数理解析研究所講究録
第 1762 巻 2011 年 56-66

Symmetries of Julia sets of polynomial skew products on $\mathbb{C}^2$

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1 Introduction

Any kind of Julia sets of a polynomial map can have symmetries. We say that a Julia set has symmetries if some transformations preserve it. Beardon [1] investigated the symmetries of the Julia sets of polynomials on $\mathbb{C}$. He considered conformal functions as symmetries. To generalize the results in one-dimension to those in higher dimensions, we [3] previously investigated the symmetries of the Julia sets of nondegenerate polynomial skew products on $\mathbb{C}^2$. We defined the Julia sets as the supports of the Green measures, which are compact, and considered suitable polynomial automorphisms as the symmetries. In this paper, we investigate the symmetries of Julia sets of polynomial skew products on $\mathbb{C}^2$, which generalize some of these previous results in [1] and [3]. We define the Julia sets by the fiberwise Green functions, which are close to the supports of the Green measures. However, the Julia sets may no longer be compact.

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$. More precisely, let $p(z) = a_\delta z^\delta + O(z^{\delta-1})$ and $q(z,w) = q_z(w) = b_d(z)w^d + O_z(w^{d-1})$. We assume that $\delta \geq 2$ and $d \geq 2$. Our results are as follows. First, we define the centroids of f as defined in [1], and show that the symmetries of the Julia set of f are birationally conjugate to rotational products. The tools of the proof are the fiberwise Green and Böttcher functions of f, which also relate to the centroids of f. Next, under some assumptions, we characterize the group of symmetries by functional equations including the iterates of f. The assumptions are, for example, the normality of f and the special form of the polynomial b_d. The normality of f, assuming f is in normal form, means that the centroids are at the origin. Finally, we classify the polynomial skew products whose Julia sets have infinitely many symmetries. Our main result claims that these maps are classified into four types.

This paper is organized into five sections, including this one. In Section 2, we briefly recall the dynamics of polynomials and the relevant results on the symmetries of the Julia sets of polynomials. In Section 3, we recall the dynamics of polynomial skew products. In particular, we review the existence of the fiberwise Green and Böttcher functions, and give the definition of Julia sets. The study of the symmetries of Julia sets begins in Section 4. We show that the symmetries are birationally conjugate to rotational products, and characterize the group of symmetries by functional equations. This section concludes with several examples. These examples include polynomial skew products that are semiconjugate to polynomial products whose Julia sets have infinitely many symmetries. We classify the polynomial skew products whose Julia sets have infinitely many symmetries in Section 5. We have two main theorems for the classification: the case when the map is in normal form and the case when it is not in normal form.

2 Symmetries of Julia sets of polynomials

In this section, we recall the dynamics of polynomials on $\mathbb{C}$ and the relevant results on the symmetries of the Julia sets of polynomials.

Let $p(z) = a_\delta z^\delta + a_{\delta-1}z^{\delta-1} + \cdots + a_0$ be a polynomial of degree $\delta \geq 2$. We denote by p_2p_1 the composition of polynomials p_1 and p_2: $p_2p_1(z) = p_2(p_1(z))$. Let p^n be the n-th iterate of p. A useful tool for the study of the dynamics of p is the Green function of p,

$$G_p(z) = \lim_{n\to\infty} \delta^{-n} \log^+ |p^n(z)|.$$

It is well known that the limit G_p is a nonnegative, continuous and subharmonic function on $\mathbb{C}$. By definition, $G_p(p(z)) = \delta G_p(z)$. Moreover, G_p is harmonic on $\mathbb{C} \setminus K_p$ and zero on K_p, where $K_z = \{z : G_p(z) = 0\}$, and $G_p(z) = \log|z| + \frac{1}{\delta-1}\log|a_\delta| + o(1)$ as $z \to \infty$. This is the Green function for K_p with a pole at infinity, determined only by the the compact set K_p. This function induces the Böttcher function φ_p defined near infinity such that $\varphi_p(z) = z + O(1)$ as $z \to \infty$, $\log|c\varphi_p(z)| = G_p(z)$, where $c = \sqrt[\delta-1]{a_\delta}$, and $\varphi_p(p(z)) = a_\delta(\varphi_p(z))^\delta$.

Let us recall some objects and results of the symmetries of the Julia sets of polynomials on $\mathbb{C}$. For further details, see [1]. We define the Julia set J_p of p as the boundary ∂K_p, and consider conformal functions as the symmetries of J_p. Since J_p is compact, such functions are conformal Euclidean isometries.

