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INTRODUCTION TO NON-DEGENERATE MIXED FUNCTIONS

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1. COMPLEX ANAYTIC HYPERSURFACE SINGULARITIES

Contents

1.1. Milnor fibration.

We first recall the theory of Milnor fibrations for holomorphic functions.

Let $f(\mathbf{z})$ be a holomorphic function of n-variables $z_1, \dots, z_n$ such that $f(\mathbf{0}) = 0$. As is well-known, J. Milnor proved that

Theorem 1. ([15]) *There exists a positive number ε_0 such that the argument mapping*

$$\varphi := f/|f| : S_\varepsilon^{2n-1} \setminus K_\varepsilon \to S^1$$

is a locally trivial fibration for any positive ε with $\varepsilon \le \varepsilon_0$ where $K_\varepsilon = f^{-1}(0) \cap S_\varepsilon^{2n-1}$.

We call this **the first description of Milnor fibration**. Topologically $S_\varepsilon \setminus K_\varepsilon \cong F \times I/h : F \to F$. The *characteristic polynomial* is defined by $P_{n-1}(t) = \det(h_* - t\,\mathrm{id})$, where $h_* : H_{n-1}(F) \to H_{n-1}(F)$. h_* is called *monodromy homomorphism.*

Theorem 2. ([15]) *Suppose that ε is sufficiently small as in the above theorem. Then K is $(n-3)$- connected. Suppose further that O is an isolated singularity. Then F has the homotopy type of a bouquet of spheres of S^{n-1}*

Define

$$\mathrm{grad}\, \mathrm{f}(\mathbf{z}) := \left(\frac{\overline{\partial \mathrm{f}}}{\partial \mathrm{z}_1}, \dots, \frac{\overline{\partial \mathrm{f}}}{\partial \mathrm{z}_\mathrm{n}}\right)$$

Euclidean inner product and hermitian inner product:

$$\mathbf{z} = (z_1, \dots, z_n) = \mathbf{x} + i\mathbf{y},\ \mathbf{w} = (w_1, \dots, w_n) = \mathbf{u} + i\mathbf{v}$$
$$(\mathbf{z}, \mathbf{w}) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$
$$(\mathbf{z}, \mathbf{w})_{\mathbb{R}} = (\mathbf{x}, \mathbf{u}) + (\mathbf{y}, \mathbf{v}) = \Re(\mathbf{z}, \mathbf{w})$$
$$\mathbb{C}^n \iff \mathbb{R}^{2n}, \quad \mathbf{z} \iff (\mathbf{x}, \mathbf{y})$$
$$f(\mathbf{z}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$$
$$\mathrm{grad}\, \mathrm{g} = (\frac{\partial \mathrm{g}}{\partial \mathrm{x}_1}, \dots, \frac{\partial \mathrm{g}}{\partial \mathrm{x}_n}, \frac{\partial \mathrm{g}}{\partial \mathrm{y}_1}, \dots, \frac{\partial \mathrm{g}}{\partial \mathrm{y}_n})$$

Tangent spaces:

$$T_{\mathbf{z}}S_\varepsilon = \{\mathbf{w} \in \mathbb{C}^n \mid \Re(\mathbf{z}, \mathbf{w}) = 0\}$$
$$\begin{aligned} T_{\mathbf{z}}f^{-1}(t) &= \{\mathbf{w} \in \mathbb{C}^n \mid (\mathbf{w}, \mathrm{grad}\, \mathrm{f(z)}) = 0\} \\ &= \{\mathbf{w} \in \mathbb{C}^n \mid w_1\frac{\partial f}{\partial z_1} + \cdots + w_n\frac{\partial f}{\partial z_n} = 0\} \\ &= \{\mathbf{w} \in \mathbb{R}^{2n} \mid (\mathbf{w}, \mathrm{grad}\, \mathrm{g}(\mathbf{z}))_{\mathbb{R}} = (\mathbf{w}, \mathrm{grad}\, \mathrm{h}(\mathbf{z}))_{\mathbb{R}} = 0\} \end{aligned}$$
$$T_{\mathbf{z}}F = \{\mathbf{w} \in \mathbb{C}^n \mid \Re(\mathbf{w}, \mathbf{z}) = \Re(\mathbf{w}, i\mathrm{grad}\log(\mathrm{f(z)})) = 0\},\ \mathrm{F} = \varphi^{-1}(1)$$

Note that

$$\log f(\mathbf{z}) = \log |f(\mathbf{z})| + i \arg(f(\mathbf{z})),$$
$$f(\mathbf{z})/|f(\mathbf{z})| = \exp(i \arg(f(\mathbf{z})).$$

Lemma 3. ([15]) $\varphi = f/|f| : S_\varepsilon \to S^1$ *is a submersion if and only if* $\{\mathbf{z}, i\mathrm{grad} \log\, \mathrm{f}(\mathbf{z})\}$ *are linearly independent over* $\mathbb{R}$.

