $\alpha_0 = 0.00036$, $\alpha_1 = 0.1$ and $\beta_1 = 0$ are the same as those obtained for Gaussian case, because the unconditional variance is $\alpha_1/(1-\alpha_1-\beta_1) = 0.02^2$. As β_1 increases, the option values become to be large.

Figure 2 shows plots of the price of European call options on coupon bond as a function of β_1, with $\alpha_1 = 0.01, 0.03, 0.05, 0.08, 0.1$. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The option with a maturity of 1 year and a strike price of 100 are written on a 5 year coupon bond with a face value of 100 and coupon rate of 5%. Coupons are paid semiannually. The parameter σ^2 is changed to the corresponding GARCH variance such that $\sigma^2 = \alpha_0/(1-\alpha_1-\beta_1)$, and α_0 is fixed at 0.01^2. As shown in the figures, option prices increase as β_1 increases, because large values of β_1 indicate that the second- and the fourth-order cumulants tend to be large.

Table 2: Prices of call options on 15 years bond with 5 year option maturity

r_0	Skewed t distribution			AR(1) model			GARCH(1,1) model			Vasicek
	$\eta = 4.5$			$\phi = -0.8$	$\phi = 0$	$\phi = 0.8$	$\alpha_0 = 3.6 \times 10^{-4}$		$\alpha_1 = 0.1$	
	$\lambda = -0.8$	$\lambda = 0$	$\lambda = 0.8$				$\beta_1 = 0$	$\beta_1 = 0.3$	$\beta_1 = 0.6$	
0.04	12.5410	12.4726	12.3788	10.5213	12.4634	34.4723	12.4634	13.7447	17.6239	12.5188
0.06	9.9049	9.8328	9.7210	7.9007	9.8300	29.9569	9.8300	11.0498	14.6506	9.8515
0.08	7.6679	7.5925	7.4730	5.6180	7.5978	25.9648	7.5978	8.7556	12.0856	7.5933
0.10	5.8009	5.7252	5.6112	3.6656	5.7386	22.4428	5.7386	6.8272	9.8876	5.7155
0.12	4.2735	4.2035	4.1079	2.0899	4.2231	19.3428	4.2231	5.2304	8.0183	4.1885
0.14	3.0533	2.9970	2.9286	0.9771	3.0193	16.6210	3.0193	3.9306	6.4416	2.9792
0.16	2.1056	2.0702	2.0323	0.3503	2.0910	14.2375	2.0910	2.8929	5.1237	2.0507
0.18	1.3939	1.3833	1.3735	0.0907	1.3989	12.1562	1.3989	2.0821	4.0328	1.3620
0.20	0.8806	0.8932	0.9056	0.0162	0.9018	10.3439	0.9018	1.4633	3.1393	0.8706
0.22	0.5278	0.5573	0.5845	0.0019	0.5588	8.7710	0.5588	1.0028	2.4156	0.5342
0.24	0.2987	0.3363	0.3711	0.0001	0.3321	7.4102	0.3321	0.6693	1.8364	0.3139
0.26	0.1594	0.1966	0.2332	0.0000	0.1889	6.2370	0.1889	0.4345	1.3787	0.1764
0.28	0.0806	0.1115	0.1458	0.0000	0.1027	5.2291	0.1027	0.2741	1.0217	0.0945
0.30	0.0395	0.0614	0.0910	0.0000	0.0532	4.3666	0.0532	0.1678	0.7469	0.0483

The bond has a face value of 100 and a coupon rate of 10%. Coupons are paid annually. Interest rates are assumed to follow discretely sampled Vasicek model with $\kappa = 0.20, \mu = 0.085, \sigma = 0.02$, and $\Delta = 1/12$. For AR(1) model, the parameter σ^2_{AR} is chosen such that the long-run variance $\sigma^2_{\text{AR}}/(1-\phi^2)$ corresponds to 0.02^2. For GARCH(1,1) model, the parameter α_0 set at 0.00036 and the unconditional variance of GARCH is changed according to $\sigma^2_{\text{GARCH}} = \alpha_0/1 - \alpha 1 - \beta_1$

