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# Cooperation principle and density of stable systems in random complex dynamics *

Dedicated to Professor Shigehiro Ushiki on the occasion of his 60th birthday

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**Abstract**

We investigate the random dynamics of rational maps and the dynamics of semigroups of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. We show that regarding random complex dynamics of polynomials, generically, the chaos of the averaged system disappears, due to the cooperation of the generators. We investigate the iteration and spectral properties of transition operators acting on the space of (Hölder) continuous functions on $\hat{\mathbb{C}}$. We also investigate the stability and bifurcation of random complex dynamics. We show that the set of stable systems is open and dense in the space of random dynamics of polynomials. Moreover, we prove that for a stable system, there exist only finitely many minimal sets, each minimal set is attracting, and the orbit of a Hölder continuous function on $\hat{\mathbb{C}}$ under the transition operator tends exponentially fast to the finite-dimensional space $U$ of finite linear combinations of unitary eigenvectors of the transition operator.

## 1 Introduction

This is a research announcement article. Many results of this article has been written in [36].

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of rational semigroups (i.e., semigroups of non-constant rational maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$.

One motivation for research in complex dynamical systems is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described by the dynamical system associated with iteration of a polynomial $f(z) = az(1-z)$ (cf. [8]). However, when there is a change in the natural environment,

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some species have several strategies to survive in nature. From this point of view, it is very natural and important not only to consider the dynamics of iteration, where the same survival strategy (i.e., function) is repeatedly applied, but also to consider random dynamics, where a new strategy might be applied at each time step. Another motivation for research in complex dynamics is Newton's method to find a root of a complex polynomial, which often is expressed as the dynamics of a rational map $g$ on $\hat{\mathbb{C}}$ with $\deg(g) \geq 2$, where $\deg(g)$ denotes the degree of $g$. We sometimes use computers to analyze such dynamics, and since we have some errors at each step of the calculation in the computers, it is quite natural to investigate the random dynamics of rational maps. In various fields, we have many mathematical models which are described by the dynamical systems associated with polynomial or rational maps. For each model, it is natural and important to consider a randomized model, since we always have some kind of noise or random terms in nature. The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([9]). They mainly investigated random dynamics generated by small perturbations of a single rational map. For research on random complex dynamics of quadratic polynomials, see [3, 4, 5, 6, 7, 10]. For research on random dynamics of polynomials or rational maps (of general degrees), see the author's works [30, 29, 31, 32, 33, 35, 34, 36, 37].

In order to investigate random complex dynamics, it is very natural to study the dynamics of associated rational semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of rational semigroups are related to each other very deeply. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([13]), who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([11]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated rational semigroup $G = \langle h_1, \ldots, h_m \rangle$ has "backward self-similarity," i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [22, Lemma 1.1.4]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of rational semigroups, see the author's papers [22]–[38], and [20, 21, 39, 40, 41, 42].

In this paper, by combining several results from [34] and many new ideas, we investigate the random complex dynamics and the dynamics of rational semigroups. In the usual iteration dynamics of a single rational map $g$ with $\deg(g) \geq 2$, we always have a non-empty chaotic part, i.e., in the Julia set $J(g)$ of $g$, which is a perfect set, we have sensitive initial values and dense orbits. Moreover, for any ball $B$ with $B \cap J(g) \neq \emptyset$, $g^n(B)$ expands as $n \to \infty$. Regarding random complex dynamics, it is natural to ask the following question. Do we have a kind of "chaos" in the averaged system? Or do we have no chaos? How do many kinds of maps in the system interact? What can we say about stability and bifurcation? Since the chaotic phenomena hold even for a single rational map, one may expect that in random dynamics of rational maps, most systems would exhibit a great amount of chaos. However, it turns out that this is not true. One of the main purposes of this paper is to prove that for a generic system of random complex dynamics of polynomials, many kinds of maps in the system "automatically" cooperate so that they make the chaos of the averaged system disappear, even though the dynamics of each map in the system have a chaotic part. We call this phenomenon the "**cooperation**

**principle**". Moreover, we prove that for a generic system, we have a kind of stability (see Theorems 1.7, 3.23). We remark that the chaos disappears in the $C^0$ "sense", but under certain conditions, the chaos remains in the $C^\beta$ "sense", where $C^\beta$ denotes the space of $\beta$-Hölder continuous functions with exponent $\beta \in (0,1)$ (see Remark 1.11).

To introduce the main idea of this paper, we let $G$ be a rational semigroup and denote by $F(G)$ the Fatou set of $G$, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where $G$ is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the Julia set of $G$. The Julia set is backward invariant under each element $h \in G$, but might not be forward invariant. This is a difficulty of the theory of rational semigroups. Nevertheless, we utilize this as follows. The key to investigating random complex dynamics is to consider the following **kernel Julia set** of $G$, which is defined by $J_{\mathrm{ker}}(G) = \bigcap_{g \in G} g^{-1}(J(G))$. This is the largest forward invariant subset of $J(G)$ under the action of $G$. Note that if $G$ is a group or if $G$ is a commutative semigroup, then $J_{\mathrm{ker}}(G) = J(G)$. However, for a general rational semigroup $G$ generated by a family of rational maps $h$ with $\deg(h) \geq 2$, it may happen that $\emptyset = J_{\mathrm{ker}}(G) \neq J(G)$.

Let Rat be the space of all non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$, endowed with the distance $\kappa$ which is defined by $\kappa(f,g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where $d$ denotes the spherical distance on $\hat{\mathbb{C}}$. Let $\mathrm{Rat}_+$ be the space of all rational maps $g$ with $\deg(g) \geq 2$. Let $\mathcal{P}$ be the space of all polynomial maps $g$ with $\deg(g) \geq 2$. Let $\tau$ be a Borel probability measure on Rat with compact support. We consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathrm{Rat}$ according to $\tau$. Thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset $A$ of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ of the Markov process is defined as $p(x,A) = \tau(\{g \in \mathrm{Rat} \mid g(x) \in A\})$. Let $G_\tau$ be the rational semigroup generated by the support of $\tau$. Let $C(\hat{\mathbb{C}})$ be the space of all complex-valued continuous functions on $\hat{\mathbb{C}}$ endowed with the supremum norm $\|\cdot\|_\infty$. Let $M_\tau$ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) = \int \varphi(g(z))d\tau(g)$. This $M_\tau$ is called the transition operator of the Markov process induced by $\tau$. For a metric space $X$, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on $X$ endowed with the topology induced by weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if $X$ is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\mathrm{supp}\,\tau$ the topological support of $\tau$. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures $\tau$ on $X$ such that $\mathrm{supp}\,\tau$ is compact. Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of $M_\tau$. This $M_\tau^*$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$ (see Remark 2.14). We define the "Julia set" $J_{meas}(\tau)$ of the dynamics of $M_\tau^*$ as the set of all elements $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that for each neighborhood $B$ of $\mu$, $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is not equicontinuous on $B$ (see Definition 2.11). For each sequence $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$, we denote by $J_\gamma$ the set of non-equicontinuity of the sequence $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n \in \mathbb{N}}$ with respect to the spherical distance on $\hat{\mathbb{C}}$. This $J_\gamma$ is called the Julia set of $\gamma$. Let $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$. For a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we denote by $U_\tau$ the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is equal to one. Moreover, we set $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$. For a metric space $X$, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of $X$ endowed with the Hausdorff metric. For a rational semigroup $G$, we say that a non-empty compact subset $L$ of $\hat{\mathbb{C}}$ is a minimal set for $(G, \hat{\mathbb{C}})$ if $L$ is minimal in $\{C \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid \forall g \in G, g(C) \subset C\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, \hat{\mathbb{C}}) := \{L \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid L \text{ is a minimal set for } (G, \hat{\mathbb{C}})\}$. For a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, let $S_\tau := \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$. For a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, let $\Gamma_\tau := \mathrm{supp}\, \tau (\subset \mathrm{Rat})$. In [34], the following two theorems were obtained.

**Theorem 1.1** (Cooperation Principle I, see Theorem 3.14 in [34]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then $J_{meas}(\tau) = \emptyset$. Moreover, for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, the 2-dimensional Lebesgue measure of $J_\gamma$ is equal to zero.*

**Theorem 1.2** (Cooperation Principle II: Disappearance of Chaos, see Theorem 3.15 in [34]).
*Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then all of the following statements hold.*

(1) *There exists a direct sum decomposition $C(\hat{\mathbb{C}}) = U_\tau \oplus \mathcal{B}_{0,\tau}$. Moreover, $\dim_{\mathbb{C}} U_\tau < \infty$ and $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$. Furthermore, each element of $U_\tau$ is locally constant on $F(G_\tau)$. Therefore each element of $U_\tau$ is a continuous function on $\hat{\mathbb{C}}$ which varies only on the Julia set $J(G_\tau)$.*

(2) *For each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ with the following property. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, there exists a number $\delta = \delta(z, \gamma) > 0$ such that diam$(\gamma_n \cdots \gamma_1(B(z, \delta))) \to 0$ as $n \to \infty$, where diam denotes the diameter with respect to the spherical distance on $\hat{\mathbb{C}}$, and $B(z, \delta)$ denotes the ball with center $z$ and radius $\delta$.*

(3) *We have $1 \leq \sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.*

(4) *For each $z \in \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{C}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{C}_z$, $d(\gamma_n \cdots \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$.*

**Remark 1.3.** If $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ and $\Gamma_\tau \cap \mathrm{Rat}_+ \neq \emptyset$, then $\sharp J(G_\tau) = \infty$.

Theorems 1.1 and 1.2 mean that if all the maps in the support of $\tau$ cooperate, the chaos of the averaged system disappears, even though the dynamics of each map of the system has a chaotic part. Moreover, Theorems 1.1 and 1.2 describe new phenomena which can hold in random complex dynamics but cannot hold in the usual iteration dynamics of a single $h \in \mathrm{Rat}_+$. For example, for any $h \in \mathrm{Rat}_+$, if we take a point $z \in J(h)$, where $J(h)$ denotes the Julia set of the semigroup generated by $h$, then the Dirac measure $\delta_z$ at $z$ belongs to $J_{meas}(\delta_h)$, and for any ball $B$ with $B \cap J(h) \neq \emptyset$, $h^n(B)$ expands as $n \to \infty$. Moreover, for any $h \in \mathrm{Rat}_+$, we have infinitely many minimal sets (periodic cycles) of $h$.

Considering these results, we have the following natural question: "When is the kernel Julia set empty?" In order to give several answers to this question, we say that a family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational (resp. polynomial) maps is a holomorphic family of rational (resp. polynomial) maps if $\Lambda$ is a finite dimensional complex manifold and the map $(z, \lambda) \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. In [34], the following result was proved.