1.1.1. *Cone theorem and the second fibration.* Suppose that $V = f^{-1}(0)$ has an isolated singularity at the origin. Then there exists a positive $\varepsilon > 0$ such that $S_r \pitchfork V$ for any $0 < r \leq \varepsilon$. Note that

$$\begin{aligned} S_r \pitchfork V &\iff T_{\mathbf{z}}S_r \supset T_{\mathbf{z}}V \\ &\iff \forall \mathbf{z} \in S_r \cap V,\ \{\mathbf{z}, \mathrm{grad}\, \mathrm{f}(\mathbf{z})\} : \text{linearly independent over}\ \mathbb{C} \end{aligned}$$

Theorem 4. ([15]) *Assume O is an isolated singularity of V. Then there exists a positive number $r_0 > 0$ so that for any r, $0 < r \leq r_0$, $S_r \pitchfork V$. In particular,* $(B_r, B_r \cap V) \cong \mathrm{Cone}\,(\mathrm{S_r}, \mathrm{K_r})$.

We call such r_0 a *stable radius.*

Proof. Suppose there does not exists such r_0. By Curve Selection lemma, one can find a real analytic path

$$p : [0, \varepsilon] \to V,\ p(t) \in V \setminus \{O\},\ t \neq 0$$

such that $p(t)$ is tangent to V. Thus we can write

$$p(t) = \lambda(t)\mathrm{grad}\, \mathrm{f(p(t)), t > 0}.$$

Then we have a contradiction as follows.

Put $\varphi(\mathbf{z}) = (\mathbf{z}, \mathbf{z}) = \|\mathbf{z}\|^2$ and $\phi = \sqrt{\varphi}$.

$$\begin{aligned} \frac{d}{dt}\varphi(p(t)) &= 2\Re(\frac{dp(t)}{dt}, p(t)) \\ &= 2\Re(\frac{dp(t)}{dt}, \lambda(t)\mathrm{grad}\, \mathrm{f(p(t))}) = 2\Re\bar{\lambda}(\mathrm{t})\frac{\mathrm{df(p(t))}}{\mathrm{dt}} = 0. \end{aligned}$$

For the proof of the second assertion, we construct a vector field χ on $B_r \setminus \{O\}$ such that

For any $\mathbf{z} \in V \setminus \{O\}$, there exists an open neighborhood $U(\mathbf{z})$ such that $\chi(\mathbf{z}) \in T_{\mathbf{z}} f^{-1}(f(\mathbf{z}))$ for any $\mathbf{z} \in U(\mathbf{z})$. More precisely, for any $0 < \varepsilon_1 < r_0$, there exists $\delta(\varepsilon_1) > 0$ such that $f^{-1}(\eta) \pitchfork S_r$ for any $|\eta| \leq \delta(\varepsilon_1)$, $\varepsilon_1 \leq r \leq r_0$. Take $U(\mathbf{z})$ so that

$$U(\mathbf{z}) \subset f^{-1}(D_{\delta(\varepsilon_1)}) \cap B_{r_0} \setminus B^{\circ}_{\varepsilon_1}$$

$\Re(\chi(\mathbf{z}), \operatorname{grad} \phi(\mathbf{z})) = -1$.

Then using the integration $\psi(\mathbf{z}, t)$ ($\psi(\mathbf{z}, 0) = \mathbf{z}$) of χ, we see that $\frac{d\psi(\psi(\mathbf{z},t)}{dt} = -1$ and thus starting from $\mathbf{z} \in S_r$, $\lim_{t \to r} |\psi(\mathbf{z}, t)| = 0$. Using this, we get a homeomorphism:

$$\Psi : (S_r, K_r) \times [0, r) \to B_r \setminus \{O\},\ \Psi(\mathbf{z}, t) = \psi(\mathbf{z}, t)$$

which extends to the homeomorphism $\mathrm{Cone(S_r, K_r)} \cong \mathrm{(B_r, V)}$. □

1.1.2. *Second description of Milnor fibration.* Take r_0 as before. Fix $0 < r \leq r_0$.

