数理解析研究所講究録
第 1760 巻 2011 年 1-9

SCATTERING THEORY FOR KLEIN-GORDON EQUATIONS WITH NON-POSITIVE ENERGY CONFERENCE IN HONOR OF HIROSHI ISOZAKI RIMS, KYOTO, 16-18 FEBRUARY 2011

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ABSTRACT. We study the scattering theory for charged Klein-Gordon equations:

$$\begin{cases} (\partial_t - \mathrm{i}v(x))^2\phi(t,x) + \epsilon^2(x,D_x)\phi(t,x) = 0, \\ \phi(0,x) = f_0, \\ \mathrm{i}^{-1}\partial_t\phi(0,x) = f_1, \end{cases}$$

where:

$$\epsilon^2(x,D_x) = -\sum_{1\le j,k\le n} \left(\partial_{x_j} - \mathrm{i}b_j(x)\right) A^{jk}(x) \left(\partial_{x_k} - \mathrm{i}b_k(x)\right) + m^2(x),$$

describing a Klein-Gordon field minimally coupled to an external electromagnetic field described by the electric potential $v(x)$ and magnetic potential $\vec{b}(x)$. The flow of the Klein-Gordon equation preserves the energy:

$$h[f,f] := \int_{\mathbb{R}^n} \overline{f}_1(x)f_1(x) + \overline{f}_0(x)\epsilon^2(x,D_x)f_0(x) - \overline{f}_0(x)v^2(x)f_0(x)\mathrm{d}x.$$

We consider the situation when the energy is not positive. In this case the flow cannot be written as a unitary group on a Hilbert space, and the Klein-Gordon equation may have complex eigenfrequencies.

Using the theory of definitizable operators on Krein spaces and time-dependent methods, we prove the existence and completeness of wave operators, both in the short- and long-range cases. The range of the wave operators are characterized in terms of the spectral theory of the generator, as in the usual Hilbert space case.

1. INTRODUCTION

1.1. Klein-Gordon equations with non-positive energy. Klein-Gordon field equations coupled with an external electromagnetic field appear in several problems of mathematical physics. It was realized since the forties by Schiff, Snyder and Weinberg [SSW] that for the Klein-Gordon equation on Minkowski space:

$$(\partial_t - \mathrm{i}v(x))^2\phi(t,x) - \Delta_x\phi(t,x) + m^2\phi(t,x) = 0, \tag{1.1}$$

complex eigenfrequencies appear if the electrostatic potential becomes too large, which causes difficulties with the quantization of this field equation. This phenomenon is usually called the *Klein paradox*. It can be traced back to the fact that the conserved energy

$$\int_{\mathbb{R}^d} |\partial_t\phi(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla_x\phi(t,x)|^2 + (m^2 - v^2(x))|\phi(t,x)|^2\mathrm{d}x$$

is not positive definite if $\|v\|_\infty$ is too large.

1991 *Mathematics Subject Classification.* 34L25, 35P25, 81U, 81Q05.

Key words and phrases. Klein-Gordon equations, Krein spaces, scattering theory, wave operators, asymptotic completeness.

A related problem appear when one considers the Klein-Gordon equation on some curved space-times of general relativity, like the Kerr space-time describing a rotating black hole. Again the conserved energy is not positive definite. A nice reference describing these problems is the appendix of the book by Fulling [Fu].