Hence the group of the symmetries of J_p is defined by

$$\Sigma_p = \{\sigma \in E : \sigma(J_p) = J_p\},$$

where $E = \{\sigma(z) = c_1 z + c_2 : |c_1| = 1, c_1, c_2 \in \mathbb{C}\}$.

The centroid of p is defined by

$$\zeta = \frac{-a_{\delta-1}}{\delta a_\delta}.$$

If the solutions of $p(z) = Z$ are $z_1, z_2, \cdots, z_\delta$, then $p(z) = a_\delta(z - z_1)(z - z_2)\cdots(z - z_\delta) + Z$ and so the center of gravity of the points z_j coincides with ζ. It is known that each symmetry σ is a rotation about the centroid of p.

Proposition 2.1 ([1, Theorem 5]). *For any symmetry σ in Σ_p, there is μ in the unit circle S^1 such that $\sigma(z) = \mu(z - \zeta) + \zeta$.*

We can characterize Σ_p by the unique equation.

Proposition 2.2 ([1, Lemma 7]). *It follows that $\Sigma_p = \{\sigma \in E : p\sigma = \sigma^\delta p\}$.*

By Proposition 2.1, the group Σ_p is identified with a subgroup of the unit circle S^1. This group is trivial, finite cyclic or infinite. We have a sufficient and necessary condition for Σ_p to be infinite.

Proposition 2.3 ([1, Lemma 4]). *The group Σ_p is infinite if and only if p is affinely conjugate to z^δ, or equivalently, if J_p is a circle. In this case, Σ_p consists of all rotations about ζ.*

We say that p is in normal form if $a_\delta = 1$ and $a_{\delta-1} = 0$, so that the centroid is at the origin. We may assume that p is in normal form without loss of generality because p is conjugate to a polynomial in normal form by the affine function $z \to c(z - \zeta)$, where $c = \sqrt[\delta-1]{a_\delta}$. With this terminology, we can restate Proposition 2.2 as follows.

Proposition 2.4. *Let p be in normal form. Then Σ_p is infinite if and only if $p(z) = z^\delta$, or equivalently, if $J_p = S^1$. In this case, $\Sigma_p \simeq S^1$.*

We can completely determine the group Σ_p even if it is finite.

Proposition 2.5 ([1, Theorem 5]). *Let p be in normal form. Then the order of Σ_p is equal to the largest integer m such that p can be written in the form $p(z) = z^r Q(z^m)$ for some polynomial Q.*

The tools for the proofs of these facts are the Green and Böttcher functions of p. We generalize Propositions 2.1 and 2.2 in Section 4, and Propositions 2.3 and 2.4 in Section 5. We use Proposition 2.5 to prove a lemma in Section 5.

3 Dynamics of polynomial skew products

In this section, we recall the dynamics of polynomial skew products on $\mathbb{C}^2$ and give the definition of Julia sets.

3.1 Polynomial skew products

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$. Let

$$\begin{cases} p(z) = a_\delta z^\delta + a_{\delta-1} z^{\delta-1} + \cdots + a_0, \\ q(z,w) = q_z(w) = b_d(z) w^d + b_{d-1}(z) w^{d-1} + \cdots + b_0(z), \end{cases}$$

and let b_d be a polynomial of degree $l \geq 0$. We assume that $\delta \geq 2$ and $d \geq 2$. As in [3], we say that f is nondegenerate if b_d is a nonzero constant.

Let us briefly recall the dynamics of polynomial skew products. Roughly speaking, the dynamics of f consists of the dynamics on the base space and on the fibers. The first component p defines the dynamics on the base space $\mathbb{C}$. Note that f preserves the set of vertical lines in $\mathbb{C}^2$. In this sense, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to vertical line $\{z\} \times \mathbb{C}$ is viewed as the composition of n polynomials on $\mathbb{C}$, $q_{p^{n-1}(z)} \cdots q_{p(z)} q_z$. Therefore, the n-th iterate of f is written as follows:

$$f^n(z,w) = (p^n(z), Q_z^n(w)),$$

$$\text{where } Q_z^n(w) = q_{p^{n-1}(z)} \cdots q_{p(z)} q_z(w).$$

3.2 Green and Böttcher functions

It is well known that for a polynomial p, the Green function of p is well defined and useful for studying the dynamics of p. In a similar fashion, we define the fiberwise Green function of f as follows:

$$G_z(w) = \lim_{n \to \infty} d^{-n} \log^+ |Q_z^n(w)|.$$

Favre and Guedj [2] showed that the limit G_z defines a local bounded function on $K_p \times \mathbb{C}$ such that $G_{p(z)}(q_z(w)) = dG_z(w)$. In fact, they used the limit $\lim_{n\to\infty} d^{-n} \log \|Q_z^n(w)\|$, where $\|w\| = |w| + 1$, which coincides with G_z on $K_p \times \mathbb{C}$. However, it is not continuous in general. If $b_d^{-1}(0) \cap K_p = \emptyset$, then it is continuous on $K_p \times \mathbb{C}$. To describe G_z more precisely, define