Theorem 5. *Fix $r, 0 < r \leq r_0$. Take $\delta > 0$ sufficiently small so that for any η $|\eta| \leq \delta$, $S_r \pitchfork f^{-1}(\eta)$. Put $E(r, \delta)^* = \{\mathbf{z} \in B_r \setminus V \,|\, |f(\mathbf{z})| \leq \delta\}$. Then $f : E(r, \delta)^* \to \Delta^*_\delta$ is a locally trivial fibration, where $\Delta^*_\delta = \{\eta \in \mathbb{C} | 0 < |\eta| \leq \delta\}$.*

Theorem 6. *Two fibrations $\varphi : S_r \setminus K_r \to S^1$ and $f : \partial E(r, \delta)^* \to S^1_\delta$ are equivalent.*

1.2. **Weighted homogeneous polynomials.** Let $a_1, \ldots, a_n$ and c be given positive integers with $\gcd(a_1, \ldots, a_n) = 1$. An analytic function $f(z_1, \ldots, z_n)$ is called *a weighted homogeneous polynomial of type $(a_1, \ldots, a_n; c)$* or *a weighted homogeneous polynomial of degree c with the weight vector $(a_1, \ldots, a_n)$* if f satisfies the functional equality

$$f(t^{a_1} z_1, \ldots, t^{a_n} z_n) = t^c f(z_1, \ldots, z_n), \quad \mathbf{z} \in \mathbb{C}^n,\ t \in \mathbb{C}^*$$

Definition 7. *The $\mathbb{C}^*$-action associated with a weighted homogeneous polynomial:*

$$\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n, \quad (t, \mathbf{z}) \mapsto t \circ \mathbf{z} := (t^{a_1} z_1, \ldots, t^{a_n} z_n))$$

Then weighted homogeneous $\iff$ $f(t \circ \mathbf{z}) = t^c f(\mathbf{z})$.

Note that $V = f^{-1}(0)$ is $\mathbb{C}^*$ stable.

Example 8. *1. Let $f(\mathbf{z})$ be a homogeneous polynomial of degree c. Then $f(\mathbf{z})$ satisfies the obvious equality: $f(tz_1, \cdots, tz_n) = t^c f(z_1, \cdots, z_n)$. Thus $f(\mathbf{z})$ is a weighted homogeneous polynomial of type $(1, \cdots, 1; c)$.*

2. Let $f(\mathbf{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$ (Pham-Brieskorn polynomial). Then it is weighted homogeneous polynomial of type $(p_1, \ldots, p_n, c)$ where $c = \mathrm{lcm(a_1, \ldots, a_n)}$ and $p_j = c/a_j$.

Theorem 9. *Assume that f is a generalized weighted homogeneous polynomial of type $(a_1, \ldots, a_n; c)$. Then*

(1) (Euler equality). *We have the equality:*

$$cf(\mathbf{z}) = \sum_{i=1}^{n} a_i z_i \frac{\partial f}{\partial z_i}.$$

(2) *Assume that $c \neq 0$. The only possible critical value of f is 0 and $f : \mathbb{C}^n - f^{-1}(0) \to \mathbb{C}^*$ is a locally trivial fibration.*

(3) *Assume that* $a_1, \ldots, a_n, c > 0$. *Then the Milnor fibration of* f *at the origin is defined on any sphere* S_ε, $\varepsilon > 0$ *andt he restriction of the above fibration over* S^1: $f : f^{-1}(S^1) \to S^1$ *is equivalent to the Milnor fibration of* f *at the origin:* $f/|f| : S_\varepsilon^{2n-1} - K_\varepsilon \to S^1$ *for any* $\varepsilon > 0$. *In particular, the Milnor fiber is diffeomorphic to the affine hypersurface* $f^{-1}(1)$. *The hypersurface* $f^{-1}(0)$ *is contractible to the origin.*

(4) ([Or-Mil]) *Assume that the origin is an isolated singular point and* $a_1, \ldots, a_n, c > 0$.

(a) *For any* $\varepsilon > 0$, *the sphere* S_ε *and the hypersurface* $f^{-1}(0)$ *intersect transversely.*

(b) *Putting* $c/a_i = u_i/v_i$ *with* $u_i, v_i \in \mathbb{N}$, $(u_i, v_i) = 1$, $i = 1, \ldots, n$, *the divisor of the zeta-function is given as follows.*

$$(\zeta(t)) = (\frac{1}{v_1}\Lambda_{u_1} - 1)\cdots(\frac{1}{v_n}\Lambda_{u_n} - 1)$$

In particular,

$$\mu = \prod_{i=1}^{n}(c/a_i - 1) = \prod_{i=1}^{n}(\frac{u_i}{v_i} - 1).$$

Proof. (4-a): Euler equality is given by differentiating $t^c f(z) = f(t \circ \mathbf{z})$. Assume that

$$\lambda z_i = \overline{\frac{\partial f}{\partial z_i}}, \; i = 1, \ldots, n$$

Then by Euler equality, we have

$$cf(\mathbf{z}) = \sum_{i=1}^{n} a_i z_i \frac{\partial f}{\partial z_i} = \bar{\lambda}\sum_{i=1}^{n} a_i z_i \bar{z}_i \neq 0.$$