We describe in this report the results of [G] concerning the scattering theory for a class of Klein-Gordon equations generalizing (1.1). In [G] we consider the charged Klein-Gordon equation:

$$\left\{\begin{array}{l}(\partial_t - \mathrm{i}v(x))^2\phi(t,x) + \epsilon^2(x,D_x)\phi(t,x) = 0,\\ \phi(0,x) = f_0,\\ \mathrm{i}^{-1}\partial_t\phi(0,x) = f_1,\end{array}\right. \tag{1.2}$$

in $\mathbb{R}_t \times \mathbb{R}^n_x$ where

$$\epsilon^2(x,D_x) = -\sum_{1\le j,k\le n}\left(\partial_{x_j} - \mathrm{i}b_j(x)\right)a^{jk}(x)\left(\partial_{x_k} - \mathrm{i}b_k(x)\right) + m^2(x),$$

describing a Klein-Gordon field minimally coupled to an external electromagnetic field described by the electric potential $v(x)$ and magnetic potential $\vec{b}(x)$. The function $x \mapsto m(x)$ corresponds to a variable mass term, incorporating for example a scalar curvature term.

Precise hypotheses on $[a^{jk}(x)]$, $\vec{b}(x)$ and $m(x)$ are given in [G], they essentially mean that the second order differential operator ϵ^2 is a long-range perturbation of $-\Delta_x + m^2$ for some $m > 0$.

For the sake of simplicity, in this report we will consider only the simple case (1.1), where:

$$\epsilon^2 := D_x^2 + m^2,\ m > 0.$$

The external electric potential $v(x)$ is assumed to satisfy:

$$\text{(A2) } v^k\epsilon^{-k} : \mathfrak{h} \to \mathfrak{h} \text{ is compact for } k = 1,2,$$

$$\text{(A4) } v(x) = v_s(x) + v_l(x),$$

where:

$$v_l(x) \in S^{-\mu_l}(\mathbb{R}^d)\ \mu_l > 0, \tag{1.3}$$

$$\langle x\rangle^{\mu_s}v_s^k\epsilon^{-k} \text{ is bounded for } k = 1,2,\ \mu_s > 1. \tag{1.4}$$

Here $S^\delta(\mathbb{R}^d)$ is the standard symbol class:

$$S^\delta(\mathbb{R}^d) := \{u \in C^\infty(\mathbb{R}^d)\ :\ \partial_x^\alpha u(x) \in O(\langle x\rangle^{\delta-|\alpha|}),\ \alpha \in \mathbb{N}^d\}.$$

In analogy to the scattering theory for Schrödinger operators, the case $1 < \mu_l$ (resp. $0 < \mu_l \le 1$) will be called the *short-range* (resp. *long-range*) case.

The Cauchy problem (1.2) can be rewritten as

$$f_t = \mathrm{e}^{-\mathrm{i}tB}f,\ B = -\begin{pmatrix}0 & 1\!\!1\\ \epsilon^2 - v^2 & 2v\end{pmatrix},\ \text{for } f_t = \begin{pmatrix}\phi(t)\\ -\mathrm{i}\partial_t\phi(t)\end{pmatrix}.$$

The evolution $\mathrm{e}^{-\mathrm{i}tB}$ preserves the *energy*:

$$h[f,f] := \int_{\mathbb{R}^n}\overline{f}_1(x)f_1(x) + \overline{f}_0(x)\epsilon^2 f_0(x) - \overline{f}_0(x)v^2(x)f_0(x)\mathrm{d}x.$$

We are interested in this paper in the scattering theory, i.e. in the complete classification of the asymptotic behavior of $\mathrm{e}^{-\mathrm{i}tB}f$ for all initial data f, when $t \to \pm\infty$.

1.2. **Scattering theory.** If the energy h is *positive*, i.e. the electric potential is not too large, one can use it to equip the space of initial data with a Hilbert space structure.

Under typical assumptions one obtains the *energy space* $\mathcal{E} = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, and the group $\mathrm{e}^{-\mathrm{i}tB}$ becomes a strongly continuous unitary group on $\mathcal{E}$, whose scattering theory can be studied by Hilbert space methods. We mention among many others the papers [E, Lu, N, S, VW, W]. In this paper we are interested in the situation when the energy is *not* positive. In this case the generator B may have complex eigenvalues, or real eigenvalues with non trivial Jordan blocks.

It follows that in general the energy norm $\|\mathrm{e}^{-\mathrm{i}tB}f\|_{\mathcal{E}}$ may be polynomially or exponentially growing in t.