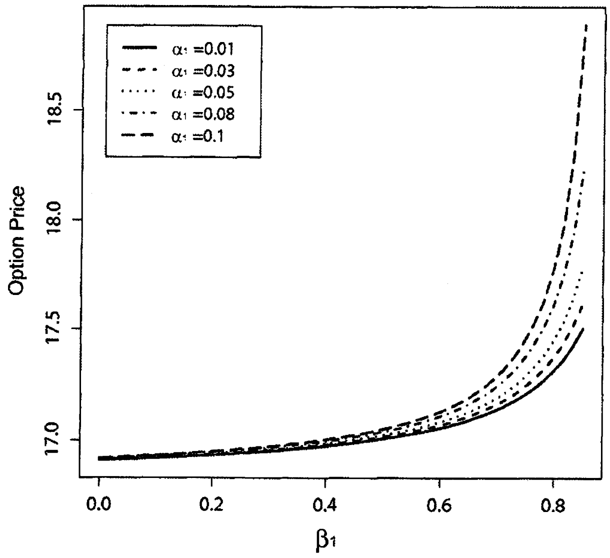

Figure 2: The option prices for GARCH(1,1) process. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The option is 1 year option with a strike price of 100. The parameter α_0 is fixed at 0.01^2. An unconditional variance of GARCH(1,1) model is $\alpha_0/(1-\alpha_1-\beta_1)$.

Figure 3 and 4 plot the price and the price difference between continuous Vasicek and AR(1) process of European call options on a coupon bond as a function of ϕ. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The option is 1 year option with a strike price of 100. The parameter σ_{AR}^2 is chosen such that the long-run variance $\sigma_{AR}^2/(1-\phi^2)$ corresponds to 0.01^2. As shown in figure, when $|\phi|$ approaches 1, option prices increase.

Figure 5 illustrates the coupon bond price of AR(1) distributed options (14) as a function of K. The parameter σ_{AR}^2 is chosen such that the long-run variance $\sigma_{\mathrm{AR}}^2/(1-\phi^2)$ is equal to 0.01^2. The rest of the parameters are the same as those investigated by using GARCH(1,1) distributed options. From Figure 5, we can observe that the effects of first-order dependence are significant for at-the-money call options. A negative autocorrelation results in lower option prices, while a positive autocorrelation leads to higher option prices. These findings can also be confirmed via Figure 6. As the values of $|\phi|$ increases , the price differences tend to be large, and these differences are significant for the at-the-money call options.

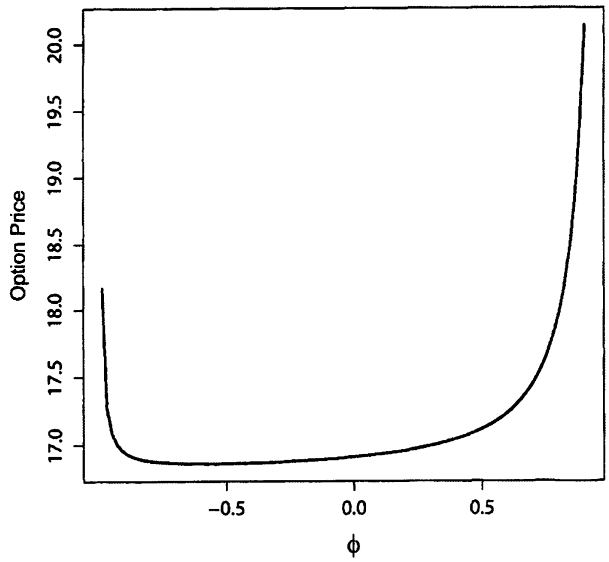

Figure 3: The option price for AR(1) process as a function of ϕ in the domain $[-0.99, 0.9]$. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The option is 1 year option with a strike price of 100. The parameter σ_{AR}^2 is chosen such that the long-run variance $\sigma_{AR}^2/(1-\phi^2)$ corresponds to 0.01^2.

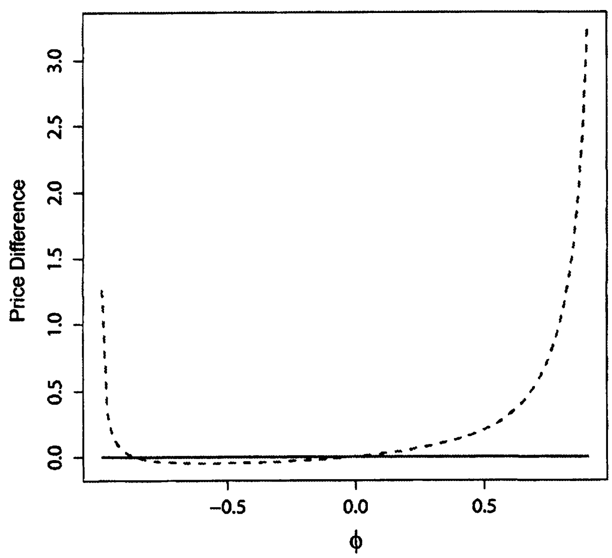

Figure 4: The option price difference for AR(1) process as a function of ϕ in the domain $[-0.99, 0.9]$. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The option is 1 year option with a strike price of 100. The parameter σ_{AR}^2 is chosen such that the long-run variance $\sigma_{AR}^2/(1-\phi^2)$ corresponds to 0.01^2.