□

1.2.1. *Equivalence of global and local fibrations.* $E = f^{-1}(S^1)$ and define

$$\psi : E \to S_r \setminus V, \quad \psi(\mathbf{z}) = \tau \circ \mathbf{z}, \; \|\tau \circ \mathbf{z}\| = r, \; \tau > 0$$

or

$$\xi = \psi^{-1} : S_r \setminus V \to E, \quad \xi(\mathbf{z}) = s(\mathbf{z}) \circ \mathbf{z}, \; s(\mathbf{z}) = |1/|f(\mathbf{z})|^{1/c}$$

Then we have the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & S_r \setminus K \\ \downarrow f & & \downarrow f/|f| \\ S^1 & \xrightarrow{\text{id}} & S^1 \end{array}$$

2. Mixed function

Now we consider the situation of real algebraic variety of codimension 2:

$$V = \{g(\mathbf{x},\mathbf{y}) = h(\mathbf{x},\mathbf{y}) = 0 | \mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n\}$$

where $z_j = x_j + iy_j$ and $g, h \in \mathbb{R}[\mathbf{x},\mathbf{y}]$. **We study when the link is fibered over the circle.** It can be writte as

$$V = \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z},\bar{\mathbf{z}}) = 0\}, \quad f(\mathbf{z},\bar{\mathbf{z}}) = g(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}) + ih(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i})$$

We call f **a mixed polynomial.**

2.1. Weighted homogeneous polynomials.

A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ is called *polar weighted homogeneous* (respectively *radially weighted homogeneous*) if there exist integers positive integers $p_1,\dots,p_n$ and a non-zero integer m_p (resp. positive integers $q_1,\dots,q_n$ and a non-zero integer m_r) such that

$$\gcd(p_1,\dots,p_n) = 1, \ \sum_{j=1}^n p_j(\nu_j - \mu_j) = m_p, \quad \text{if } c_{\nu,\mu} \neq 0$$

$$\text{(resp.} \qquad \gcd(q_1,\dots,q_n) = 1, \ \sum_{j=1}^n q_j(\nu_j + \mu_j) = m_r)$$

We say $f(\mathbf{z},\bar{\mathbf{z}})$ is *a rad-polar weighted homogeneous* if f is radially weighted homogeneous of type, say $(q_1,\dots,q_n;m_r)$, and f is also polar weighted homogeneous of type, say $(p_1,\dots,p_n;m_p)$([23]). We define vectors of rational numbers $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ by $u_i = q_i/m_r$, $v_i = p_i/m_p$ and we call them *the normalized radial (respectively polar) weights.*

Example 10. *Let* $f = z_1^3\bar{z}_1 + z_2^4\bar{z}_2$. *Polar weight:(3,2;6), Radial weight: (5,4;20)*

Using a polar coordinate (r,η) of $\mathbb{C}^*$ where $r > 0$ and $\eta \in S^1$ with $S^1 = \{\eta \in \mathbb{C} \mid |\eta| = 1\}$, we define *a polar* $\mathbb{R}^+ \times S^1$*-action* on $\mathbb{C}^n$ by

$$(r,\eta)\circ\mathbf{z} = (r^{q_1}\eta^{p_1}z_1,\dots,r^{q_n}\eta^{p_n}z_n), \quad re^{i\eta} = (r,\eta) \in \mathbb{R}^+ \times S^1$$

$$(r,\eta)\circ\bar{\mathbf{z}} = \overline{(r,\eta)\circ\mathbf{z}} = (r^{q_1}\eta^{-p_1}\bar{z}_1,\dots,r^{q_n}\eta^{-p_n}\bar{z}_n).$$

Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is rad-polar weighted homogeneous polynomial. Then f satisfies the functional equality

$$f((r,\eta)\circ(\mathbf{z},\bar{\mathbf{z}})) = r^{m_r}\eta^{m_p}f(\mathbf{z},\bar{\mathbf{z}}). \tag{1}$$

This notion was introduced by Ruas-Seade-Verjovsky [27] implicitly and then by Cisneros-Molina [5].

It is easy to see that such a polynomial defines a global fibration

$$f : \mathbb{C}^n - f^{-1}(0) \to \mathbb{C}^*.$$

For example, put $U_{r,\theta} = \{\rho e^{i\eta} \mid 1/r \le \rho \le r, \ -\theta \le \eta \le \theta\}$

$$\Psi : [1/r,r]\times[-\theta,\theta]\times f^{-1}(1) \to f^{-1}(U_{r,\theta}), \quad \Psi(\rho,\theta,\mathbf{z}) = (\rho^{1/m_r}, \exp(i\theta/m_p))\circ\mathbf{z}$$

Theorem 11. $\varphi = f/|f| : S_r^{2n-1}\setminus K_r \to S^1$ *is a locally trivial fibration for any* $r > 0$ *and it is equivalent to* $f : f^{-1}(S^1) \to S^1$.