To our knowledge the only result about scattering theory in this situation is due to Kako [K] where the case $v(x) \in O(\langle x\rangle^{-\mu})$, $\mu > 2$ is treated. In [K], spectral projections $\mathbb{1}_I(B)$ for bounded intervals I such that $\pm m \notin I^{\mathrm{cl}}$ are constructed by stationary arguments, and local wave operators

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\,\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tB_\infty}\mathbb{1}_I(B_\infty) = W_I^{\pm}$$

are shown to exist, for B_∞ being the generator of the free Klein-Gordon equation obtained for $\epsilon^2 = -\Delta + m^2$ and $v(x) \equiv 0$.

Their ranges are shown to be equal to the range of $\mathbb{1}_I(B)$, which is a result of local asymptotic completeness of wave operators.

In this paper we reconsider this problem using two tools:

the first tool is the theory of selfadjoint operators on *Krein spaces* . Krein spaces are complete, hilbertizable vector spaces equipped with a bounded, non-degenerate but *non-positive* hermitian sesquilinear form $h[\cdot,\cdot]$, the adjoint of a densely defined linear operator being defined with respect to h.

The idea of using Krein space theory to study the Klein-Gordon equation with a non-positive energy is of course not new. Equations coming from classical mechanics (like the Klein-Gordon equation) are actually typical applications of Krein space theory. We mention among others the papers [J2, J3, LNT1, LNT2].

Our second tool is an adaptation to the framework of definitizable selfadjoint operators on Krein spaces of the *time-dependent* approach to Hilbert space scattering theory, in the version initiated by Sigal and Soffer [SS], based on *propagation estimates.* The method of propagation estimates proved very powerful and flexible to study scattering theory for Schrödinger operators, in particular for the problem of asymptotic completeness of wave operators.

Its adaptation to the Krein space setup requires some care, because one needs to work with two sesquilinear forms, the non-positive one defining the Krein scalar product, and a positive one defining the hilbertizable topology, the dynamics $\mathrm{e}^{-\mathrm{i}tB}$ preserving the first, but of course not the second.

2. Definitizable operators on Krein spaces

2.1. **Krein spaces.** If $\mathcal{H}$ is a topological complex vector space, we denote by $\mathcal{H}^\#$ the space of continuous linear forms on $\mathcal{H}$ and by $\langle w, u\rangle$, for $u \in \mathcal{H}$, $w \in \mathcal{H}^\#$ the duality bracket between $\mathcal{H}$ and $\mathcal{H}^\#$.

Definition 2.1. *A* Krein space $\mathcal{K}$ *is a hilbertizable vector space* $\mathcal{H}$ *equipped with a bounded hermitian sesquilinear form* $[u, v]$ *non-degenerate in the sense that if* $w \in \mathcal{H}^\#$ *there exists a unique* $u \in \mathcal{H}$ *such that*

$$[u, v] = \langle w, v\rangle, \ v \in \mathcal{H}.$$

If we fix a scalar product $(\cdot|\cdot)$ on $\mathcal{H}$ endowing $\mathcal{H}$ with its hilbertizable topology, then by the Riesz theorem there exists a bounded, invertible selfadjoint operator

B such that

$$[u,v] = (u|Bv), \ u,v \in \mathcal{H}.$$

If A is a densely defined linear operator on $\mathcal{H}$, we will denote by $A^* \in (\mathcal{H})$ the adjoint of A on $(\mathcal{H},(\cdot|\cdot))$ and by $A^\dagger \in B(\mathcal{H})$ the adjoint of A on $(\mathcal{K},[\cdot,\cdot])$ defined by

$$[A^\dagger u, v] := [u, Av], \ u \in \mathrm{Dom}A, v \in \mathrm{Dom}A^\dagger.$$

Definition 2.2. *A Krein space $(\mathcal{K},[\cdot,\cdot])$ is a* Pontryagin space *if either $\mathbb{1}_{\mathbb{R}^-}(B)$ or $\mathbb{1}_{\mathbb{R}^+}(B)$ has finite rank.*

Replacing $[\cdot,\cdot]$ by $-[\cdot,\cdot]$ we can assume that $\mathbb{1}_{\mathbb{R}^-}(B)$ has finite rank, which is the usual convention for Pontryagin spaces.