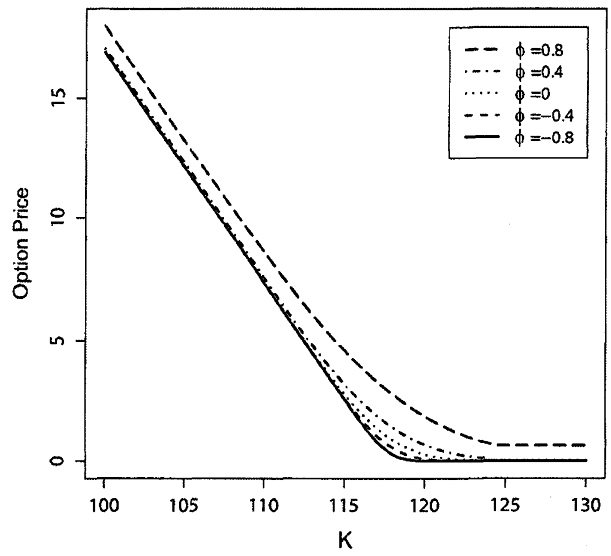

Figure 5: Relationship between K and the option prices for AR(1) process, with $\phi = -0.8, -0.4, 0, 0.4$, and 0.8. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The option maturity is 1. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The parameter σ^2_{AR} is chosen such that the long-run variance $\sigma^2_{AR}/(1-\phi^2)$ corresponds to 0.01^2.

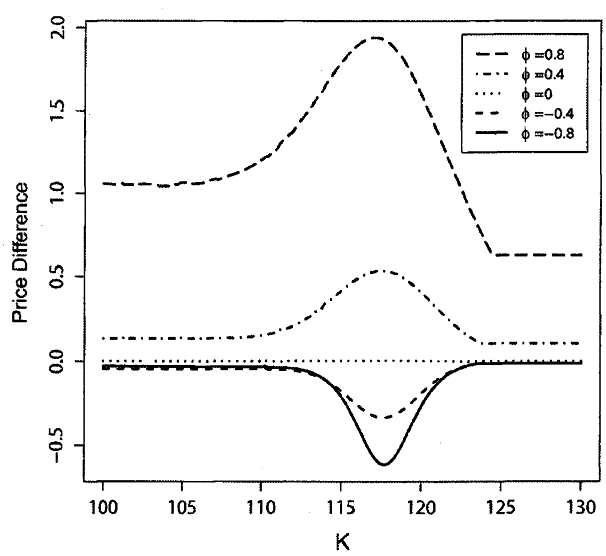

Figure 6: Relationship between K and the option prices differences between continuous Vasicek models and AR(1) process, with $\phi = -0.8, -0.4, 0, 0.4$, and 0.8. We set $\kappa = 0.20, \mu = 0.05, r_0 = 0.05$, and $\Delta = 1/12$. The option maturity is 1. The bond is 5 year bond with a face value of 100 and a coupon rate of 5%. Coupons are paid semiannually. The parameter σ^2_{AR} is chosen such that the long-run variance $\sigma^2_{AR}/(1-\phi^2)$ corresponds to 0.01^2.

6 Conclusion

In this paper, we provide closed-form expressions for European call options on zero-coupon bonds and coupon bonds in a discretized Vasicek model where the innovations are non-Gaussian and dependent. The resulting bond and option pricing formulas reveal the relationship between these prices and the cumulants of the underlying innovation distribution. We present some numerical examples and demonstrate the effects on the bond and option prices. Option values are sensitive to the skewness and the kurtosis of the assumed innovation processes of the short rates for at-the-money options. The findings in this study strongly suggest the importance of incorporating the non-Gaussianity and dependency of the short rate models to hedge the interest rate risk and price the bonds and options adequately.

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