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} \log |b_d(p^n(z))|.$$

It belongs to $L^1(\mu_p)$, where μ_p is the Green measure of p. For fixed z in $K_p \backslash \{\Phi = -\infty\}$, the function G_z is nonnegative, continuous and subharmonic on $\mathbb{C}$. More precisely, it is harmonic on $\mathbb{C} \setminus K_z$ and zero on K_z, where $K_z = \{w : G_z(w) = 0\}$, and $G_z(w) = \log|w| + \Phi(z) + o_z(1)$ as $w \to \infty$. This is the Green function for the compact set K_z with a pole at infinity. We remark that $K_p \setminus \{\Phi = -\infty\}$ is forward invariant under p; that is, $p(K_p \setminus \{\Phi = -\infty\}) \subset K_p \setminus \{\Phi = -\infty\}$.

The fiberwise Green function G_z induces the fiberwise Böttcher function φ_z, which is useful to investigate the symmetries of Julia sets.

Lemma 3.1. *For every z in $K_p \backslash \{\Phi = -\infty\}$, there exists a unique conformal function φ_z defined near infinity such that*

(i) $\varphi_z(w) = w + O_z(1)$ *as* $w \to \infty$,

(ii) $\log|c_z \varphi_z(w)| = G_z(w)$, *where* $c_z = \exp(\Phi(z))$,

(iii) $\varphi_{p(z)}(q_z(w)) = b_d(z)(\varphi_z(w))^d$.

3.3 Julia sets

In this paper, we consider the following Julia set:

$$J_f = \bigcup_{z \in J_p} \{z\} \times \partial K_z.$$

Here we define $\partial K_z = \emptyset$ if $K_z = \mathbb{C}$. We call ∂K_z the fiberwise Julia set. Hence J_f is the union of the fiberwise Julia sets over the base Julia set J_p. It follows that J_f is forward invariant under f; that is, $f(J_f) \subset J_f$. If $b_d^{-1}(0) \cap J_p = \emptyset$, then J_f is completely invariant under f. Moreover, J_f is compact if and only if $b_d^{-1}(0) \cap J_p = \emptyset$.

The following subset of J_p plays an important role in the proofs:

$$J_p^* = J_p \setminus \{\Phi = -\infty\}.$$

Note that J_p^* is dense in J_p because it contains most periodic points. For any z in J_p^*, the limits G_z and φ_z are well defined. In addition, J_p^* is forward invariant under p, and $J_p^* \setminus p(J_p^*) \subset p(b_d^{-1}(0))$.

There is another Julia set of f that might be appropriately called the Julia set of f. Favre and Guedj [2] showed that the closure

$$\overline{\bigcup_{z \in J_p^*} \{z\} \times \partial K_z}$$

coincides with the support of the Green measure of f. Similar to J_f, this Julia set is compact if and only if $b_d^{-1}(0) \cap J_p = \emptyset$.

Remark 3.2. *The same results hold for the symmetries of the last Julia set if $b_d^{-1}(0) \cap J_p = \emptyset$, or if it holds that K_z contains the restriction of the last Julia set to $\{z\} \times \mathbb{C}$ for any periodic point z in J_p^*.*

4 Symmetries of Julia sets

In this section, we consider suitable symmetries of the Julia set of a polynomial skew product f.