**Theorem 1.4** (Cooperation Principle III, see Theorem 1.7 in [34]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that for each $z \in \mathbb{C}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda \in \Lambda}$ of polynomial maps with $\bigcup_{\lambda \in \Lambda}\{g_\lambda\} \subset \operatorname{supp}\tau$ such that $\lambda \mapsto g_\lambda(z)$ is non-constant on $\Lambda$, then $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$ and all statements in Theorems 1.1 and 1.2 hold.*

In this paper, regarding the previous question, we prove the following very strong results. To state the results, let $\mathcal{Y}$ be a subset of Rat. We say that $\mathcal{Y}$ satisfies condition $(*)$ if $\mathcal{Y}$ is closed in Rat and at least one of the following (1) and (2) holds: (1) for each $(z_0, h_0) \in \hat{\mathbb{C}} \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational maps with $\bigcup_{\lambda \in \Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$, such that, $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of $\lambda_0$. (2) $\mathcal{Y} \subset \mathcal{P}$ and for each $(z_0, h_0) \in \mathbb{C} \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda \in \Lambda}$ of polynomial maps with $\bigcup_{\lambda \in \Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of $\lambda_0$. For example, Rat, $\text{Rat}_+$, $\mathcal{P}$, and $\{z^d + c \mid c \in \mathbb{C}\}$ $(d \in \mathbb{N}, d \geq 2)$ satisfy condition $(*)$. For a subset $\Gamma$ of Rat, we denote by $\langle \Gamma \rangle$ the rational semigroup generated by $\Gamma$. Let $\Gamma \in \text{Cpt(Rat)}$ and let $G = \langle \Gamma \rangle$. We say that $G$ is **mean stable** if there exist non-empty open subsets $U, V$ of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following (I)(II)(III) hold: (I) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$. (II) For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, $\gamma_n \cdots \gamma_1(\overline{U}) \subset V$. (III) For each point $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ such that $g(z) \in U$. Note that this definition does not depend on the choice of a compact set $\Gamma$ which generates $G$. Moreover, for a $\Gamma \in \text{Cpt(Rat)}$, we say that $\Gamma$ is mean stable if $\langle \Gamma \rangle$ is mean stable. Furthermore, for a $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$, we say that $\tau$ is mean stable if $G_\tau$ is mean stable. We remark that if $\Gamma \in \text{Cpt(Rat)}$ is mean stable, then $J_{\ker}(\langle \Gamma \rangle) = \emptyset$. Thus if $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then $J_{\ker}(G_\tau) = \emptyset$ and all statements in Theorems 1.1 and 1.2 hold. Note also that it is not so difficult to see that $\Gamma$ is mean stable if and only if the cardinality of the set of all minimal sets for $(\langle \Gamma \rangle, \hat{\mathbb{C}})$ is finite and each minimal set $L$ is "attracting", i.e., there exists an open subset $W_L$ of $F(\langle \Gamma \rangle)$ with $L \subset W_L$ and an $\epsilon > 0$ such that for each $z \in W_L$ and for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, $d(\gamma_n \cdots \gamma_1(z), L) \to 0$ and $\operatorname{diam}(\gamma_n \cdots \gamma_1(B(z, \epsilon))) \to 0$ as $n \to \infty$ (see Remark 3.6). Thus, the notion "mean stability" of random complex dynamics can be regarded as an analogy of "hyperbolicity" of the usual iteration dynamics of a single rational map. For a metric space $(X, d)$, let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(X)$ such that $\mu_n \to \mu$ in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$ as $n \to \infty$ if and only if (i) $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{C}$, and (ii) $\operatorname{supp}\mu_n \to \operatorname{supp}\mu$ with respect to the Hausdorff metric in the space $\text{Cpt}(X)$. Under these notations, we prove the following theorems.

**Theorem 1.5** (Cooperation Principle IV, Density of Mean Stable Systems, see Theorem 3.19). *Let $\mathcal{Y}$ be a subset of $\mathcal{P}$ satisfying condition $(*)$. Then, we have the following.*

(1) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$ is open and dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. Moreover, the set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid J_{\ker}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ contains $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$.*

(2) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable, $\sharp\Gamma_\tau < \infty\}$ is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.*

We remark that in the study of iteration of a single rational map, we have a very famous conjecture (HD conjecture, see [17, Conjecture 1.1]) which states that hyperbolic rational maps are dense in the space of rational maps. Theorem 1.5 solves this kind of problem in the study of random dynamics of complex polynomials. We also prove the following result.

**Theorem 1.6** (see Corollary 3.22). *Let $\mathcal{Y}$ be a subset of* $\text{Rat}_+$ *satisfying condition* $(*)$. *Then, the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable }\} \cup \{\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \text{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

*is dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

For the proofs of Theorems 1.5 and 1.6, we need to investigate and classify the minimal sets for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$, where $\Gamma \in \text{Cpt(Rat)}$ (Lemmas 3.7,3.15). In particular, it is important to analyze the reason of instability for a non-attracting minimal set.

For each $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ and for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, let $T_{L,\tau}$ be the function of probability of tending to $L$. We set $C(\hat{\mathbb{C}})^* := \{\rho : C(\hat{\mathbb{C}}) \to \mathbb{C} \mid \rho \text{ is linear and continuous}\}$ endowed with the weak*-topology. We prove the following stability result.

**Theorem 1.7** (Cooperation Principle V, $\mathcal{O}$-Stability for Mean Stable Systems, see Theorem 3.23). *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ be mean stable. Suppose $J(G_\tau) \neq \emptyset$. Then there exists a neighborhood $\Omega$ of $\tau$ in $(\mathfrak{M}_{1,c}(\text{Rat}), \mathcal{O})$ such that all of the following hold.*

(1) *For each $\nu \in \Omega$, $\nu$ is mean stable, $\sharp(J(G_\nu)) \geq 3$, and $\sharp\text{Min}(G_\nu, \hat{\mathbb{C}}) = \sharp\text{Min}(G_\tau, \hat{\mathbb{C}})$.*

(2) *For each $\nu \in \Omega$, $\dim_{\mathbb{C}}(U_\nu) = \dim_{\mathbb{C}}(U_\tau)$.*

(3) *The map $\nu \mapsto \pi_\nu$ and $\nu \mapsto U_\nu$ are continuous on $\Omega$, where $\pi_\nu : C(\hat{\mathbb{C}}) \to U_\nu$ denotes the canonical projection (see Theorem 1.2). More precisely, for each $\nu \in \Omega$, there exists a family $\{\varphi_{j,\nu}\}_{j=1}^q$ of unitary eigenvectors of $M_\nu : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where $q = \dim_{\mathbb{C}}(U_\tau)$, and a finite family $\{\rho_{j,\nu}\}_{j=1}^q$ in $C(\hat{\mathbb{C}})^*$ such that all of the following hold.*

    (a) *$\{\varphi_{j,\nu}\}_{j=1}^q$ is a basis of $U_\nu$.*

    (b) *For each $j$, $\nu \mapsto \varphi_{j,\nu} \in C(\hat{\mathbb{C}})$ is continuous on $\Omega$.*

    (c) *For each $j$, $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on $\Omega$.*

    (d) *For each $(i,j)$ and each $\nu \in \Omega$, $\rho_{i,\nu}(\varphi_{j,\nu}) = \delta_{ij}$.*

    (e) *For each $\nu \in \Omega$ and each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{j=1}^q \rho_{j,\nu}(\varphi) \cdot \varphi_{j,\nu}$.*

(4) *For each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, there exists a continuous map $\nu \mapsto L_\nu \in \text{Min}(G_\nu, \hat{\mathbb{C}}) \subset \text{Cpt}(\hat{\mathbb{C}})$ on $\Omega$ with respect to the Hausdorff metric such that $L_\tau = L$. Moreover, for each $\nu \in \Omega$, $\{L_\nu\}_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}})} = \text{Min}(G_\nu, \hat{\mathbb{C}})$. Moreover, for each $\nu \in \Omega$ and for each $L, L' \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, we have $L_\nu \cap L'_\nu = \emptyset$. Furthermore, for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, the map $\nu \mapsto T_{L_\nu,\nu} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is continuous on $\Omega$.*

By applying these results, we give a characterization of mean stability (Theorem 3.24).

We remark that if $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$ is mean stable and $\sharp\text{Min}(G_\tau, \hat{\mathbb{C}}) > 1$, then the averaged system of $\tau$ is stable (Theorem 1.7) and the system also has a kind of variety. Thus such a $\tau$ can describe a stable system which does not lose variety. This fact (with Theorems 1.5, 1.1, 1.2) might be useful when we consider mathematical modeling in various fields.

Let $\mathcal{Y}$ be a subset of Rat satisfying $(*)$. Let $\{\mu_t\}_{t\in[0,1]}$ be a continuous family in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. We consider the bifurcation of $\{M_{\mu_t}\}_{t\in[0,1]}$ and $\{G_{\mu_t}\}_{t\in[0,1]}$. We prove the following result.

**Theorem 1.8** (Bifurcation: see Theorem 3.25 and Lemmas 3.7, 3.15). *Let $\mathcal{Y}$ be a subset of* $\mathrm{Rat}_+$ *satisfying condition* $(*)$. *For each* $t \in [0,1]$, *let* $\mu_t$ *be an element of* $\mathfrak{M}_{1,c}(\mathcal{Y})$. *Suppose that all of the following conditions (1)–(4) hold.*

(1) $t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ *is continuous on* $[0,1]$.

(2) *If* $t_1, t_2 \in [0,1]$ *and* $t_1 < t_2$, *then* $\Gamma_{\mu_{t_1}} \subset \mathrm{int}(\Gamma_{\mu_{t_2}})$ *with respect to the topology of* $\mathcal{Y}$.

(3) $\mathrm{int}(\Gamma_{\mu_0}) \neq \emptyset$ *and* $F(G_{\mu_1}) \neq \emptyset$.

(4) $\sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.

*Let* $B := \{t \in [0,1) \mid \mu_t$ *is not mean stable*$\}$. *Then, we have the following.*

(a) *For each* $t \in [0,1]$, $J_{\mathrm{ker}}(G_{\mu_t}) = \emptyset$ *and* $\sharp J(G_{\mu_t}) \geq 3$, *and all statements in* [34, Theorem 3.15] *(with* $\tau = \mu_t$*) hold.*

(b) *We have* $1 \leq \sharp B \leq \sharp\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}}) - \sharp\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}) < \infty$. *Moreover, for each* $t \in B$, *either* (i) *there exists an element* $L \in \mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, *a point* $z \in L$, *and an element* $g \in \partial\Gamma_{\mu_t} (\subset \mathcal{Y})$ *such that* $z \in L \cap J(G_{\mu_t})$ *and* $g(z) \in L \cap J(G_{\mu_t})$, *or* (ii) *there exist an element* $L \in \mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, *a point* $z \in L$, *and finitely many elements* $g_1, \ldots, g_r \in \partial\Gamma_{\mu_t}$ *such that* $L \subset F(G_{\mu_t})$ *and* $z$ *belongs to a Siegel disk or a Hermann ring of* $g_r \circ \cdots \circ g_1$.

In Example 3.26, an example to which we can apply the above theorem is given.

We also investigate the spectral properties of $M_\tau$ acting on Hölder continuous functions on $\hat{\mathbb{C}}$ and stability (see subsection 3.2). For each $\alpha \in (0,1)$, let $C^\alpha(\hat{\mathbb{C}}) := \{\varphi \in C(\hat{\mathbb{C}}) \mid \sup_{x,y \in \hat{\mathbb{C}}, x \neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha < \infty\}$ be the Banach space of all complex-valued $\alpha$-Hölder continuous functions on $\hat{\mathbb{C}}$ endowed with the $\alpha$-Hölder norm $\|\cdot\|_\alpha$, where $\|\varphi\|_\alpha := \sup_{z \in \hat{\mathbb{C}}} |\varphi(z)| + \sup_{x,y \in \hat{\mathbb{C}}, x \neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha$ for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$.

Regarding the space $U_\tau$, we prove the following.

**Theorem 1.9.** *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. *Suppose that* $J_{\mathrm{ker}}(G_\tau) = \emptyset$ *and* $J(G_\tau) \neq \emptyset$. *Then, there exists an* $\alpha \in (0,1)$ *such that* $U_\tau \subset C^\alpha(\hat{\mathbb{C}})$. *Moreover, for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, *the function* $T_{L,\tau} : \hat{\mathbb{C}} \to [0,1]$ *of probability of tending to* $L$ *belongs to* $C^\alpha(\hat{\mathbb{C}})$.

Thus each element of $U_\tau$ has a kind of regularity. For the proof of Theorem 1.9, the result "each element of $U_\tau$ is locally constant on $F(G_\tau)$" (Theorem 1.2 (1)) is used.