Proof. First,

$$\psi_\theta : \varphi^{-1}(1) \to \varphi(\exp(i\theta)), \quad \psi_\theta(\mathbf{z}) = (1,\exp(i\theta/m_p))\circ\mathbf{z}$$

gives the trivialization. Define

$$\Phi : S_r^{2n-1}\setminus K_r \to f^{-1}(S^1) \quad \Phi(\mathbf{z}) = (1/|f(\mathbf{z})|^{1/m_r},0)\circ\mathbf{z}$$

□

2.2. **Mixed non-singular point.** Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial and we consider a hypersurface $V = \{\mathbf{z} \in \mathbb{C}^n; f(\mathbf{z},\bar{\mathbf{z}}) = 0\}$. Put $z_j = x_j + iy_j$. Then $f(\mathbf{z},\bar{\mathbf{z}})$ is a real analytic function of $2n$ variables $(\mathbf{x},\mathbf{y})$ with $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$. Put $f(\mathbf{z},\bar{\mathbf{z}}) = g(\mathbf{x},\mathbf{y}) + i\,h(\mathbf{x},\mathbf{y})$ where g, h are real analytic functions. Recall that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Thus for a complex valued function f, we define

$$\frac{\partial f}{\partial z_j} = \frac{\partial g}{\partial z_j} + i\frac{\partial h}{\partial z_j}, \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{\partial g}{\partial \bar{z}_j} + i\frac{\partial g}{\partial \bar{z}_j}$$

We assume that g, h are non-constant polynomials. Then V is a real codimension two subvariety. Put

$$d_{\mathbb{R}}g(\mathbf{x},\mathbf{y}) = \left(\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_n},\frac{\partial g}{\partial y_1},\dots,\frac{\partial g}{\partial y_n}\right) \in \mathbb{R}^{2n}$$
$$d_{\mathbb{R}}h(\mathbf{x},\mathbf{y}) = \left(\frac{\partial h}{\partial x_1},\dots,\frac{\partial h}{\partial x_n},\frac{\partial h}{\partial y_1},\dots,\frac{\partial h}{\partial y_n}\right) \in \mathbb{R}^{2n}$$

For a complex valued mixed polynomial, we use the notation:

$$df(\mathbf{z},\bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right) \in \mathbb{C}^n, \quad \bar{d}f(\mathbf{z},\bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial \bar{z}_1},\dots,\frac{\partial f}{\partial \bar{z}_n}\right) \in \mathbb{C}^n$$

We say that a point $\mathbf{z} \in V$ is **a mixed-singular point** of V if and only if $df_z : T_{\mathbf{z}}|BC^n \to T_{f(\mathbf{z})}\mathbb{C}$ is surjective or equivalently the two vectors $dg_{\mathbb{R}}(\mathbf{x},\mathbf{y})$, $dh_{\mathbb{R}}(\mathbf{x},\mathbf{y})$ are linearly dependent over $\mathbb{R}$.

Proposition 12. [23] *The following two conditions are equivalent.*

(1) $\mathbf{z} \in V$ *is a mixed singular point.*
(2) $dg_{\mathbb{R}}, dh_{\mathbb{R}}$ *are linearly dependent over* $\mathbb{R}$.
(3) *There exists a complex number* α, $|\alpha| = 1$ *such that* $\overline{df(\mathbf{z},\bar{\mathbf{z}})} = \alpha\,\bar{d}f(\mathbf{z},\bar{\mathbf{z}})$.

2.3. **Transversality.** We assume again $f(\mathbf{z},\bar{\mathbf{z}})$ is a rad-polar weighted homogeneous polynomial as before. First we observe that $f^{-1}(t)$ is mixed non-singular for any $t \neq 0$ as $df : T_{\mathbf{z}}f^{-1}(t) \to T_t\mathbb{C}$ is surjective.

Proposition 13. [23] *Let* $V = f^{-1}(0)$. *Assume that the radial weight* $q_j > 0$ *for any* j. *Then* V *is contractible to the origin* O. *If further* O *is an isolated mixed singularity of* V, $V\backslash\{O\}$ *is smooth.*

Proposition 14. [23] *(Transversality) Under the same assumption as in Proposition 13, the sphere* $S_\tau = \{\mathbf{z} \in \mathbb{C}^n; \|\mathbf{z}\| = \tau\}$ *intersects transversely with* V *for any* $\tau > 0$.

2.4. **Two special cases.** There are two cases foe which we know more about the topology of Milnor fiber.