As a symmetry, we consider a polynomial automorphism of the form $\gamma(z, w) = (\gamma_1(z), \gamma_2(z, w))$ that preserves J_f. Since γ_1 is conformal, $\gamma_1(z) = c_1 z + c_2$, where c_1 and c_2 are complex numbers. Since J_p is compact, $|c_1| = 1$. Since γ_2 is conformal on each fiber, $\gamma_2(z, w) = c_3 w + c_4(z)$, where c_3 is a complex number and c_4 is a polynomial in z. Since K_z is compact for some z in J_p, it follows that $|c_3| = 1$. Therefore, we define

$$\Gamma_f = \{\gamma \in S : \gamma(J_f) = J_f\},$$

where

$$S = \left\{ \gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} c_1 z + c_2 \\ c_3 w + c_4(z) \end{pmatrix} : |c_1| = |c_3| = 1 \right\}.$$

Let us denote γ in Γ_f by $(\sigma(z), \gamma_z(w))$. Since σ preserves J_p, it follows that σ belongs to Σ_p. By definition, $\gamma_z(\partial K_z) = \partial K_{\sigma(z)}$ and so $\gamma_z(K_z) = K_{\sigma(z)}$ for any z in J_p.

4.1 Centroids

As defined in Section 2, we define the centroids of f as

$$\zeta = \frac{-a_{\delta-1}}{\delta a_\delta} \text{ and } \zeta_z = \frac{-b_{d-1}(z)}{d b_d(z)}.$$

Although ζ is a constant, ζ_z is a rational function in z, If f is nondegenerate, then ζ_z is a polynomial.

The fiberwise Böttcher function φ_z relates to the centroid ζ_z. The following proposition follows from (i) and (iii) in Lemma 3.1.

Lemma 4.1. *It follows that $\varphi_z(w) = w - \zeta_z + o_z(1)$ for any z in J_p^*.*

We first show that a symmetry γ is birationally conjugate to a rotational product, which generalizes Proposition 2.1.

Proposition 4.2. *For any γ in Γ_f, there are μ and ν in S^1 such that*

$$\gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix},$$

where $\sigma(z) = \mu(z-\zeta)+\zeta$ belongs to Σ_p.

Corollary 4.3. *It follows that σ, the first component of γ in Γ_f, preserves the set $\{z \in J_p : \zeta_z = \infty\}$.*

By Proposition 4.2, we can identify Γ_f with a subgroup of the torus:

$$\Gamma_f = \left\{ \gamma_{\mu,\nu} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix} : \gamma_{\mu,\nu}(J_f) = J_f \right\}$$

$$\simeq \{(\mu,\nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma_f\} \subset S^1 \times S^1.$$

Hereafter, we use the notation $=$ instead of $\simeq$. By definition, Γ_f is a subset of $\Sigma_p \times S^1$. More practically, the birational map $(z,w) \to (z-\zeta, w-\zeta_z)$ conjugates the symmetry γ in Γ_f to a rotational product $\tilde{\gamma}(z,w) = (\mu z, \nu w)$.

4.2 Normal form

As in Section 2, we say that f is in normal form if p and b_d are monic and $a_{\delta-1}$ and b_{d-1} are the constant 0. Roughly speaking, we define the normality of f by the normality of p and q_z. Hence if f is in normal form, then the centroids are at the origin.

Unlike the cases of polynomials and nondegenerate polynomial skew products, we may not assume that f is in normal form without loss of generality. However, we can normalize f to a rational map as follows. Define $h(z,w) = (c_1(z-\zeta), c_2(w-\zeta_z))$, where $c_1^{\delta-1}$ is equal to a_δ, the coefficient of the leading term of p, and $c_1^l c_2^{d-1}$ is equal to the coefficient of the leading term of b_d. Then h is a birational map. Let $\tilde{f}$ be the conjugation of f by h: $hf = \tilde{f}h$. The rational map $\tilde{f}$ satisfies all conditions in the definition of normality. Hence we call $\tilde{f}$ the normalized rational skew product of f.

4.3 Functional equations

Under some assumptions, we characterize Γ_f by functional equations, which generalizes Proposition 2.2. Although the group Σ_p of a polynomial p is characterized by the unique equation $p\sigma = \sigma^\delta p$, our characterization of Γ_f needs infinitely many equations as in [3, Lemma 3.2]. Moreover, unlike the nondegenerate case, we need some assumptions for Γ_f to coincide with $\mathcal{F}$, which may be removable.

Let us provide some definitions. We saw in Proposition 4.2 that γ in Γ_f can be written as

$$\gamma\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix}.$$

Thus define $\mathcal{F} = \{\gamma \in S : f^n\gamma = \gamma_n f^n \text{ for } \forall n \geq 1\}$, where

$$\gamma_n\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu^{\delta^n}(z-\zeta)+\zeta \\ \mu^{l_n}\nu^{d^n}(w-\zeta_{p^n(z)})+\zeta_{p^n(\sigma(z))} \end{pmatrix} \text{ and } l_n = \frac{\delta^n - d^n}{\delta - d} l.$$

In addition, let us provide a lemma about certain symmetries of b_d.