If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then by [34, Proposition 3.65], we have $S_\tau \subset F(G_\tau)$. From this point of view, we consider the situation that $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ satisfies $J_{\mathrm{ker}}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Under this situation, we have several very strong results. Note that there exists an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ such that $J_{\mathrm{ker}}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$, and $\tau$ is not mean stable (see Example 4.3).

**Theorem 1.10** (Cooperation Principle VI, Exponential Rate of Convergence: see Theorem 3.29). *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. *Suppose that* $J_{\mathrm{ker}}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, *and* $S_\tau \subset F(G_\tau)$. *Then, there exists a constant* $\alpha \in (0,1)$, *a constant* $\lambda \in (0,1)$, *and a constant* $C > 0$ *such that for each* $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha$ *for each* $n \in \mathbb{N}$.

For the proof of Theorem 1.10, we need some careful arguments on the hyperbolic metric on each connected component of $F(G_\tau)$.

We remark that in 1983, by numerical experiments, K. Matsumoto and I. Tsuda ([16]) observed that if we add some uniform noise to the dynamical system associated with iteration of a chaotic map on the unit interval $[0,1]$, then under certain conditions, the quantities which represent chaos (e.g., entropy, Lyapunov exponent, etc.) decrease. More precisely, they observed that the entropy decreases and the Lyapunov exponent turns negative. They called this phenomenon "noise-induced order", and many physicists have investigated it by numerical experiments, although there has been only a few mathematical supports for it.

**Remark 1.11.** Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be mean stable and suppose $J(G_\tau) \neq \emptyset$. Then by [34, Theorem 3.15], the chaos of the averaged system of $\tau$ disappears (Cooperation Principle II), and by Theorem 1.10, there exists an $\alpha_0 \in (0,1)$ such that for each $\alpha \in (0,1)$ the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\alpha(\hat{\mathbb{C}})$ is well-behaved. However, [34, Theorem 3.82] tells us that under certain conditions on a mean stable $\tau$, there exists a $\beta \in (0,1)$ such that any non-constant element $\varphi \in U_\tau$ does not belong to $C^\beta(\hat{\mathbb{C}})$ (note: for the proof of this result, we use the Birkhoff ergodic theorem and potential theory). Hence, there exists an element $\psi \in C^\beta(\hat{\mathbb{C}})$ such that $\|M_\tau^n(\psi)\|_\beta \to \infty$ as $n \to \infty$. Therefore, the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\beta(\hat{\mathbb{C}})$ is not well behaved. In other words, regarding the dynamics of the averaged system of $\tau$, there still exists a kind of chaos (or complexity) in the space $(C^\beta(\hat{\mathbb{C}}), \|\cdot\|_\beta)$ even though there exists no chaos in the space $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. From this point of view, in the field of random dynamics, we have a kind of gradation or stratification between chaos and non-chaos. It may be nice to investigate and reconsider the chaos theory and mathematical modeling from this point of view.

We now consider the spectrum $\mathrm{Spec}_\alpha(M_\tau)$ of $M_\tau : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. From Theorem 1.10, denoting by $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ the set of unitary eigenvalues of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ (note: by Theorem 1.9, $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) \subset \mathrm{Spec}_\alpha(M_\tau)$ for some $\alpha \in (0,1)$), we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\mathrm{Spec}_\alpha(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

**Theorem 1.12** (see Theorem 3.30). *Under the assumptions of Theorem 1.10,* $\mathrm{Spec}_\alpha(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, *where* $\lambda \in (0,1)$ *denotes the constant in Theorem 1.10.*

Combining Theorem 1.12 and perturbation theory for linear operators ([15]), we obtain the following theorem. We remark that even if $g_n \to g$ in Rat, for a $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|\varphi \circ g_n - \varphi \circ g\|_\alpha$ does not tend to zero in general. Thus when we perturb generators $\{h_j\}$ of $\Gamma_\tau$, we cannot apply perturbation theory for $M_\tau$ on $C^\alpha(\hat{\mathbb{C}})$. However, for a fixed generator system $(h_1, \dots h_m) \in \mathrm{Rat}^m$, the map $(p_1, \dots, p_m) \in \mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \mapsto M_{\sum_{j=1}^m p_j \delta_{h_j}} \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm. Thus we can apply perturbation theory for the above real-analytic family of operators.

**Theorem 1.13** (see Theorem 3.31). *Let* $m \in \mathbb{N}$ *with* $m \geq 2$. *Let* $h_1, \dots, h_m \in \mathrm{Rat}$. *Let* $G = \langle h_1, \dots, h_m \rangle$. *Suppose that* $J_{\ker}(G) = \emptyset, J(G) \neq \emptyset$ *and* $\cup_{L \in \mathrm{Min}(G, \hat{\mathbb{C}})} L \subset F(G)$. *Let* $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \dots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. *For each* $a = (a_1, \dots, a_m) \in \mathcal{W}_m$, *let* $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. *Then we have all of the following.*

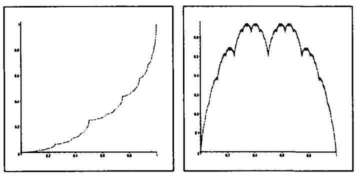
(1) *For each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that $a \mapsto (\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic in an open neighborhood of $b$ in $\mathcal{W}_m$.*

(2) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then, for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that the map $a \mapsto T_{L,\tau_a} \in (C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$ is real-analytic in an open neighborhood of $b$ in $\mathcal{W}_m$. Moreover, the map $a \mapsto T_{L,\tau_a} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is real-analytic in $\mathcal{W}_m$. In particular, for each $z \in \hat{\mathbb{C}}$, the map $a \mapsto T_{L,\tau_a}(z)$ is real-analytic in $\mathcal{W}_m$. Furthermore, for any multi-index $n = (n_1, \dots, n_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}$ and for any $b \in \mathcal{W}_m$, the function $z \mapsto [(\frac{\partial}{\partial a_1})^{n_1} \cdots (\frac{\partial}{\partial a_{m-1}})^{n_{m-1}}(T_{L,\tau_a}(z))]|_{a=b}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$.*

(3) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and let $b \in \mathcal{W}_m$. For each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,b}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ and let $\zeta_{i,b}(z) := T_{L,\tau_b}(h_i(z)) - T_{L,\tau_b}(h_m(z))$. Then, $\psi_{i,b}$ is the unique solution of the functional equation $(I - M_{\tau_b})(\psi) = \zeta_{i,b}, \psi|_{S_{\tau_b}} = 0, \psi \in C(\hat{\mathbb{C}})$, where $I$ denotes the identity map. Moreover, there exists a number $\alpha \in (0,1)$ such that $\psi_{i,b} = \sum_{n=0}^\infty M_{\tau_b}^n(\zeta_{i,b})$ in $(C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$.*

**Remark 1.14.** The function $z \mapsto \psi_{i,b}(z) = [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function $T(x) := \sum_{n=0}^\infty \frac{1}{2^n} \min_{m \in \mathbb{Z}} |2^n x - m|$ where $x \in \mathbb{R}$ (for more details of the Takagi function, see [43]). In order to explain the details, let $g_1(x) := 2x, g_2(x) := 2(x-1)+1$ $(x \in \mathbb{R})$ and let $0 < a < 1$ be a constant. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose the map $g_1$ with probability $a$ and the map $g_2$ with probability $1-a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then, as the author of this paper pointed out in [34], we can see that the function $T_{+\infty,a}|_{[0,1]} : [0,1] \to [0,1]$ is equal to Lebesgue's singular function $L_a$ with respect to the parameter $a$. (For the definition of $L_a$, see [43]. See Figure 1, [34].) It is well-known (see [43, 19]) that for each $x \in [0,1]$, $a \mapsto L_a(x)$ is real-analytic in $(0,1)$, and that $x \mapsto (1/2)[\frac{\partial}{\partial a}(L_a(x))]|_{a=1/2}$ is equal to the Takagi function restricted to $[0,1]$ (Figure 1). From this point of view, the function $z \mapsto \psi_{i,b}(z)$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function. For the figure of the graph of $\psi_{i,b}$, see Example 4.2 and Figure 5.

Figure 1: From left to right, the graphs of Lebesgue's singular function and the Takagi function



We also present a result on the non-differentiability of the function $\psi_{i,b}(z)$ of Theorem 1.13 at points in $J(G_\tau)$ (Theorem 3.39), which is obtained by the application of the Birkhoff ergodic theorem, potential theory and some results from [34].

Combining these results, we can say that for a generic $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, the chaos of the averaged system associated with $\tau$ disappears, $\sharp\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$, each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting, there exists a stability on $U_\tau$ and $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ in a neighborhood of $\tau$ in

$(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$, and there exists an $\alpha \in (0,1)$ such that for each $\varphi \in C^{\alpha}(\hat{\mathbb{C}})$, $M_{\tau}^{n}(\varphi)$ tends to the space $U_{\tau}$ exponentially fast. Note that these phenomena can hold in random complex dynamics but cannot hold in the usual iteration dynamics of a single rational map $h$ with $\deg(h) \geq 2$. We systematically investigate these phenomena and their mechanisms. As the author mentioned in Remark 1.11, these results will stimulate the chaos theory and the mathematical modeling in various fields, and will lead us to a new interesting field. Moreover, these results are related to fractal geometry very deeply.

In section 2, we give some basic notations and definitions. In section 3, we present the main results of this paper. In section 4, we present several examples which describe the main results.

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## 2 Preliminaries

In this section, we give some fundamental notations and definitions.
**Notation:** Let $(X, d)$ be a metric space, $A$ a subset of $X$, and $r > 0$. We set $B(A, r) := \{z \in X \mid d(z, A) < r\}$. Moreover, for a subset $C$ of $\mathbb{C}$, we set $D(C, r) := \{z \in \mathbb{C} \mid \inf_{a \in C} |z - a| < r\}$. Moreover, for any topological space $Y$ and for any subset $A$ of $Y$, we denote by $\text{int}(A)$ the set of all interior points of $A$. Furthermore, we denote by $\text{Con}(A)$ the set of all connected components of $A$.

**Definition 2.1.** Let $Y$ be a metric space. We set $C(Y) := \{\varphi : Y \to \mathbb{C} \mid \varphi \text{ is continuous }\}$. When $Y$ is compact, we endow $C(Y)$ with the supremum norm $\|\cdot\|_{\infty}$. Moreover, for a subset $\mathcal{F}$ of $C(Y)$, we set $\mathcal{F}_{nc} := \{\varphi \in \mathcal{F} \mid \varphi \text{ is not constant}\}$.

**Definition 2.2.** A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([13, 11]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps. We set $\text{Rat} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the distance $\kappa$ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where $d$ denotes the spherical distance on $\hat{\mathbb{C}}$. Moreover, we set $\text{Rat}_{+} := \{h \in \text{Rat} \mid \deg(h) \geq 2\}$ endowed with the relative topology from Rat. Furthermore, we set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

**Remark 2.3** ([1]). For each $d \in \mathbb{N}$, let $\text{Rat}_d := \{g \in \text{Rat} \mid \deg(g) = d\}$ and for each $d \in \mathbb{N}$ with $d \geq 2$, let $\mathcal{P}_d := \{g \in \mathcal{P} \mid \deg(g) = d\}$. Then for each $d$, $\text{Rat}_d$ (resp. $\mathcal{P}_d$) is a connected component of Rat (resp. $\mathcal{P}$). Moreover, $\text{Rat}_d$ (resp. $\mathcal{P}_d$) is open and closed in Rat (resp. $\mathcal{P}$) and is a finite dimensional complex manifold. Furthermore, $h_n \to h$ in $\mathcal{P}$ if and only if $\deg(h_n) = \deg(h)$ for each large $n$ and the coefficients of $h_n$ tend to the coefficients of $h$ appropriately as $n \to \infty$.