2.2. **Selfadjoint operators on Krein spaces.** A densely defined operator A on $\mathcal{K}$ is called *selfadjoint* if $A = A^\dagger$. Not much can be said about selfadjoint operators on Krein spaces except for the obvious fact that $\sigma(A) = \overline{\sigma(A)}$. However there exists a class of selfadjoint operators, called *definitizable* which share some properties of selfadjoint operators on Hilbert spaces.

Definition 2.3. *A selfadjoint operator A is* definitizable *if*
(1) $\rho(A) \neq \emptyset$;
(2) *there exists a real polynomial $p(\lambda)$ such that*

$$[u, P(A)u] \geq 0, \ \forall u \in \mathrm{Dom}A^k, \ k := \mathrm{deg}p.$$

A real polynomial p satisfying condition (2) above is called *definitizing* for A.

Definition 2.4. *Let A a definitizable selfadjoint operator and p a definitizing polynomial for A. The set*

$$c_p(A) := p^{-1}(\{0\}) \cap \sigma(A) \cap \mathbb{R}$$

is called the set of (finite) critical points *of A.*

The usefulness of the notion of Pontryagin spaces in this context comes from the following theorem.

Theorem 2.5. *A selfadjoint operator A on a Pontryagin space is definitizable with a definitizing polynomial p of even degree.*

The following result is due to Langer [La].

Proposition 2.6. *Let A be a definitizable selfadjoint operator with definitizing polynomial p. Then:*
(1) *$\sigma(A)/\mathbb{R}$ is the union of pairs $\{\lambda_i, \overline{\lambda}_i\}$ of eigenvalues of finite algebraic multiplicity;*
(2) *Let $I \subset \mathbb{R}$ be a compact interval with $\partial I \cap c_p(A) = \emptyset$, and $k(I)$ be the maximal multiplicity of critical points of A in I (as roots of $p(\lambda)$). Then there exist constants $C(I)$, $\delta(I)$ such that:*

$$\|(A-z)^{-1}\| \leq C(I)|\mathrm{Im}z|^{-1-k(I)}, \ \textit{uniformly for } \mathrm{Re}z \in I, \ 0 < |\mathrm{Im}z| \leq \delta(I)$$

(3) *Set now*

$$E_0 = \sum_{\lambda\in\sigma(A), \ \mathrm{Im}\lambda>0} E(\lambda, A) + E(\overline{\lambda}, A), \ \mathcal{K}_0 := E_0\mathcal{K},$$

where $E(z,A)$ is the Riesz spectral projection on an isolated eigenvalue $z \in \mathbb{C}$ of A. Then E_0 is an orthogonal projector, hence $\mathcal{K}_0$ is a Krein space and

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_0^\perp =: \mathcal{K}_0 \oplus \mathcal{K}_1.$$

2.3. **Functional calculus for definitizable operators.** Because of the power-like growth of its resolvent near the real axis, a definitizable operator admits a smooth functional calculus. A convenient way to construct it is through *almost analytic extensions.*

Proposition 2.7 (Smooth functional calculus). (1) *let* $f \in S^{\rho}(\mathbb{R})$ *for* $\rho < 0$ *if* $\deg p$ *is even and* $\rho < -1$ *if* $\deg p$ *is odd. Then the integral:*

$$f(A) := \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z-A)^{-1}\mathrm{d}z\wedge \mathrm{d}\overline{z} \tag{2.5}$$

is norm convergent in $B(\mathcal{H})$ *and independent on the choice of the almost analytic extension* $\tilde{f}$;