Lemma 4.4. *It follows that $|b_d(\sigma(z))| = |b_d(z)|$ for any symmetry σ and for any z in $J_p^* \setminus \{b_d(\sigma(z)) = 0\}$, where σ is the first component of γ in Γ_f.*

We use this lemma to prove the main theorems in the next section. It is natural to ask whether the equation $b_d(\sigma(z)) = \mu^l b_d(z)$, where l is the degree of b_d, holds or not. In the following proposition, we assume some conditions that guarantee this equation.

Proposition 4.5. *If p is in normal form and $b_d(z) = z^l$, then $\Gamma_f \subset \mathcal{F}$. Moreover, σ preserves J_p^*, where σ is the first component of γ in Γ_f.*

With a slight change in the proof, we can replace the assumption in this proposition with the assumption that f is in normal form and q is not divisible by any polynomial in z.

The following corollary of Proposition 4.5 is useful to determine Γ_f for a given map f. In fact, we use this corollary to calculate the groups of symmetries of some examples in Section 4.4 and to prove the main theorems in Sections 5.1 and 5.2.

Corollary 4.6. *If f is in normal form and $b_d(z) = z^l$, then*

$$q(\mu z, \nu w) = \mu^l \nu^d q(z, w)$$

for any $\gamma(z, w) = (\mu z, \nu w)$ in Γ_f.

For the inverse inclusion, we have the following statement.

Proposition 4.7. *If $b_d^{-1}(0) \cap J_p = \emptyset$ or $b_{d-1}(z) \equiv 0$, then $\Gamma_f \supset \mathcal{F}$.*

Combining Propositions 4.5 and 4.7, we get sufficient conditions for Γ_f to coincide with $\mathcal{F}$.

Corollary 4.8. *Assume that f satisfies one of the following conditions: (i) f is in normal form and q is not divisible by any polynomial in z, (ii) f is in normal form and $b_d(z) = z^l$, (iii) $p(z) = z^\delta$ and $b_d(z) = z^l$. Then $\Gamma_f = \mathcal{F}$ and so γ_n belongs to Γ_f for any $n \geq 1$ if γ belongs to Γ_f.*

4.4 Examples

Let us provide some examples of the groups of the symmetries of the Julia sets of polynomial skew products that are not nondegenerate. For a map of these examples, if it is in normal form, then the symmetries have to satisfy the equation in Corollary 4.6. Moreover, we look for the symmetries, i.e., the pairs of the two numbers in the torus, which satisfy the infinitely many equations in Proposition 4.5.

Example 4.9. *Let* $f(z,w) = (z^3, zw^2 + z)$*. Then* $\Gamma_f \simeq \{(\mu,\nu) : \mu^2 = \nu^2 = 1\} = \{(1,1),(-1,-1),(1,-1),(-1,1)\}$*. Moreover, let* $g(z,w) = (z^3, zw^2 + 2z^2w + z)$*. Then it is conjugate to* f *by* $h(z,w) = (z, w - z) : hf = gh$*. Hence* $\Gamma_g = \{(z,w),(-z,-w),(z,-w-2z),(-z,w+2z)\}$*.*

Example 4.10. *Let* $f(z,w) = (z^2, (z-1)w^2)$*. Then* $\Gamma_f \simeq \{1\} \times S^1$*.*

Example 4.11. *Let* $f(z,w) = (z^3, zw^2 + z^3)$*. Then* $\Gamma_f \simeq \{(\mu,\nu) : \mu^2 = \nu^2 \in S^1\}$*. Moreover,* f *is semiconjugate to* $f_0(z,w) = (z^3, w^2 + 1)$ *by* $\pi(z,w) = (z, zw) : \pi f_0 = f\pi$*.*

Example 4.12. *Let* $f(z,w) = (z^2, z^3w^5 + zw^3 + w^2)$*. Then* $\Gamma_f \simeq \{(\mu,\nu) : \mu = \nu^{-1} \in S^1\}$*. Moreover,* f *is semiconjugate to* $f_0(z,w) = (z^2, w^5 + w^3 + w^2)$ *by* $\pi(z,w) = (z, w/z) : \pi f_0 = f\pi$*.*

In particular, the groups of symmetries of Examples 4.10, 4.11 and 4.12 are infinite.