**Definition 2.4.** Let $G$ be a rational semigroup. The **Fatou set** of $G$ is defined to be $F(G) :=$
$\{z \in \hat{\mathbb{C}} \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{g|_U : U \to \hat{\mathbb{C}}\}_{g \in G} \text{ is equicontinuous on } U\}$. (For the definition of equicontinuity, see [1].) The **Julia set** of $G$ is defined to be $J(G) := \hat{\mathbb{C}} \setminus F(G)$.

If $G$ is generated by $\{g_i\}_i$, then we write $G = \langle g_1, g_2, \ldots \rangle$. If $G$ is generated by a subset $\Gamma$ of Rat, then we write $G = \langle \Gamma \rangle$. For finitely many elements $g_1, \ldots, g_m \in$ Rat, we set $F(g_1, \ldots, g_m) := F(\langle g_1, \ldots, g_m \rangle)$ and $J(g_1, \ldots, g_m) := J(\langle g_1, \ldots, g_m \rangle)$. For a subset $A$ of $\hat{\mathbb{C}}$, we set $G(A) := \bigcup_{g \in G} g(A)$ and $G^{-1}(A) := \bigcup_{g \in G} g^{-1}(A)$. We set $G^* := G \cup \{\mathrm{Id}\}$, where Id denotes the identity map.

**Lemma 2.5** ([13, 11]). *Let $G$ be a rational semigroup. Then, for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that the equality does not hold in general.*

The following is the key to investigating random complex dynamics.

**Definition 2.6.** Let $G$ be a rational semigroup. We set $J_{\ker}(G) := \bigcap_{g \in G} g^{-1}(J(G))$. This is called the **kernel Julia set** of $G$.

**Remark 2.7.** Let $G$ be a rational semigroup. (1) $J_{\ker}(G)$ is a compact subset of $J(G)$. (2) For each $h \in G$, $h(J_{\ker}(G)) \subset J_{\ker}(G)$. (3) If $G$ is a rational semigroup and if $F(G) \neq \emptyset$, then $\mathrm{int}(J_{\ker}(G)) = \emptyset$. (4) If $G$ is generated by a single map or if $G$ is a group, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup $G$, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$ (see [34]).

It is sometimes important to investigate the dynamics of sequences of maps.

**Definition 2.8.** For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$ and each $m, n \in \mathbb{N}$ with $m \geq n$, we set $\gamma_{m,n} = \gamma_m \circ \cdots \circ \gamma_n$ and we set

$$F_\gamma := \{z \in \hat{\mathbb{C}} \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{\gamma_{n,1}\}_{n \in \mathbb{N}} \text{ is equicontinuous on } U\}$$

and $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$. The set $F_\gamma$ is called the **Fatou set** of the sequence $\gamma$ and the set $J_\gamma$ is called the **Julia set** of the sequence $\gamma$.

**Remark 2.9.** Let $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$. Then by [1, Theorem 2.8.2], $J_\gamma \neq \emptyset$. Moreover, if $\Gamma$ is a non-empty compact subset of $\mathrm{Rat}_+$ and $\gamma \in \Gamma^{\mathbb{N}}$, then by [25], $J_\gamma$ is a perfect set and $J_\gamma$ has uncountably many points.

We now give some notations on random dynamics.

**Definition 2.10.** For a metric space $Y$, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on $Y$ endowed with the topology such that $\mu_n \to \mu$ in $\mathfrak{M}_1(Y)$ if and only if for each bounded continuous function $\varphi : Y \to \mathbb{C}$, $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$. Note that if $Y$ is a compact metric space, then $\mathfrak{M}_1(Y)$ is a compact metric space with the metric $d_0(\mu_1, \mu_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}{1 + |\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}$, where $\{\phi_j\}_{j \in \mathbb{N}}$ is a dense subset of $C(Y)$. Moreover, for each $\tau \in \mathfrak{M}_1(Y)$, we set $\mathrm{supp}\, \tau := \{z \in Y \mid \forall \text{ neighborhood } U \text{ of } z,\ \tau(U) > 0\}$. Note that $\mathrm{supp}\, \tau$ is a closed subset of $Y$. Furthermore, we set $\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \mathrm{supp}\, \tau \text{ is compact}\}$.

For a complex Banach space $\mathcal{B}$, we denote by $\mathcal{B}^*$ the space of all continuous complex linear functionals $\rho : \mathcal{B} \to \mathbb{C}$, endowed with the weak* topology.

For any $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, we will consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $g \in$ Rat according to $\tau$ (thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset $A$ of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ of the Markov process is defined as $p(x, A) = \tau(\{g \in \mathrm{Rat} \mid g(x) \in A\})$).

**Definition 2.11.** Let $\tau \in \mathfrak{M}_1(\text{Rat})$.

1. We set $\Gamma_\tau := \operatorname{supp} \tau$ (thus $\Gamma_\tau$ is a closed subset of Rat). Moreover, we set $X_\tau := (\Gamma_\tau)^{\mathbb{N}}$ ($= \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \gamma_j \in \Gamma_\tau\ (\forall j)\}$) endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is the unique Borel probability measure on $X_\tau$ such that for each cylinder set $A = A_1 \times \cdots \times A_n \times \Gamma_\tau \times \Gamma_\tau \times \cdots$ in $X_\tau$, $\tilde{\tau}(A) = \prod_{j=1}^{n} \tau(A_j)$. We denote by $G_\tau$ the subsemigroup of Rat generated by the subset $\Gamma_\tau$ of Rat.

2. Let $M_\tau$ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) := \int_{\Gamma_\tau} \varphi(g(z))\, d\tau(g)$. $M_\tau$ is called the **transition operator** of the Markov process induced by $\tau$. Moreover, let $M_\tau^* : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$ be the dual of $M_\tau$, which is defined as $M_\tau^*(\mu)(\varphi) = \mu(M_\tau(\varphi))$ for each $\mu \in C(\hat{\mathbb{C}})^*$ and each $\varphi \in C(\hat{\mathbb{C}})$. Remark: we have $M_\tau^*(\mathfrak{M}_1(\hat{\mathbb{C}})) \subset \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and each open subset $V$ of $\hat{\mathbb{C}}$, we have $M_\tau^*(\mu)(V) = \int_{\Gamma_\tau} \mu(g^{-1}(V))\, d\tau(g)$.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that there exists a neighborhood $B$ of $\mu$ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ such that the sequence $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equicontinuous on $B$. We set $J_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$.

**Remark 2.12.** Let $\Gamma$ be a closed subset of Rat. Then there exists a $\tau \in \mathfrak{M}_1(\text{Rat})$ such that $\Gamma_\tau = \Gamma$. By using this fact, we sometimes apply the results on random complex dynamics to the study of the dynamics of rational semigroups.

**Definition 2.13.** Let $Y$ be a compact metric space. Let $\Phi : Y \to \mathfrak{M}_1(Y)$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where $\delta_z$ denotes the Dirac measure at $z$. Using this topological embedding $\Phi : Y \to \mathfrak{M}_1(Y)$, we regard $Y$ as a compact subset of $\mathfrak{M}_1(Y)$.

**Remark 2.14.** If $h \in \text{Rat}$ and $\tau = \delta_h$, then we have $M_\tau^* \circ \Phi = \Phi \circ h$ on $\hat{\mathbb{C}}$. Moreover, for a general $\tau \in \mathfrak{M}_1(\text{Rat})$, $M_\tau^*(\mu) = \int h_*(\mu) d\tau(h)$ for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Therefore, for a general $\tau \in \mathfrak{M}_1(\text{Rat})$, the map $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$.

**Remark 2.15.** If $\tau = \delta_h \in \mathfrak{M}_1(\text{Rat}_+)$ with $h \in \text{Rat}_+$, then $J_{meas}(\tau) \neq \emptyset$. In fact, using the embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we have $\emptyset \neq \Phi(J(h)) \subset J_{meas}(\tau)$.

The following is an important and interesting object in random dynamics.

**Definition 2.16.** Let $A$ be a subset of $\hat{\mathbb{C}}$. Let $\tau \in \mathfrak{M}_1(\text{Rat})$. For each $z \in \hat{\mathbb{C}}$, we set $T_{A,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau \mid d(\gamma_{n,1}(z), A) \to 0 \text{ as } n \to \infty\})$. This is the probability of tending to $A$ starting with the initial value $z \in \hat{\mathbb{C}}$. For any $a \in \hat{\mathbb{C}}$, we set $T_{a,\tau} := T_{\{a\},\tau}$.

**Definition 2.17.** Let $\mathcal{B}$ be a complex vector space and let $M : \mathcal{B} \to \mathcal{B}$ be a linear operator. Let $\varphi \in \mathcal{B}$ and $a \in \mathbb{C}$ be such that $\varphi \neq 0$, $|a| = 1$, and $M(\varphi) = a\varphi$. Then we say that $\varphi$ is a unitary eigenvector of $M$ with respect to $a$, and we say that $a$ is a unitary eigenvalue.

**Definition 2.18.** Let $\tau \in \mathfrak{M}_1(\text{Rat})$. Let $K$ be a non-empty subset of $\hat{\mathbb{C}}$ such that $G_\tau(K) \subset K$. We denote by $\mathcal{U}_{f,\tau}(K)$ the set of all unitary eigenvectors of $M_\tau : C(K) \to C(K)$. Moreover, we denote by $\mathcal{U}_{v,\tau}(K)$ the set of all unitary eigenvalues of $M_\tau : C(K) \to C(K)$.

Similarly, we denote by $\mathcal{U}_{f,\tau,*}(K)$ the set of all unitary eigenvectors of $M_\tau^* : C(K)^* \to C(K)^*$, and we denote by $\mathcal{U}_{v,\tau,*}(K)$ the set of all unitary eigenvalues of $M_\tau^* : C(K)^* \to C(K)^*$.

**Definition 2.19.** Let $V$ be a complex vector space and let $A$ be a subset of $V$. We set $\mathrm{LS}(A) := \{\sum_{j=1}^m a_j v_j \mid a_1, \dots, a_m \in \mathbb{C}, v_1, \dots, v_m \in A, m \in \mathbb{N}\}$.

**Definition 2.20.** Let $Y$ be a topological space and let $V$ be a subset of $Y$. We denote by $C_V(Y)$ the space of all $\varphi \in C(Y)$ such that for each connected component $U$ of $V$, there exists a constant $c_U \in \mathbb{C}$ with $\varphi|_U \equiv c_U$.

**Definition 2.21.** For a topological space $Y$, we denote by $\mathrm{Cpt}(Y)$ the space of all non-empty compact subsets of $Y$. If $Y$ is a metric space, we endow $\mathrm{Cpt}(Y)$ with the Hausdorff metric.

**Definition 2.22.** Let $G$ be a rational semigroup. Let $Y \in \mathrm{Cpt}(\hat{\mathbb{C}})$ be such that $G(Y) \subset Y$. Let $K \in \mathrm{Cpt}(\hat{\mathbb{C}})$. We say that $K$ is a minimal set for $(G, Y)$ if $K$ is minimal among the space $\{L \in \mathrm{Cpt}(Y) \mid G(L) \subset L\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, Y) := \{K \in \mathrm{Cpt}(Y) \mid K \text{ is a minimal set for } (G, Y)\}$.

**Remark 2.23.** Let $G$ be a rational semigroup. By Zorn's lemma, it is easy to see that if $K_1 \in \mathrm{Cpt}(\hat{\mathbb{C}})$ and $G(K_1) \subset K_1$, then there exists a $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $K \subset K_1$. Moreover, it is easy to see that for each $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and each $z \in K$, $\overline{G(z)} = K$. In particular, if $K_1, K_2 \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $K_1 \neq K_2$, then $K_1 \cap K_2 = \emptyset$. Moreover, by the formula $\overline{G(z)} = K$, we obtain that for each $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$, either (1) $\sharp K < \infty$ or (2) $K$ is perfect and $\sharp K > \aleph_0$. Furthermore, it is easy to see that if $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}), G = \langle \Gamma \rangle$, and $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$, then $K = \bigcup_{h \in \Gamma} h(K)$.