2.4.1. *Case1, Simplicial polynomial.* Let $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{j=1}^{s} c_j\,\mathbf{z}^{\mathbf{n}_j}\bar{\mathbf{z}}^{\mathbf{m}_j}$ be a mixed polynomial. Here we assume that $c_1,\dots,c_s \neq 0$. Put

$$\hat{f}(\mathbf{w}) := \sum_{j=1}^{s} c_j\,\mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}, \quad \mathbf{w} = (w_1,\dots,w_n) \in \mathbb{C}^n.$$

We call $\hat{f}$ the **the associated Laurent polynomial.** This polynomial plays an important role for the determination of the topology of the hypersurface $F = f^{-1}(1)$. Note that

Proposition 15. *If $f(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial of polar weight type $(p_1,\dots,p_n;m_p)$, $\hat{f}(\mathbf{w})$ is also a weighted homogeneous Laurent polynomial of type $(p_1,\dots,p_n;m_p)$ in the complex variables $w_1,\dots,w_n$.*

A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called **simplicial** if the exponent vectors $\{\mathbf{n}_j \pm \mathbf{m}_j \mid j=1,\dots,s\}$ are linearly independent in $\mathbb{Z}^n$ respectively. In particular, simplicity implies that $s \le n$. When $s=n$, we say that f is **full.** Put $\mathbf{n}_j=(n_{j,1},\dots,n_{j,n})$, $\mathbf{m}_j=(m_{j,1},\dots,m_{j,n})$ in $\mathbb{N}^n$. Assume that $s\le n$. Consider two integral matrix $N=(n_{i,j})$ and $M=(m_{i,j})$ where the k-th row vectors are $\mathbf{n}_k$, $\mathbf{m}_k$ respectively.

Lemma 16. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial as above. If $f(\mathbf{z},\bar{\mathbf{z}})$ is simplicial, then $f(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial. In the case $s=n$, $f(\mathbf{z},\bar{\mathbf{z}})$ is simplicial if and only if* $\det(N\pm M)\neq 0$.

2.4.2. *Example.* Let

$$B_{\mathbf{a},\mathbf{b}}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_1^{b_1}+\cdots+z_n^{a_n}\bar{z}_n^{b_n},\ a_i,b_i\ge 1,\ \forall i$$
$$f_{\mathbf{a},\mathbf{b}}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_2^{b_1}+\cdots+z_n^{a_n}\bar{z}_1^{b_n},\ a_i,b_i\ge 1,\ \forall i$$
$$k(\mathbf{z},\bar{\mathbf{z}}) = z_1^{d}(\bar{z}_1+\bar{z}_2)+\cdots+z_n^{d}(\bar{z}_n+\bar{z}_1),\ d\ge 2.$$

The associated Laurent polynomials are

$$\widehat{f_{\mathbf{a},\mathbf{b}}}(\mathbf{w}) = w_1^{a_1}w_2^{-b_1}+\cdots+w_n^{a_n}w_1^{-b_n}$$
$$\hat{k}(\mathbf{w}) = w_1^{d}(1/w_1+1/w_2)+\cdots+w_n^{d}(1/w_n+1/w_1).$$

Corollary 17. *For the polynomial $f_{\mathbf{a},\mathbf{b}}$, the following conditions are equivalent.*

(1) $f_{\mathbf{a},\mathbf{b}}$ *is simplicial.*
(2) $f_{\mathbf{a},\mathbf{b}}$ *is a polar weighted homogeneous polynomial.*
(3) (SC) $a_1\cdots a_n \neq b_1\cdots b_n$.

Let $f(\mathbf{z},\bar{\mathbf{z}})=\sum_{j=1}^{s} c_j\,\mathbf{z}^{\mathbf{n}_j}\bar{\mathbf{z}}^{\mathbf{m}_j}$ be a polar weighted homogeneous polynomial of radial weight type $(q_1,\dots,q_n;m_r)$ and of polar weight type $(p_1,\dots,p_n;m_p)$. Let $F=f^{-1}(1)$ be the fiber.