(2) *For* ρ *as in* (1), *the map* $S^{\rho}(\mathbb{R}) \ni f \mapsto f(A) \in B(\mathcal{H})$ *is a homomorphism of algebras with:*

$$f(A)^{\dagger} = \overline{f}(A),$$
$$\|f(A)\| \leq \|f\|_{m}, \text{ for some } m \in \mathbb{N}.$$

Here $\tilde{f}(z)$ is an almost analytic extension of f (see eg [HS], [D]) equal to f on the real line.

Due to the positivity hidden in the definition of definitizability, it is possible to extend the functional calculus to a class of Borel functions (see the survey paper by Langer [La]). If $J \subset \mathbb{R}$ is a finite union of disjoints intervals, we denote by $\mathcal{B}_c(J)$ the $*-$algebra of bounded Borel functions on J which are locally constant near $c_p(A)$.

Proposition 2.8 (Borel functional calculus). (1) *Let* $J \subset \mathbb{R}$ *a finite union of disjoint bounded intervals* I *such that* $\partial I \cap c_p(A) = \emptyset$. *Then the map* $C_0^{\infty}(\mathbb{R}) \ni f \mapsto f(A) \in B(\mathcal{H})$ *can be extended to an homomorphism of* $*-$*algebras:*

$$\mathcal{B}_c(J) \ni f \mapsto f(A) \in B(\mathcal{H}),$$

with $\overline{f}(A) = f^{\dagger}(A)$ *for all* $f \in \mathcal{B}(J)$;

(2) *Let* $\lambda_0 \in \mathbb{R}\backslash c_p(A)$. *Then:*

$$\mathbb{1}_{\{\lambda_0\}}(A) = \mathrm{s-}\lim_{\epsilon\to 0} \mathbb{1}_{[\lambda_0-\epsilon,\lambda_0+\epsilon]}(A)$$

equals the orthogonal projection on $\mathrm{Ker}(A-\lambda_0)$;

(3) *Let* I *a bounded interval with* $I^{\mathrm{cl}} \cap c_p(A) = \emptyset$. *Then there exists* $C_I \geq 0$ *such that*

$$\|f(A)\| \leq C_I\|f\|_{\infty},\ f \in \mathcal{B}_c(I);$$

(4) *Assume that* p *is of even degree. Then the above map extends to all* $f \in \mathcal{B}_c(\mathbb{R})$ *with the same properties. In particular statement (3) extends to all intervals* I *with* $I^{\mathrm{cl}} \cap c_p(A) = \emptyset$. *Moreover one has:*

$$1(A) + E_0 = \mathbb{1},$$

where the projection E_0 *is defined in Prop. 2.6.*

3. Scattering theory for Klein-Gordon equations

3.1. **Properties of eigenvalues and critical points.** The essential spectrum of B is very easy to describe:

Lemma 3.1. *One has:*

$$\sigma_{\mathrm{ess}}(B) ==]-\infty,-m] \cup [m,+\infty[.$$

Proposition 3.2. *Assume that* $v = v_1 + v_2$ *where:*

$$\text{(B1)} \quad \begin{cases} \partial_x^\alpha v_1 \in O(\langle x\rangle^{-\mu-|\alpha|}), \ |\alpha| \leq 2, \\ v_2 \text{ has compact support}, \ v_2 \in L^d(\mathbb{R}^d). \end{cases}$$

Then $\sigma_{\rm p}(B) \cap \mathbb{R} \subset [-m, m]$.

The proposition follows from the observation that $Bf = \lambda f$ iff $p(\lambda) f_0 = E f_0$ for

$$p(\lambda) = p - v^2 - 2\lambda v, \ E = \lambda^2 - m^2.$$

and well known results on absence of strictly positive eigenvalues for Schrödinger operators.