**Remark 2.24.** In [34, Remark 3.9], for the statement "for each $K \in \mathrm{Min}(G, Y)$, either (1) $\sharp K < \infty$ or (2) $K$ is perfect", we should assume that each element $g \in G$ is a finite-to-one map.

**Definition 2.25.** For each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we set $S_\tau := \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$.

In [34], the following result was proved by the author of this paper.

**Theorem 2.26** ([34], Cooperation Principle II: Disappearance of Chaos). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, all of the following statements hold.*

1. *Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$. Then, $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$ and there exists a direct sum decomposition $C(\hat{\mathbb{C}}) = \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$. Moreover, $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C_{F(G_\tau)}(\hat{\mathbb{C}})$ and $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) < \infty$.*

2. *$\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.*

3. *Let $W := \bigcup_{A \in \mathrm{Con}(F(G_\tau)), A \cap S_\tau \neq \emptyset} A$. Then $S_\tau$ is compact. Moreover, for each $z \in \hat{\mathbb{C}}$ there exists a Borel measurable subset $\mathcal{C}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ such that for each $\gamma \in \mathcal{C}_z$, there exists an $n \in \mathbb{N}$ with $\gamma_{n,1}(z) \in W$ and $d(\gamma_{m,1}(z), S_\tau) \to 0$ as $m \to \infty$.*

**Definition 2.27.** Under the assumptions of Theorem 2.26, we denote by $\pi_\tau : C(\hat{\mathbb{C}}) \to \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ the projection determined by the direct sum decomposition $C(\hat{\mathbb{C}}) = \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$.

**Remark 2.28.** Under the assumptions of Theorem 2.26, by the theorem, we have that $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\infty \to 0$ as $n \to \infty$, for each $\varphi \in C(\hat{\mathbb{C}})$.

# 3 Results

In this section, we present the main results of this paper.

## 3.1 Stability and bifurcation

In this subsection, we present some results on stability and bifurcation of $M_\tau$ or $M_\tau^*$.

**Definition 3.1.** Let $(X,d)$ be a metric space. Let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(X)$ such that $\mu_n \to \mu$ in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$ as $n \to \infty$ if and only if (1) $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{C}$, and (2) $\operatorname{supp} \mu_n \to \operatorname{supp} \mu$ with respect to the Hausdorff metric in the space $\mathrm{Cpt}(X)$.

**Definition 3.2.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle \Gamma \rangle$. We say that $G$ is **mean stable** if there exist non-empty open subsets $U, V$ of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following hold.

(1) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

(3) For each point $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ such that $g(z) \in U$.

Note that this definition does not depend on the choice of a compact set $\Gamma$ which generates $G$. Moreover, for a $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, we say that $\Gamma$ is mean stable if $\langle \Gamma \rangle$ is mean stable. Furthermore, for a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we say that $\tau$ is mean stable if $G_\tau$ is mean stable.

**Remark 3.3.** If $G$ is mean stable, then $J_{\ker}(G) = \emptyset$.

**Definition 3.4.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ and let $G = \langle \Gamma \rangle$. We say that $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting (for $(G, \hat{\mathbb{C}})$) if there exist non-empty open subsets $U, V$ of $F(G)$ and a number $n \in \mathbb{N}$ such that both of the following hold.

(1) $L \subset V \subset \overline{V} \subset U \subset \overline{U} \subset F(G)$, $\sharp(\hat{\mathbb{C}} \setminus V) \geq 3$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

**Remark 3.5.** For each $h \in G \cap \mathrm{Rat}$,

$$\sharp\{\text{attracting minimal set for } (G, \hat{\mathbb{C}})\} \leq \sharp\{\text{attracting cycles of } h\} < \infty.$$

**Remark 3.6.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle \Gamma \rangle$. Suppose that $\sharp J(G) \geq 3$. Then [34, Theorem 3.15, Remark 3.61, Proposition 3.65] imply that $\Gamma$ is mean stable if and only if $\sharp \mathrm{Min}(G, \hat{\mathbb{C}}) < \infty$ and each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting for $(G, \hat{\mathbb{C}})$. Combining this with Remark 3.5, it follows that $\Gamma$ is mean stable if and only if each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting for $(G, \hat{\mathbb{C}})$.

We now give a classification of minimal sets.

**Lemma 3.7.** *Let* $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ *and let* $G = \langle \Gamma \rangle$. *Let* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. *Then exactly one of the following holds.*

(1) $L$ *is attracting.*

(2) $L \cap J(G) \neq \emptyset$. *Moreover, for each* $z \in L \cap J(G)$, *there exists an element* $g \in \Gamma$ *with* $g(z) \in L \cap J(G)$.

(3) $L \subset F(G)$ *and there exists an element* $g \in G$ *and an element* $U \in \mathrm{Con}(F(G))$ *with* $L \cap U \neq \emptyset$ *such that* $g(U) \subset U$ *and* $U$ *is a subset of a Siegel disk or a Hermann ring of* $g$.

**Definition 3.8.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$.

- We say that $L$ is $J$-touching (for $(G, \hat{\mathbb{C}})$) if $L \cap J(G) \neq \emptyset$.
- We say that $L$ is sub-rotative (for $(G, \hat{\mathbb{C}})$) if (3) in Lemma 3.7 holds.

**Definition 3.9.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$. Suppose $L$ is $J$-touching or sub-rotative. Moreover, suppose $L \neq \hat{\mathbb{C}}$. Let $g \in \Gamma$. We say that $g$ is a bifurcation element for $(\Gamma, L)$ if one of the following statements (1)(2) holds.

(1) $L$ is $J$-touching and there exists a point $z \in L \cap J(\langle \Gamma \rangle)$ such that $g(z) \in J(\langle \Gamma \rangle)$.

(2) There exist an open subset $U$ of $\hat{\mathbb{C}}$ with $U \cap L \neq \emptyset$ and finitely many elements $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ such that $g \circ \gamma_{n-1} \cdots \circ \gamma_1(U) \subset U$ and $U$ is a subset of a Siegel disk or a Hermann ring of $g \circ \gamma_{n-1} \cdots \circ \gamma_1$.

Furthermore, we say that an element $g \in \Gamma$ is a bifurcation element for $\Gamma$ if there exists an $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ such that $g$ is a bifurcation element for $(\Gamma, L)$.

We now consider families of rational maps.

**Definition 3.10.** Let $\Lambda$ be a finite dimensional complex manifold and let $\{g_\lambda\}_{\lambda \in \Lambda}$ be a family of rational maps on $\hat{\mathbb{C}}$. We say that $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of rational maps if the map $(z, \lambda) \in \hat{\mathbb{C}} \times \Lambda \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. We say that $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of polynomials if $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of rational maps and each $g_\lambda$ is a polynomial.

**Definition 3.11.** Let $\mathcal{Y}$ be a subset of Rat and let $U$ be a non-empty open subset of $\hat{\mathbb{C}}$. We say that $\mathcal{Y}$ is strongly $U$-admissible if for each $(z_0, h_0) \in U \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational maps with $\bigcup_{\lambda \in \Lambda} \{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of $\lambda_0$.

**Example 3.12.** $\mathrm{Rat}_+$ is strongly $\hat{\mathbb{C}}$-admissible. $\mathcal{P}$ is strongly $\mathbb{C}$-admissible. Let $f_0 \in \mathcal{P}$. Then $\{f_0 + c \mid c \in \mathbb{C}\}$ is strongly $\mathbb{C}$-admissible.

**Definition 3.13.** Let $\mathcal{Y}$ be a subset of Rat. We say that $\mathcal{Y}$ satisfies condition $(*)$ if $\mathcal{Y}$ is a closed subset of Rat and at least one of the following (1) and (2) holds. (1): $\mathcal{Y}$ is strongly $\hat{\mathbb{C}}$-admissible. (2) $\mathcal{Y} \subset \mathcal{P}$ and $\mathcal{Y}$ is strongly $\mathbb{C}$-admissible.

**Example 3.14.** The sets Rat, $\text{Rat}_+$ and $\mathcal{P}$ satisfy $(*)$. For an $h_0 \in \mathcal{P}$, the set $\{h_0 + c \mid c \in \mathbb{C}\}$ is a subset of $\mathcal{P}$ and satisfies $(*)$.

We now present a result on bifurcation elements.

**Lemma 3.15.** *Let $\mathcal{Y}$ be a subset of* $\text{Rat}_+$ *satisfying condition* $(*)$. *Let* $\Gamma \in \text{Cpt}(\mathcal{Y})$ *and let* $L \in \text{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})$. *Suppose that $L$ is $J$-touching or sub-rotative. Moreover, suppose* $L \neq \hat{\mathbb{C}}$. *Then, there exists a bifurcation element for* $(\Gamma, L)$. *Moreover, each bifurcation element* $g \in \Gamma$ *for* $(\Gamma, L)$ *belongs to* $\partial\Gamma$, *where the boundary* $\partial\Gamma$ *of* $\Gamma$ *is taken in the topological space* $\mathcal{Y}$.

We now present several results on the density of mean stable systems.

**Theorem 3.16.** *Let $\mathcal{Y}$ be a subset of* $\text{Rat}_+$ *satisfying condition* $(*)$. *Let* $\Gamma \in \text{Cpt}(\mathcal{Y})$. *Suppose that there exists an attracting* $L \in \text{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})$. *Let* $\{L_j\}_{j=1}^{r}$ *be the set of attracting minimal sets for* $(\langle\Gamma\rangle, \hat{\mathbb{C}})$ *such that* $L_i \neq L_j$ *if* $i \neq j$ *(Remark: by Remark 3.5, the set of attracting minimal sets is finite). Let $\mathcal{U}$ be a neighborhood of $\Gamma$ in* $\text{Cpt}(\mathcal{Y})$. *For each* $j = 1, \dots, r$, *let $\mathcal{V}_j$ be a neighborhood of $L_j$ with respect to the Hausdorff metric in* $\text{Cpt}(\mathcal{Y})$. *Suppose that* $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *for each* $(i,j)$ *with* $i \neq j$. *Then, there exists an open neighborhood* $\mathcal{U}'$ *of $\Gamma$ in $\mathcal{U}$ such that for any element* $\Gamma' \in \mathcal{U}'$ *satisfying that* $\Gamma \subset \text{int}(\Gamma')$ *with respect to the topology in $\mathcal{Y}$, both of the following statements hold.*

(1) $\langle\Gamma'\rangle$ *is mean stable and*

$$\sharp\text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \sharp\{L' \in \text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) \mid L' \textit{ is attracting for } (\langle\Gamma'\rangle, \hat{\mathbb{C}})\} = r.$$

(2) *For each* $j = 1, \dots, r$, *there exists a unique element* $L'_j \in \text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ *with* $L'_j \in \mathcal{V}_j$. *Moreover,* $L'_j$ *is attracting for* $(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ *for each* $j = 1, \dots, r$.