2.4.3. *Canonical stratification of F and the topology of each stratum.* For any subset $I\subset\{1,2,\dots,n\}$, we define

$$\mathbb{C}^I=\{\mathbf{z}\mid z_j=0,\ j\notin I\},\ \mathbb{C}^{*I}=\{\mathbf{z}\mid z_i\neq 0 \text{ iff } i\in I\},\ \mathbb{C}^{*n}=\mathbb{C}^{*\{1,\dots,n\}}$$

and we define mixed polynomials f^I by the restriction: $f^I=f|_{\mathbb{C}^I}$. For simplicity, we write a point of $\mathbb{C}^I$ as $\mathbf{z}_I$. Put $F^{*I}=\mathbb{C}^{*I}\cap F$. Note that F^{*I} is a non-empty proper subset of $\mathbb{C}^{*I}$ if and only if $f^I(\mathbf{z}_I,\bar{\mathbf{z}}_I)$ is not constantly zero. Now we observe that the hypersurface $F=f^{-1}(1)$ has the canonical stratification

$$F=\amalg_I F^{*I}.$$

Thus it is essential to determine the topology of each stratum F^{*I}. Put $F^*:=F\cap\mathbb{C}^{*n}$, the open dense stratum and put $\hat{F}^*:=\hat{f}^{-1}(1)\cap\mathbb{C}^{*n}$ where $\hat{f}(\mathbf{w})$ is the associated Laurent weighted homogeneous polynomial.

Theorem 18. [23] *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a simplicial polar weighted homogeneous polynomial and let $\hat{f}(\mathbf{w})$ be the associated Laurent weighted homogeneous polynomial. Then there exists a canonical diffeomorphism $\varphi:\mathbb{C}^{*n}\to\mathbb{C}^{*n}$ which gives an* *isomorphism of the two Milnor fibrations defined by $f(\mathbf{z},\bar{\mathbf{z}})$ and $\hat{f}(\mathbf{w})$:*

$$\begin{array}{ccc} \mathbb{C}^{*n} - f^{-1}(0) & \xrightarrow{f} & \mathbb{C}^* \\ \downarrow \varphi & & \downarrow \mathrm{id} \\ \mathbb{C}^{*n} - \hat{f}^{-1}(0) & \xrightarrow{\hat{f}} & \mathbb{C}^* \end{array}$$

and it satisfies $\varphi(F^{*n}) = \hat{F}^{*n}$ and φ is compatible with the respective canonical monodromy maps.

Proof. Assume first that $s = n$ for simplicity. Recall that

$$\hat{f}(\mathbf{w}) = \sum_{j=1}^{n} c_j \mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}.$$

Let $\mathbf{w} = (w_1, \dots, w_n)$ be the complex coordinates of $\mathbb{C}^n$ which is the ambient space of $\hat{F}$. We construct $\varphi : \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ so that $\varphi(\mathbf{z}) = \mathbf{w}$ satisfies

$$\mathbf{w}(\varphi(\mathbf{z}))^{\mathbf{n}_j - \mathbf{m}_j} = \mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j}, \quad \text{thus} \quad \hat{f}(\varphi(\mathbf{z})) = f(\mathbf{z}).$$

For the construction of φ, we use the polar coordinates (ρ_j, θ_j) for $z_j \in \mathbb{C}^*$ and polar coordinates (ξ_j, η_j) for $\mathbf{w}_j$. Thus $\mathbf{z}_j = \rho_j \exp(i\theta_j)$ and $\mathbf{w}_j = \xi_j \exp(i\eta_j)$.

First we take $\eta_j = \theta_j$. Put $\mathbf{n}_j = (n_{j,1}, \dots, n_{j,n})$, $\mathbf{m}_j = (m_{j,1}, \dots, m_{j,n})$ in $\mathbb{N}^n$. Consider two integral matrix $N = (n_{i,j})$ and $M = (m_{i,j})$ where the k-th row vector are $\mathbf{n}_k$, $\mathbf{m}_k$ respectively. Now taking the logarithm of the equality $\mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j} = \mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}$, we get an equivalent equality:

$$\begin{aligned} &(n_{j1} + m_{j1}) \log \rho_1 + \cdots + (n_{jn} + m_{jn}) \log \rho_n \\ &\quad = (n_{j1} - m_{j1}) \log \xi_1 + \cdots + (n_{jn} - m_{jn}) \log \xi_n \end{aligned}$$

for $j = 1, \dots, n$. This can be written as

$$(2) \qquad (N + M) \begin{pmatrix} \log \rho_1 \\ \vdots \\ \log \rho_n \end{pmatrix} = (N - M) \begin{pmatrix} \log \xi_1 \\ \vdots \\ \log \xi_n \end{pmatrix}$$

Put $(N - M)^{-1}(N + M) = (\lambda_{ij}) \in \mathrm{GL}(\mathrm{n}, \mathbb{Q})$. Now we define φ as follows.