We introduce now an important implicit condition, stating that $\pm m$ are not critical points:

$$\text{(B2)} \ \pm m \notin c_p(B).$$

For this condition to hold it suffices that there are no eigenstates of B for the eigenvalues $\pm m$ with *negative* energy. Elementary computations (which can certainly be improved) yield the following result:

Lemma 3.3. *If either*

$$\|v\|_\infty < \sqrt{2} m,$$

or

$$v \text{ has constant sign}, \ \|v\|_\infty < 2m,$$

then (B2) *holds.*

3.2. **Spectrum of B.** We first summarize what we know about the spectrum of B. We set $\sigma_{\rm pp}^{\mathbb{C}}(B) = \sigma_{\rm pp}(B) \backslash \mathbb{R}$, $\sigma_{\rm pp}^{\mathbb{R}}(B) = \sigma_{\rm pp}(B) \cap \mathbb{R}$.

Proposition 3.4. *Assume hypotheses (A), (B). Then:*

(1) $\sigma_{\rm ess}(B) =]-\infty, -m] \cup [m, +\infty[$;
(2) $\sigma_{\rm pp}^{\mathbb{C}}(B) = \bigcup_{j=1}^N \{z_j, \overline{z}_j\}$, *where* z_j, $\overline{z}_j$ *are eigenvalues of finite algebraic multiplicities;*
(3) $\sigma_{\rm pp}^{\mathbb{R}}(B) \subset [-m, m]$ *is a (finite or infinite) sequence* $(\lambda_i)_{i\in\mathbb{N}}$ *of eigenvalues which can accumulate only at* $\pm m$*, the eigenvalues in* $]-m, m[$ *have finite algebraic multiplicities;*
(4) $\sigma_{\rm pp}^{\mathbb{R}}/c_p(B)$ *have trivial Jordan blocks.*

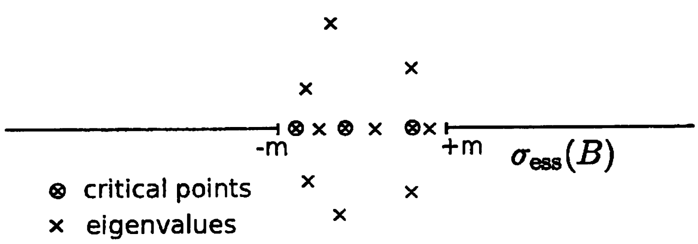

FIGURE 1. The spectrum of B

3.3. **Bound and scattering states.** We set

$$\begin{aligned} 1\!\!1^{\mathbb{C}}_{\rm pp}(B) &:= \textstyle\sum_{z\in\sigma^{\mathbb{C}}_{\rm pp}(B)} E(z,B),\\ 1\!\!1^{\mathbb{R}}_{\rm pp}(B) &:= \textstyle\sum_{\lambda\in\sigma^{\mathbb{R}}_{\rm pp}(B)} 1\!\!1_{\{\lambda\}}(B),\\ 1\!\!1_{\rm pp}(B) &:= 1\!\!1^{\mathbb{C}}_{\rm pp}(B) + 1\!\!1^{\mathbb{R}}_{\rm pp}(B). \end{aligned}$$

Here $E(z,B)$ for $z \in \sigma^{\mathbb{C}}_{\rm pp}(B)$ is the Riesz spectral projection on z. If $\lambda \in \sigma^{\mathbb{R}}_{\rm pp}(B)\backslash c_p(B)$, then $1\!\!1_{\{\lambda\}}(B)$ is defined in Prop. 2.8. If $\lambda \in c_p(B)$ then $1\!\!1_{\{\lambda\}}(B) = 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(B)$ for all $\epsilon > 0$ small enough.

Note that the first sum is finite, the second strongly convergent, since $\pm m$ are not critical points of B.