5 Infinite symmetries

In this section, we classify the polynomial skew products whose Julia sets have infinitely many symmetries. We first show that these maps in normal form are classified into four types in Section 5.1. We then remove the assumption of normality and show that the normalized rational skew products of these maps are also classified into four types in Section 5.2.

These maps include polynomial skew products that are semiconjugate to polynomial products such as those given in Examples 4.11 and 4.12. The following lemma gives a sufficient condition of the polynomial map $(z^\delta, q(z,w))$ to be semiconjugate to a polynomial product.

Lemma 5.1. *Let* $q(z,w)$ *be a polynomial. If there exist nonzero integers* s *and* r *and positive integer* δ *such that* $q(z^r, z^s w) = z^{s\delta} q(1,w)$*, then* $(z^\delta, q(z,w))$

is semiconjugate to $(z^\delta, q(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$,

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "{(z^\delta, q(1,w))}"] \arrow[d, "\pi"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\ \mathbb{C}^2 \arrow[r, "{(z^\delta, q(z,w))}"] & \mathbb{C}^2. \end{tikzcd}$$

Remark 5.2. *This lemma holds even if* q *is a rational function; we apply this lemma for the normalized rational skew products in Section 5.2.*

5.1 Classification of the maps in normal form

We first assume that polynomial skew products are in normal form and classify the maps whose Julia sets have infinitely many symmetries.

Theorem 5.3. *Let* f *be in normal form. Then* Γ_f *is infinite if and only if one of the following holds:*

(i) $f(z,w) = (z^\delta, z^l w^d)$,

(ii) $f(z,w) = (z^\delta, q(w))$,

(iii) $f(z,w) = (p(z), b_d(z)w^d)$,

(iv) $f(z,w) = (z^\delta, q(z,w))$ *and it is semiconjugate to* $(z^\delta, q(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$ *for some nonzero coprime integers* r *and* s. *If* $l = 0$, *then* $\delta = d$ *and* $s/r > 0$. *If* $l \neq 0$, *then* $\delta \neq d$ *and* $s/r = l/(\delta - d)$.

To avoid overlap, we assume that $q(w) \neq w^d$ *in (ii),* $p(z) \neq z^\delta$ *or* $b_d(z) \neq z^l$ *in (iii), and* $q(z,w) \neq b_d(z)w^d$ *in (iv).*

In [2, Section 6.2], Favre and Guedj studied the dynamics of polynomial skew products of the form *(iii)*.

5.2 Classification of normalized rational skew products

Now we classify the polynomial skew products whose Julia sets have infinitely many symmetries.

We saw that the birational map h conjugates f to the normalized rational skew product $\tilde{f}$: $hf = \tilde{f}h$. Note that h also conjugates a symmetry γ, which corresponds to μ and ν, to a rotational product $\tilde{\gamma}(z,w) = (\mu z, \nu w)$. Let $\tilde{f}(z,w) = (\tilde{p}(z), \tilde{q}(z,w))$ and let $\tilde{q}(z,w) = \tilde{b}_d(z)w^d + \tilde{b}_{d-1}(z)w^{d-1} + \cdots + \tilde{b}_0(z)$. Then $\tilde{p}$ and $\tilde{b}_d$ are polynomial and $\tilde{b}_{d-1} \equiv 0$.

Theorem 5.4. *Let f be a polynomial skew product whose Julia set has infinitely many symmetries. Then $\tilde{f}$ is one of the following:*

(i) $\tilde{f}(z,w) = (z^\delta, z^l w^d)$,

(ii) $\tilde{f}(z,w) = (z^\delta, \tilde{q}(w))$,

(iii) $\tilde{f}(z,w) = (\tilde{p}(z), \tilde{b}_d(z)w^d)$,

(iv) $\tilde{f}(z,w) = (z^\delta, \tilde{q}(z,w))$ *and it is semiconjugate to* $(z^\delta, \tilde{q}(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$ *for some nonzero coprime integers r and s. If $l = 0$, then $\delta = d$ and $s/r > 0$. If $l \neq 0$, then $\delta \neq d$ and $s/r = l/(\delta - d)$.*

In the cases from (i) to (iii), the maps h and $\tilde{f}$ are polynomial. To avoid overlap, we assume that $\tilde{q}(w) \neq w^d$ in (ii), $\tilde{p}(z) \neq z^\delta$ or $\tilde{b}_d(z) \neq z^l$ in (iii), and $\tilde{q}(z,w) \neq \tilde{b}_d(z)w^d$ in (iv).

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