**Remark 3.17.** Theorem 3.16 (with [34, Theorem 3.15]) generalizes [9, Theorem 0.1].

**Theorem 3.18.** *Let $\mathcal{Y}$ be a subset of* $\text{Rat}_+$ *satisfying condition* $(*)$. *Let* $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$. *Suppose that there exists an attracting* $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$. *Let* $\{L_j\}_{j=1}^{r}$ *be the set of attracting minimal sets for* $(G_\tau, \hat{\mathbb{C}})$ *such that* $L_i \neq L_j$ *if* $i \neq j$. *Let $\mathcal{U}$ be a neighborhood of $\tau$ in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. *For each* $j = 1, \dots, r$, *let $\mathcal{V}_j$ be a neighborhood of $L_j$ with respect to the Hausdorff metric in* $\text{Cpt}(\mathcal{Y})$. *Suppose that* $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *for each* $(i,j)$ *with* $i \neq j$. *Then, there exists an element* $\rho \in \mathcal{U}$ *with* $\sharp\Gamma_\rho < \infty$ *such that all of the following hold.*

(1) $G_\rho$ *is mean stable and*

$$\sharp\text{Min}(G_\rho, \hat{\mathbb{C}}) = \sharp\{L' \in \text{Min}(G_\rho, \hat{\mathbb{C}}) \mid L' \textit{ is attracting for } (\Gamma_\rho, \hat{\mathbb{C}})\} = r.$$

(2) *For each* $j = 1, \dots, r$, *there exists a unique element* $L'_j \in \text{Min}(G_\rho, \hat{\mathbb{C}})$ *with* $L'_j \in \mathcal{V}_j$. *Moreover,* $L'_j$ *is attracting for* $(G_\rho, \hat{\mathbb{C}})$ *for each* $j = 1, \dots, r$.

**Theorem 3.19** (Cooperation Principle IV: Density of Mean Stable Systems)**.** *Let $\mathcal{Y}$ be a subset of $\mathcal{P}$ satisfying condition* $(*)$. *Then, we have the following.*

(1) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ *is mean stable*$\}$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. *Moreover, the set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid J_{\ker}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ *contains* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid$ $\tau$ *is mean stable*$\}$.

(2) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ *is mean stable,* $\sharp\Gamma_\tau < \infty\}$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

**Theorem 3.20.** *Let* $\mathcal{Y}$ *be a subset of* $\mathrm{Rat}_+$ *satisfying condition* $(*)$. *Let* $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$. *Suppose that there exists no attracting minimal set for* $(\langle\Gamma\rangle, \hat{\mathbb{C}})$. *Then we have the following.*

(1) *For any element* $\Gamma' \in \mathrm{Cpt}(\mathrm{Rat})$ *such that* $\Gamma \subset \mathrm{int}(\Gamma')$ *with respect to the topology in* $\mathcal{Y}$, *we have that* $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ *and* $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$.

(2) *For any neighborhood* $\mathcal{U}$ *of* $\Gamma$ *in* $\mathrm{Cpt}(\mathcal{Y})$, *there exists an element* $\Gamma' \in \mathcal{U}$ *with* $\Gamma' \supset \Gamma$ *such that* $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ *and* $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$.

**Corollary 3.21.** *Let* $\mathcal{Y}$ *be a subset of* $\mathrm{Rat}_+$ *satisfying condition* $(*)$. *Let* $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$. *Suppose that there exists no attracting minimal set for* $(G_\tau, \hat{\mathbb{C}})$. *Let* $\mathcal{U}$ *be a neighborhood of* $\tau$ *in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. *Then, there exists an element* $\rho \in \mathcal{U}$ *such that* $\mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ *and* $J(G_\rho) = \hat{\mathbb{C}}$.

**Corollary 3.22.** *Let* $\mathcal{Y}$ *be a subset of* $\mathrm{Rat}_+$ *satisfying condition* $(*)$. *Then, the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable }\} \cup \{\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

*is dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

We now present a result on the stability of mean stable systems.

**Theorem 3.23** (Cooperation Principle V: $\mathcal{O}$-stability of mean stable systems). *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ *be mean stable. Suppose* $J(G_\tau) \neq \emptyset$. *Then there exists a neighborhood* $\Omega$ *of* $\tau$ *in* $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ *such that all of the following statements hold.*

1. *For each* $\nu \in \Omega$, $\nu$ *is mean stable,* $\sharp(J(G_\nu)) \geq 3$, *and* $\sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}}) = \sharp\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$.

2. *For each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, *there exists a continuous map* $\nu \mapsto Q_{L,\nu} \in \mathrm{Cpt}(\hat{\mathbb{C}})$ *on* $\Omega$ *with respect to the Hausdorff metric such that* $Q_{L,\tau} = L$. *Moreover, for each* $\nu \in \Omega$, $\{Q_{L,\nu}\}_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} = \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$. *Moreover, for each* $\nu \in \Omega$ *and for each* $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L \neq L'$, *we have* $Q_{L,\nu} \cap Q_{L',\nu} = \emptyset$.

3. *For each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *and* $\nu \in \Omega$, *let* $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$, $\Lambda_{r_L,\nu} := \{h_{r_L} \circ \cdots \circ h_1 \mid h_j \in \Gamma_\nu (\forall j)\}$, *and* $G_\nu^{r_L} := \langle \Lambda_{r_L,\nu} \rangle$. *Let* $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, L)$ (Remark: *by* [34, Theorem 3.15-12], *we have* $r_L = \sharp\mathrm{Min}(G_\tau^{r_L}, L)$). *Then, for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *and for each* $j = 1, \ldots, r_L$, *there exists a continuous map* $\nu \mapsto L_{j,\nu} \in \mathrm{Cpt}(\hat{\mathbb{C}})$ *with respect to the Hausdorff metric such that, for each* $\nu \in \Omega$, $\{L_{j,\nu}\}_{j=1}^{r_L} = \mathrm{Min}(G_\nu^{r_L}, Q_{L,\nu})$ *and* $L_{i,\nu} \neq L_{j,\nu}$ *whenever* $i \neq j$. *Moreover, for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, *for each* $j = 1, \ldots, r_L$, *and for each* $\nu \in \Omega$, *we have* $L_{j+1,\nu} = \bigcup_{h \in \Gamma_\nu} h(L_{j,\nu})$, *where* $L_{r_L+1,\nu} := L_{1,\nu}$.

4. *For each* $\nu \in \Omega$, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} r_L$. *For each* $\nu \in \Omega$ *and for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, *we have* $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) = r_L$, $\mathcal{U}_{v,\nu}(Q_{L,\nu}) = \{a_L^i\}_{i=1}^{r_L}$, *and* $\mathcal{U}_{v,\nu}(\hat{\mathbb{C}}) = \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \{a_L^i\}_{i=1}^{r_L}$, *where* $a_L := \exp(2\pi i / r_L)$.

5. *The maps $\nu \mapsto \pi_\nu$ and $\nu \mapsto \mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$ are continuous on $\Omega$. More precisely, for each $\nu \in \Omega$, there exists a finite family $\{\varphi_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \dots, r_L\}$ in $\mathcal{U}_{f,\nu}(\hat{\mathbb{C}})$ and a finite family $\{\rho_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \dots, r_L\}$ in $C(\hat{\mathbb{C}})^*$ such that all of the following hold.*

   (a) *$\{\varphi_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \dots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$ and $\{\rho_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \dots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(\hat{\mathbb{C}}))$.*

   (b) *Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $i = 1, \dots, r_L$. Let $\nu \in \Omega$. Then $M_\tau(\varphi_{L,i,\nu}) = a_L^i \varphi_{L,i,\nu}$, $\varphi_{L,i,\nu}|_{Q_{L,\nu}} = (\varphi_{L,1,\nu}|_{Q_{L,\nu}})^i$, $\varphi_{L,i,\nu}|_{Q_{L',\nu}} \equiv 0$ for any $L' \in \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ with $L' \neq L$, and $\mathrm{supp}\, \rho_{L,i,\nu} = Q_{L,\nu}$. Moreover, $\{\varphi_{L,i,\nu}|_{Q_{L,\nu}}\}_{i=1}^{r_L}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))$ and $\{\rho_{L,i,\nu}|_{C(Q_{L,\nu})} \mid i = 1, \dots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(Q_{L,\nu}))$. In particular,*
   $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) = r_L$.

   (c) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \dots r_L$, $\nu \mapsto \varphi_{L,i,\nu} \in C(\hat{\mathbb{C}})$ is continuous on $\Omega$ and $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on $\Omega$.*

   (d) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each $(i,j)$ and each $\nu \in \Omega$, $\rho_{L,i,\nu}(\varphi_{L,j,\nu}) = \delta_{ij}$. Moreover, For each $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, for each $(i,j)$, and for each $\nu \in \Omega$, $\rho_{L,i,\nu}(\varphi_{L',j,\nu}) = 0$.*

   (e) *For each $\nu \in \Omega$ and for each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{i=1}^{r_L} \rho_{L,i,\nu}(\varphi) \cdot \varphi_{L,i,\nu}$.*

6. *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, the map $\nu \mapsto T_{Q_{L,\nu},\nu} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is continuous on $\Omega$.*

We now present a result on a characterization of mean stability.

**Theorem 3.24.** *Let $\mathcal{Y}$ be a subset of $\mathrm{Rat}_+$ satisfying condition $(*)$. We consider the following subsets $A, B, C, D, E$ of $\mathfrak{M}_{1,c}(\mathcal{Y})$ which are defined as follows.*

(1) *$A := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$.*

(2) *Let $B$ be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood $\Omega$ of $\tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on $\Omega$.*

(3) *Let $C$ be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood $\Omega$ of $\tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $F(G_\nu) \neq \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on $\Omega$.*

(4) *Let $D$ be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood $\Omega$ of $\tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$ and $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$.*

(5) *Let $E$ be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that for each $\varphi \in C(\hat{\mathbb{C}})$, there exists a neighborhood $\Omega$ of $\tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$, and* (b) *the map $\nu \mapsto \pi_\nu(\varphi) \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ defined on $\Omega$ is continuous at $\tau$.*

*Then, $A = B = C = D = E$.*

We now present a result on bifurcation of dynamics of $G_\tau$ and $M_\tau$ regarding a continuous family of measures $\tau$.

**Theorem 3.25.** *Let $\mathcal{Y}$ be a subset of* $\text{Rat}_+$ *satisfying condition* $(*)$. *For each $t \in [0,1]$, let $\mu_t$ be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all of the following conditions (1)–(4) hold.*

(1) *$t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ is continuous on $[0,1]$.*

(2) *If $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $\Gamma_{\mu_{t_1}} \subset \text{int}(\Gamma_{\mu_{t_2}})$ with respect to the topology of $\mathcal{Y}$.*

(3) *$\text{int}(\Gamma_{\mu_0}) \neq \emptyset$ and $F(G_{\mu_1}) \neq \emptyset$.*

(4) *$\sharp(\text{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\text{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.*

*Let $B := \{t \in [0,1) \mid$ there exists a bifurcation element $g \in \Gamma_{\mu_t}$ for $\Gamma_{\mu_t}\}$. Then, we have the following.*

(a) *For each $t \in [0,1]$, $J_{\text{ker}}(G_{\mu_t}) = \emptyset$ and $\sharp J(G_{\mu_t}) \geq 3$, and all statements in* [34, Theorem 3.15] *(with $\tau = \mu_t$) hold.*

(b) *We have*
$$1 \leq \sharp B \leq \sharp\text{Min}(G_{\mu_0}, \hat{\mathbb{C}}) - \sharp\text{Min}(G_{\mu_1}, \hat{\mathbb{C}}) < \infty.$$
*Moreover, for each $t \in B$, $\mu_t$ is not mean stable. Furthermore, for each $t \in [0,1) \setminus B$, $\mu_t$ is mean stable.*

**Example 3.26.** Let $c$ be a point in the interior of the Mandelbrot set $\mathcal{M}$. Suppose $z \mapsto z^2 + c$ is hyperbolic. Let $r_0 > 0$ be a small number. Let $r_1 > 0$ be a large number such that $D(c, r_1) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. For each $t \in [0,1]$, let $\mu_t \in \mathfrak{M}_1(\overline{D(c, (1-t)r_0 + tr_1)})$ be the normalized 2-dimensional Lebesgue measure on $\overline{D(c, (1-t)r_0 + tr_1)}$. Then $\{\mu_t\}_{t \in [0,1]}$ satisfies the conditions (1)–(4) in Theorem 3.25 (for example, $2 = \sharp\text{Min}(G_{\mu_0}, \hat{\mathbb{C}}) > \sharp\text{Min}(G_{\mu_1}, \hat{\mathbb{C}}) = 1$). Thus
$$\sharp\{t \in [0,1] \mid \text{there exists a bifurcation element } g \in \Gamma_{\mu_t} \text{ for } \Gamma_{\mu_t}\} = 1.$$

## 3.2 Spectral properties of $M_\tau$ and stability

In this subsection, we present some results on spectral properties of $M_\tau$ acting on the space of Hölder continuous functions on $\hat{\mathbb{C}}$ and the stability.