We set:

$$\mathcal{E}_{\rm pp}(B) := 1\!\!1_{\rm pp}(B)\mathcal{E},\ \mathcal{E} =: \mathcal{E}_{\rm pp}(B) \oplus^{\perp} \mathcal{E}_{\rm scatt}(B).$$

The properties of $\mathcal{E}_{\rm pp}(B)$ and $\mathcal{E}_{\rm scatt}(B)$ are summarized in the following proposition:

Proposition 3.5. (1) *$\mathcal{E}_{\rm pp}(B)$ and $\mathcal{E}_{\rm scatt}(B)$ are Krein subspaces of $\mathcal{E}$, invariant under $(\mathrm{e}^{-\mathrm{i}tB})_{t\in\mathbb{R}}$;*

(2) *$\mathcal{E}_{\rm pp}(B)$ and $\mathcal{E}_{\rm scatt}(B)$ are closed symplectic subspaces of $\mathcal{E}$ and are symplectically orthogonal;*

(3) *Let $u \in \mathcal{E}_{\rm pp}(B)$. Then*

$$\mathrm{e}^{-\mathrm{i}tB}u = \sum_{z\in\sigma^{\mathbb{C}}_{\rm pp}(B)} \mathrm{e}^{-\mathrm{i}tB}E(z,B)u + \sum_{\lambda\in\sigma^{\mathbb{R}}_{\rm pp}(B)} \mathrm{e}^{-\mathrm{i}tB}1\!\!1_{\{\lambda\}}(B)u,$$

where the sum is strongly convergent, uniformly for $t \in \mathbb{R}$;

(4) *one has*

$$\mathcal{E}_{\rm scatt}(B) = \mathcal{E}^{-}_{\rm scatt}(B) \oplus^{\perp} \mathcal{E}^{+}_{\rm scatt}(B),$$

for

$$\mathcal{E}^{-}_{\rm scatt}(B) := 1\!\!1_{]-\infty,-m[}(B)\mathcal{E},\ \mathcal{E}^{+}_{\rm scatt}(B) := 1\!\!1_{]m,+\infty[}(B)\mathcal{E};$$

The space $\mathcal{E}_{\rm scatt}(B)$ will be called the space of *scattering states* for B.

Remark 3.6. *Since the projections $E(z,B)$ and $1\!\!1_{\{\lambda\}}(B)$ are finite rank, it follows from Prop. 3.5 (3) that $\mathrm{e}^{-\mathrm{i}tB}u$ for $u \in \mathcal{E}_{\rm pp}(B)$ can be explicitly computed modulo an error of size $\epsilon > 0$, uniformly in $t \in \mathbb{R}$.*

3.4. **Existence and completeness of short-range wave operators.** In this subsection we assume hypotheses (A1) for $\mu_0 > 1$, (A2), (A3), (A4) for $v_l = 0$, and (B). In other words we are in the short-range case. We set $\mathcal{E}_\infty := H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, equipped with the usual energy scalar product:

$$h_\infty[f,f] = (f_1|f_1) + (f_0|\epsilon^2 f_0),$$

so that $\mathcal{E}_\infty = \mathcal{E}$ as topological spaces. We set also

$$B_\infty := -\begin{pmatrix} 0 & 1\!\!1 \\ \epsilon^2 & 0 \end{pmatrix},$$

which is the generator of the free Klein-Gordon evolution with mass m.

Theorem 3.7. *Assume hypotheses (A1) for $\mu_0 > 1$, (A2), (A3), (A4) for $v_l = 0$, and (B). Then:*

(1) *for all $f \in \mathcal{E}_\infty$ there exist unique $f^{\pm} \in \mathcal{E}_{\rm scatt}(B)$ such that*

$$\mathrm{e}^{-\mathrm{i}tB}f^{\pm} - \mathrm{e}^{-\mathrm{i}tB_\infty}f \to 0,\ t \to \pm\infty.$$

(2) *Let us define the* short-range wave operators $\Omega_s^{\pm}$*:*

$$\Omega_s^{\pm}: \begin{array}{ll} \mathcal{E}_\infty & \rightarrow \mathcal{E}_{\text{scatt}}(B), \\ f & \mapsto f^{\pm}. \end{array}$$

Then:

(*i*) $\Omega_s^{\pm}$ *are bounded symplectic transformations,*

(*ii*) $\Omega_s^{\pm}\mathrm{e}^{-\mathrm{i}tB_\infty} = \mathrm{e}^{-\mathrm{i}tB}\Omega_s^{\pm}$, $t \in \mathbb{R}$,

(*iii*) $\Omega_s^{\pm}$ *are unitary from* $(\mathcal{E}_\infty, h_\infty[\cdot,\cdot])$ *to* $(\mathcal{E}_{\text{scatt}}(B), h[\cdot,\cdot])$.

3.5. **Existence and completeness of long-range wave operators.** We assume now hypotheses (A), (B), i.e. we are in the long-range case. As in the case of Schrödinger operators, it is necessary to introduce a *modified free dynamics* to define the wave operators. We choose to use *time-independent modifiers* analogous to those introduced by Isozaki-Kitada for Schrödinger operators [IK]. It turns out that it is necessary to assume that the long-range potential v_l is of constant sign near infinity. This is not a serious restriction from the point of view of physical applications. Hence we introduce the hypothesis

$$\text{(C)} \quad \pm v_l(x) \geq 0 \text{ for } |x| \gg 1.$$

Let us now define the time-independent modifiers. As in [IK] we construct solutions $\varphi_{\pm}(x,\xi)$ of the eikonal equations:

$$\pm\left(|\partial_x\varphi_{\pm}(x,\xi)|^2 + m^2\right)^{\frac{1}{2}} - v_l(x) = \pm(\xi^2 + m^2)^{\frac{1}{2}},$$

in some outgoing and incoming regions. We denote by $j_{\pm}$ the associated Fourier integral operators defined as:

$$j_{\pm}u(x) = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}\varphi_{\pm}(x,\xi)-\mathrm{i}y\cdot\xi}u(y)\mathrm{d}y\mathrm{d}\xi,$$

which are bounded operators on $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$.

Definition 3.8. *The time-independent modifier T is defined as*

$$T := \pm\frac{1}{2}\begin{pmatrix} j_+ - j_- & -(j_+ + j_-)\epsilon^{-1} \\ -(j_+ + j_-)\epsilon & j_+ - j_- \end{pmatrix},$$

where we use the $\pm$ sign according to the sign of v_l in (C).

Theorem 3.9. *Assume hypotheses* (A), (B) *and* (C). *Then:*
(1) *for all $f \in \mathcal{E}_\infty$ there exist unique $f^{\pm} \in \mathcal{E}_{\text{scatt}}(B)$ such that*

$$\mathrm{e}^{-\mathrm{i}tB}f^{\pm} - T\mathrm{e}^{-\mathrm{i}tB_\infty}f \rightarrow 0, \ t \rightarrow \pm\infty.$$

(2) *Let us define the* long-range wave operators $\Omega_l^{\pm}$*:*

$$\Omega_l^{\pm}: \begin{array}{ll} \mathcal{E}_\infty & \rightarrow \mathcal{E}_{\text{scatt}}(B), \\ f & \mapsto f^{\pm}. \end{array}$$

Then:

(*i*) $\Omega_l^{\pm}$ *are bounded, symplectic transformations,*

(*ii*) $\Omega_l^{\pm}\mathrm{e}^{-\mathrm{i}tB_\infty} = \mathrm{e}^{-\mathrm{i}tB}\Omega_l^{\pm}$, $t \in \mathbb{R}$,

(*iii*) $\Omega_l^{\pm}$ *are unitary from* $(\mathcal{E}_\infty, h_\infty[\cdot,\cdot])$ *to* $(\mathcal{E}_{\text{scatt}}(B), h[\cdot|\cdot])$.

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