**Definition 3.27.** Let $K \in \text{Cpt}(\hat{\mathbb{C}})$. For each $\alpha \in (0,1)$, let $C^\alpha(K) := \{\varphi \in C(K) \mid \sup_{x,y \in K, x \neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha < \infty\}$ be the Banach space of all complex-valued $\alpha$-Hölder continuous functions on $K$ endowed with the $\alpha$-Hölder norm $\|\cdot\|_\alpha$, where $\|\varphi\|_\alpha := \sup_{z \in K} |\varphi(z)| + \sup_{x,y \in K, x \neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha$ for each $\varphi \in C^\alpha(K)$.

**Theorem 3.28.** *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$. Suppose that $J_{\text{ker}}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, there exists an $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$, $\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$. Moreover, for each $\alpha \in (0, \alpha_0)$, there exists a constant $E_\alpha > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|\pi_\tau(\varphi)\|_\alpha \leq E_\alpha \|\varphi\|_\infty$. Furthermore, for each $\alpha \in (0, \alpha_0)$ and for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, $T_{L,\tau} \in C^\alpha(\hat{\mathbb{C}})$.*

If $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then by [34, Proposition 3.65], we have $S_\tau \subset F(G_\tau)$ (see Definition 2.25). From this point of view, we consider the situation that $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ satisfies $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Under this situation, we have several very strong results. Note that there exists an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ such that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$, and $\tau$ is not mean stable (see Example 4.3).

**Theorem 3.29** (Cooperation Principle VI: Exponential rate of convergence). *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Let*

$$r := \prod_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}})} \dim_{\mathbb{C}}(\text{LS}(\mathcal{U}_{f,\tau}(L))).$$

*Then, there exists a constant $\alpha \in (0,1)$, a constant $\lambda \in (0,1)$, and a constant $C > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, we have all of the following.*

(1) $\|M_\tau^{nr}(\varphi) - \pi_\tau(\varphi)\|_\alpha \leq C\lambda^n \|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(2) $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(3) $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha$ *for each* $n \in \mathbb{N}$.

(4) $\|\pi_\tau(\varphi)\|_\alpha \leq C\|\varphi\|_\alpha$.

We now consider the spectrum $\text{Spec}_\alpha(M_\tau)$ of $M_\tau : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. By Theorem 3.28, $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) \subset \text{Spec}_\alpha(M_\tau)$ for some $\alpha \in (0,1)$. From Theorem 3.29, we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\text{Spec}_\alpha(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

**Theorem 3.30.** *Under the assumptions of Theorem 3.29, we have all of the following.*

(1) $\text{Spec}_\alpha(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, *where $\lambda \in (0,1)$ denotes the constant in Theorem 3.29.*

(2) *Let $\zeta \in \mathbb{C} \setminus (\{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}}))$. Then, $(\zeta I - M_\tau)^{-1} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$ is equal to*

$$(\zeta I - M_\tau)|_{\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))}^{-1} \circ \pi_\tau + \sum_{n=0}^{\infty} \frac{M_\tau^n}{\zeta^{n+1}}(I - \pi_\tau),$$

*where $I$ denotes the identity on $C^\alpha(\hat{\mathbb{C}})$.*

Combining Theorem 3.30 and perturbation theory for linear operators ([15]), we obtain the following. In particular, as we remarked in Remark 1.14, we obtain complex analogues of the Takagi function.

**Theorem 3.31.** *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h_1, \ldots, h_m \in \text{Rat}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that $J_{\ker}(G) = \emptyset$, $J(G) \neq \emptyset$ and $\bigcup_{L \in \text{Min}(G,\hat{\mathbb{C}})} L \subset F(G)$. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \ldots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \ldots, a_m) \in W_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\text{Rat})$. Then we have all of the following.*

(1) *For each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that $a \mapsto (\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm, is real-analytic in an open neighborhood of $b$ in $\mathcal{W}_m$.*

(2) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then, for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that the map $a \mapsto T_{L,\tau_a} \in (C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$ is real-analytic in an open neighborhood of $b$ in $\mathcal{W}_m$. Moreover, the map $a \mapsto T_{L,\tau_a} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is real-analytic in $\mathcal{W}_m$. In particular, for each $z \in \hat{\mathbb{C}}$, the map $a \mapsto T_{L,\tau_a}(z)$ is real-analytic in $\mathcal{W}_m$. Furthermore, for any multi-index $n = (n_1, \dots, n_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}$ and for any $b \in \mathcal{W}_m$, the function $z \mapsto [(\frac{\partial}{\partial a_1})^{n_1} \cdots (\frac{\partial}{\partial a_{m-1}})^{n_{m-1}} (T_{L,\tau_a}(z))]|_{a=b}$ belongs to $C_{F(G)}(\hat{\mathbb{C}})$.*

(3) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and let $b \in \mathcal{W}_m$. For each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,b}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ and let $\zeta_{i,b}(z) := T_{L,\tau_b}(h_i(z)) - T_{L,\tau_b}(h_m(z))$. Then, $\psi_{i,b}$ is the unique solution of the functional equation $(I - M_{\tau_b})(\psi) = \zeta_{i,b}, \psi|_{S_{\tau_b}} = 0, \psi \in C(\hat{\mathbb{C}})$, where $I$ denotes the identity map. Moreover, there exists a number $\alpha \in (0,1)$ such that $\psi_{i,b} = \sum_{n=0}^{\infty} M_{\tau_b}^n(\zeta_{i,b})$ in $(C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$.*

We now present a result on the non-differentiability of $\psi_{i,b}$ at points in $J(G_\tau)$. In order to do that, we need several definitions and notations.

**Definition 3.32.** For a rational semigroup $G$, we set

$$P(G) := \overline{\bigcup_{g \in G} \{\text{all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$$

where the closure is taken in $\hat{\mathbb{C}}$. This is called the postcritical set of $G$. We say that a rational semigroup $G$ is hyperbolic if $P(G) \subset F(G)$. For a polynomial semigroup $G$, we set $P^*(G) := P(G) \setminus \{\infty\}$. For a polynomial semigroup $G$, we set $\hat{K}(G) := \{z \in \mathbb{C} \mid G(z) \text{ is bounded in } \mathbb{C}\}$. Moreover, for each polynomial $h$, we set $K(h) := \hat{K}(\langle h \rangle)$.

**Remark 3.33.** Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and suppose that $\langle \Gamma \rangle$ is hyperbolic and $J_{\ker}(\langle \Gamma \rangle) = \emptyset$. Then by [34, Propositions 3.63, 3.65], there exists an neighborhood $\mathcal{U}$ of $\Gamma$ in $\mathrm{Cpt}(\mathrm{Rat})$ such that for each $\Gamma' \in \mathcal{U}$, $\Gamma'$ is mean stable, $J_{\ker}(\langle \Gamma' \rangle) = \emptyset$, $J(\langle \Gamma' \rangle) \neq \emptyset$ and $\bigcup_{L \in \mathrm{Min}(\langle \Gamma' \rangle, \hat{\mathbb{C}})} L \subset F(\langle \Gamma' \rangle)$.

**Definition 3.34.** Let $m \in \mathbb{N}$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat})^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \dots, h_m\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the map defined by $f(\gamma, y) = (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \dots) \in \Gamma^{\mathbb{N}}$ and $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map $((\gamma_1, \gamma_2, \dots) \mapsto (\gamma_2, \gamma_3, \dots))$. This map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is called the skew product associated with $\Gamma$. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\hat{\mathbb{C}}} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be an $f$-invariant Borel probability measure. Let $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\}$. For each $p = (p_1, \dots, p_m) \in \mathcal{W}_m$, we define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by $\tilde{p}(\gamma, y) := p_j$ if $\gamma_1 = h_j$ (where $\gamma = (\gamma_1, \gamma_2, \dots)$), and we set

$$u(h, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \tilde{p}(\gamma, y)\ d\mu(\gamma, y))}{\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \|(D\gamma_1)_y\|_s\ d\mu(\gamma, y)}$$

(when the integral of the denominator converges), where $\|\cdot\|_s$ denotes the norm of the derivative with respect to the spherical metric on $\hat{\mathbb{C}}$.

**Definition 3.35.** Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \dots, h_m\}$. For any $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, let $G_\gamma(y) := \lim_{n\to\infty} \frac{1}{\deg(\gamma_{n,1})} \log^+ |\gamma_{n,1}(y)|$, where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [18], for each $\gamma \in \Gamma^{\mathbb{N}}$, $G_\gamma(y)$ exists, $G_\gamma$ is subharmonic on $\mathbb{C}$, and $G_\gamma|_{A_{\infty,\gamma}}$ is equal to the Green's function on $A_{\infty,\gamma}$ with pole at $\infty$, where $A_{\infty,\gamma} := \{z \in \hat{\mathbb{C}} \mid \gamma_{n,1}(z) \to \infty \text{ as } n \to \infty\}$. Moreover, $(\gamma, y) \mapsto G_\gamma(y)$ is continuous on $\Gamma^{\mathbb{N}} \times \mathbb{C}$. Let $\mu_\gamma := dd^c G_\gamma$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Note that by the argument in [14, 18], $\mu_\gamma$ is a Borel probability measure on $J_\gamma$ such that $\operatorname{supp} \mu_\gamma = J_\gamma$. Furthermore, for each $\gamma \in \Gamma^{\mathbb{N}}$, let $\Omega(\gamma) = \sum_c G_\gamma(c)$, where $c$ runs over all critical points of $\gamma_1$ in $\mathbb{C}$, counting multiplicities.

**Remark 3.36.** Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. Let $\Gamma = \{h_1, \dots, h_m\}$ and let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product map associated with $\Gamma$. Moreover, let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma)$. Then, there exists a unique $f$-invariant ergodic Borel probability measure $\mu$ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $\pi_*(\mu) = \tilde{\tau}$ and $h_\mu(f|\sigma) = \max_{\rho \in \mathfrak{E}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}): f_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(f|\sigma) = \sum_{j=1}^m p_j \log(\deg(h_j))$, where $h_\rho(f|\sigma)$ denotes the relative metric entropy of $(f, \rho)$ with respect to $(\sigma, \tilde{\tau})$, and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures (see [24]). This $\mu$ is called the **maximal relative entropy measure** for $f$ with respect to $(\sigma, \tilde{\tau})$.

**Definition 3.37.** Let $V$ be a non-empty open subset of $\hat{\mathbb{C}}$. Let $\varphi : V \to \mathbb{C}$ be a function and let $y \in V$ be a point. Suppose that $\varphi$ is bounded around $y$. Then we set

$$\mathrm{H\ddot{o}l}(\varphi, y) := \inf\{\beta \in \mathbb{R} \mid \limsup_{z \to y} \frac{|\varphi(z) - \varphi(y)|}{d(z, y)^\beta} = \infty\},$$

where $d$ denotes the spherical distance. This is called the **pointwise Hölder exponent of $\varphi$ at $y$.**

**Remark 3.38.** If $\mathrm{H\ddot{o}l}(\varphi, y) < 1$, then $\varphi$ is non-differentiable at $y$. If $\mathrm{H\ddot{o}l}(\varphi, y) > 1$, then $\varphi$ is differentiable at $y$ and the derivative at $y$ is equal to 0.

We now present a result on the non-differentiability of $\psi_{i,b}(z) = [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ at points in $J(G_\tau)$.

**Theorem 3.39.** *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ and we set $\Gamma := \{h_1, h_2, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \dots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \dots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with $\Gamma$. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure for $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Moreover, let $\lambda := (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Suppose that $G$ is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each $(i, j)$ with $i \neq j$. For each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$, for each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,p,L}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=p}$. Then, we have all of the following.*

1. *$G_\tau = G$ is mean stable, $J_{\ker}(G) = \emptyset$, and $S_\tau \subset F(G_\tau)$. Moreover, $0 < \dim_H(J(G)) < 2$, $\operatorname{supp} \lambda = J(G)$, and $\lambda(\{z\}) = 0$ for each $z \in J(G)$.*

2. *Suppose* $\sharp \mathrm{Min}(G, \hat{\mathbb{C}}) \neq 1$. *Then there exists a Borel subset* $A$ *of* $J(G)$ *with* $\lambda(A) = 1$ *such that for each* $z_0 \in A$, *for each* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ *and for each* $i = 1, \ldots, m-1$, *exactly one of the following* (a),(b),(c) *holds.*

   (a) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1) = \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0) < u(h,p,\mu)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

   (b) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0) = u(h,p,\mu) \leq \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

   (c) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1) = u(h,p,\mu) < \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

3. *If* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$, *then*

$$u(h,p,\mu) = \frac{-(\sum_{j=1}^m p_j \log p_j)}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)}$$

   *and*

$$2 > \dim_H(\lambda) = \frac{\sum_{j=1}^m p_j \log \deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)} > 0.$$

4. *Suppose* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$. *Moreover, suppose that at least one of the following* (a), (b), *and* (c) *holds:* (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ *is bounded in* $\mathbb{C}$. (c) $m = 2$. *Then,* $u(h,p,\mu) < 1$.

# 4 Examples

In this section, we give some examples.

**Example 4.1** (Proposition 6.1 in [34]). Let $f_1 \in \mathcal{P}$. Suppose that $\mathrm{int}(K(f_1))$ is not empty. Let $b \in \mathrm{int}(K(f_1))$ be a point. Let $d$ be a positive integer such that $d \geq 2$. Suppose that $(\deg(f_1), d) \neq (2,2)$. Then, there exists a number $c > 0$ such that for each $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, setting $f_\lambda = (f_{\lambda,1}, f_{\lambda,2}) = (f_1, \lambda(z-b)^d + b)$ and $G_\lambda := \langle f_1, f_{\lambda,2} \rangle$, we have all of the following.

*(a)* $f_\lambda$ satisfies the open set condition with an open subset $U_\lambda$ of $\hat{\mathbb{C}}$ (i.e., $f_{\lambda,1}^{-1}(U_\lambda) \cup f_{\lambda,2}^{-1}(U_\lambda) \subset U_\lambda$ and $f_{\lambda,1}^{-1}(U_\lambda) \cap f_{\lambda,2}^{-1}(U_\lambda) = \emptyset$), $f_{\lambda,1}^{-1}(J(G_\lambda)) \cap f_{\lambda,2}^{-1}(J(G_\lambda)) = \emptyset$, $\mathrm{int}(J(G_\lambda)) = \emptyset$, $J_{\mathrm{ker}}(G_\lambda) = \emptyset$, $G_\lambda(K(f_1)) \subset K(f_1) \subset \mathrm{int}(K(f_{\lambda,2}))$ and $\emptyset \neq K(f_1) \subset \hat{K}(G_\lambda)$.

*(b)* If $K(f_1)$ is connected, then $P^*(G_\lambda)$ is bounded in $\mathbb{C}$.

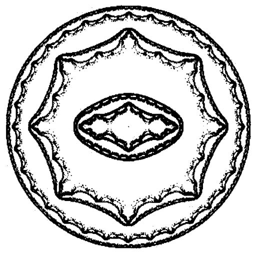
*(c)* If $f_1$ is hyperbolic and $K(f_1)$ is connected, then $G_\lambda$ is hyperbolic, $J(G_\lambda)$ is porous (for the definition of porosity, see [27]), and $\dim_H(J(G_\lambda)) < 2$.

By Example 4.1, Remark 3.33 and [34, Proposition 6.4], we can obtain many examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ to which we can apply Theorems 3.23, 3.24, 3.28, 3.29, 3.30, 3.31, 3.39.

**Example 4.2** (Devil's coliseum ([34]) and complex analogue of the Takagi function). Let $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2$, and $h_2 := g_2^2$. Let $G = \langle h_1, h_2 \rangle$ and for each $a = (a_1, a_2) \in \mathcal{W}_2 := \{(a_1, a_2) \in (0,1)^2 \mid \sum_{j=1}^2 a_j = 1\} \cong (0,1)$, let $\tau_a := \sum_{i=1}^2 a_i \delta_{h_i}$. Then by [34, Example 6.2], setting $A := K(h_2) \setminus D(0, 0.4)$, we have $\overline{D(0, 0.4)} \subset \mathrm{int}(K(h_1))$,

$h_2(K(h_1)) \subset \text{int}(K(h_1))$, $h_1^{-1}(A) \cup h_2^{-1}(A) \subset A$, and $h_1^{-1}(A) \cap h_2^{-1}(A) = \emptyset$. Therefore $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$ and $\emptyset \neq K(h_1) \subset \hat{K}(G)$. Moreover, $G$ is hyperbolic and mean stable, and for each $a \in \mathcal{W}_2$, we obtain that $T_{\infty,\tau_a}$ is continuous on $\hat{\mathbb{C}}$ and the set of varying points of $T_{\infty,\tau_a}$ is equal to $J(G)$. Moreover, by [34] $\dim_H(J(G)) < 2$ and for each non-empty open subset $U$ of $J(G)$ there exists an uncountable dense subset $A_U$ of $U$ such that for each $z \in A_U$, $T_{\infty,\tau_a}$ is not differentiable at $z$. See Figures 2 and 3. The function $T_{\infty,\tau_a}$ is called a devil's coliseum. It is a complex analogue of the devil's staircase. (Remark: as the author of this paper pointed out in [34], the devil's staircase can be regarded as the function of probability of tending to $+\infty$ regarding the random dynamics on $\mathbb{R}$ such that at every step we choose $h_1(x) = 3x$ with probability $1/2$ and we choose $h_2(x) = 3(x-1)+1$ with probability $1/2$. For the detail, see [34].) By Theorem 3.31, for each $z \in \hat{\mathbb{C}}$, $a_1 \mapsto T_{\infty,\tau_a}(z)$ is real-analytic in $(0,1)$, and for each $b \in \mathcal{W}_2$, $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b} = \sum_{n=0}^{\infty} M_{\tau_b}^n(\zeta_{1,b})$, where $\zeta_{1,b}(z) := T_{\infty,\tau_b}(h_1(z)) - T_{\infty,\tau_b}(h_2(z))$. Moreover, by Theorem 3.31, the function $\psi(z) := [\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b}$ defined on $\hat{\mathbb{C}}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$. As mentioned in Remark 1.14, the function $\psi(z)$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function. By Theorem 3.39, there exists an uncountable dense subset $A$ of $J(G)$ such that for each $z \in A$, either $\psi$ is not differentiable at $z$ or $\psi$ is not differentiable at each point $w \in h_1^{-1}(\{z\}) \cup h_2^{-1}(\{z\})$. For the graph of $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a_1=1/2}$, see Figure 5.

Figure 2: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2-1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. $P^*(G)$ is bounded in $\mathbb{C}$ and $\sharp(\text{Con}(J(G))) > \aleph_0$. $G$ is hyperbolic ([33]). $(h_1, h_2)$ satisfies the open set condition ([40]). Moreover, $\forall J \in \text{Con}(J(G)), \exists! \gamma \in \{h_1, h_2\}^{\mathbb{N}}$ s.t. $J = J_\gamma$. For almost every $\gamma \in \{h_1, h_2\}^{\mathbb{N}}$ with respect to a Bernoulli measure, $J_\gamma$ is a simple closed curve but not a quasicircle, and the basin $A_\gamma$ of infinity for the sequence $\gamma$ is a John domain ([33]).



We now give an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ such that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$ and $\tau$ is not mean stable.

**Example 4.3.** Let $h_1 \in \mathcal{P}$ be such that $J(h_1)$ is connected and $h_1$ has a Siegel disk $S$. Let $b \in S$ be a point. Let $d \in \mathbb{N}$ be such that $(\deg(h_1), d) \neq (2,2)$. Then by [34, Proposition 6.1] (or [31, Proposition 2.40]) and its proof, there exists a number $c > 0$ such that for each $\lambda \in \mathbb{C}$ with $0 < |\lambda| < c$, setting $h_2(z) := \lambda(z-b)^d + b$ and $G := \langle h_1, h_2 \rangle$, we have $J_{\ker}(G) = \emptyset$ and $h_2(K(h_1)) \subset S \subset \text{int}(K(h_1)) \subset \text{int}(K(h_2))$. Then the set of minimal sets for $(G, \hat{\mathbb{C}})$ is $\{\{\infty\}, L_0\}$, where $L_0$ is a compact subset of $S$ ($\subset F(G)$). Let $(p_1, p_2) \in \mathcal{W}_2$ be any element and let $\tau := \sum_{j=1}^{2} p_j \delta_{h_j}$. Then $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$ and $\tau$ is not mean stable. In fact, $L_0$ is sub-rotative. Even though $\tau$ is not mean stable, we can apply Theorems 3.28, 3.29, 3.30, 3.31, 3.39 to this $\tau$.

Figure 3: The graph of $z \mapsto T_{\infty,\tau_{1/2}}(z)$, where, letting $(h_1, h_2)$ be the element in Figure 2, we set $\tau_a := \sum_{j=1}^{2} a_j \delta_{h_j}$. A devil's coliseum (a complex analogue of the devil's staircase). $\tau_a$ is mean stable. The set of varying points is equal to Figure 2.

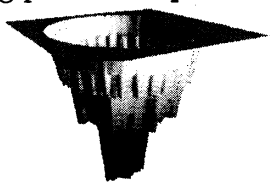


Figure 4: The graph of $z \mapsto T_{0,\tau_{1/2}}(z)$, where, letting $(h_1, h_2)$ be the element in Figure 2, we set $\tau_a := \sum_{j=1}^{2} a_j \delta_{h_j}$. Figure 3 upside down. A "fractal wedding cake".

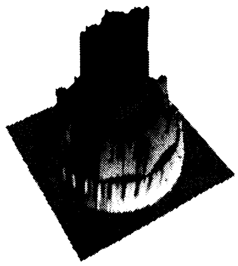
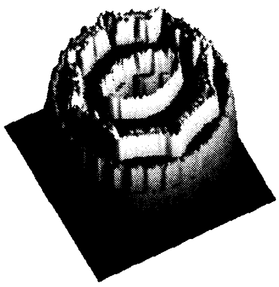


Figure 5: The graph of $z \mapsto [(\partial T_{\infty,\tau_a}(z)/\partial a_1)]|_{a_1=1/2}$, where, $\tau_a$ is the element in Figure 3. A complex analogue of the Takagi function.



**Example 4.4.** By [34, Example 6.7], we have an example $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset$ and such that there exists a $J$-touching minimal set for $(G_\tau, \hat{\mathbb{C}})$. This $\tau$ is not mean stable but we can apply Theorem 3.28 to this $\tau$.

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