$$\begin{aligned} \varphi : \mathbb{C}^{*n} \to \mathbb{C}^{*n}, \ \mathbf{z} = (\rho_1 \exp(i\theta_1), \dots, \rho_n \exp(i\theta_n)) \mapsto \\ \mathbf{w} = (\xi_1 \exp(i\theta_1), \dots, \xi_n \exp(i\theta_n)) \end{aligned}$$

where ξ_j is given by $\xi_j = \exp(\sum_{i=1}^{n} \lambda_{ji} \log \rho_i)$ for $j = 1, \dots, n$. It is obvious that φ is a real analytic isomorphism of $\mathbb{C}^{*n}$ to $\mathbb{C}^{*n}$. Let us consider the Milnor fibrations of $f(\mathbf{z}, \bar{\mathbf{z}})$ and $\hat{f}(\mathbf{w})$ in the respective ambient tori $\mathbb{C}^{*n}$.

$$f : \mathbb{C}^{*n} \setminus f^{-1}(0) \to \mathbb{C}^*, \quad \hat{f} : \mathbb{C}^{*n} \setminus \hat{f}^{-1}(0) \to \mathbb{C}^*$$

Recall that the monodromy maps h^*, $\hat{h}^*$ are given as

$$\begin{aligned} h^* &: F^* \to F^*, \quad \mathbf{z} \mapsto \exp(2\pi i/m_p) \circ \mathbf{z} \\ \hat{h}^* &: \hat{F}^* \to \hat{F}^*, \quad \mathbf{w} \mapsto \exp(2\pi i/m_p) \circ \mathbf{w}. \end{aligned}$$

Recall that the $\mathbb{C}^*$-action associated with $\hat{f}(\mathbf{w})$ is the polar action of $f(\mathbf{z}, \bar{\mathbf{z}})$. Namely $\exp i\theta \circ \mathbf{w} = (\exp(ip_1\theta)w_1, \dots, \exp(ip_n\theta)w_n)$. Thus we have the commutative diagram:

$$\begin{array}{ccc} F_\alpha^* & \xrightarrow{h^*} & F_\alpha^* \\ \downarrow{\varphi} & & \downarrow{\varphi} \\ \hat{F}_\alpha^* & \xrightarrow{\hat{h}^*} & \hat{F}_\alpha^* \end{array}$$

where $F_\alpha^* = f^{-1}(\alpha) \cap \mathbb{C}^{*n}$ and $\hat{F}_\alpha^* = \hat{f}^{-1}(\alpha) \cap \mathbb{C}^{*n}$ for $\alpha \in \mathbb{C}^*$. □

2.4.4. *Remark.* The case $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_1 + \dots + z_n^{a_n}\bar{z}_n$ is studied in [27]. In this case, $g = z_1^{a_1-1} + \dots + z_n^{a_n-1}$ and $\varphi : f^{-1}(1) \to g^{-1}(1)$ is given by

$$w_j = z_j|z_j|^{\frac{2}{a_j-1}}, \ j = 1, \dots, n$$

We can see that this is a homeomorphism.

2.4.5. *Zeta-functions.* Now we know that by [19, 20], the inclusion map $\hat{F}^* \hookrightarrow \mathbb{C}^{*n}$ is $(n-1)$-equivalence and $\chi(\hat{F}^*) = (-1)^{n-1}\det(N - M)$ for $s = n$ and 0 otherwise. In general, for a diffeomorphism $h : F \to F$, the zeta function of h is defined by

$$\zeta_h(t) = \frac{\prod_{j=1}^\infty \det(th_{2j-1} - \mathrm{id})}{\prod_{j=0}^\infty \det(th_{2j} - \mathrm{id})}$$

where $h_j = h_* : H_j(F) \to H_j(F)$.

Note also in our case the monodromy map $\hat{h} : \hat{F}^* \to \hat{F}^*$ has a period m_p. The fixed point locus of $(\hat{h})^k$ is F^* if $m_p \mid k$ and $\emptyset$ otherwise. Thus using the formula of the zeta function (see, for example [15]),

$$\zeta_{\hat{h}^*}(t) = \exp(\sum_{j=0}^\infty (-1)^{n-1} d t^{jm_p}/(jm_p)) = (1 - t^{m_p})^{(-1)^n d/m_p}$$

where $d = \det(N - M)$ if $s = n$ and $d = 0$ for $s < n$. Translating this in the monodromy $h^* : F^* \to F^*$, we obtain

Corollary 19. *F^* has a homotopy type of CW-complex of dimension $n - 1$ and the inclusion map $F^* \hookrightarrow \mathbb{C}^{*n}$ is an $(s-1)$-equivalence. The zeta function $\zeta_{h^*}(t)$ of $h^* : F^* \to F^*$ is given as $(1 - t^{m_p})^{(-1)^n d/m_p}$ with $d = \det(N - M)$ if $s = n$ and $\zeta_{h^*}(t) = 1$ for $s < n$.*

2.4.6. *Connectivity of F.* Now we are ready to patch together the information of the strata F^{*I} for the topology of F. First we introduce the notion of *k-convenience* which is introduced for holomorphic functions ([20]). We say $f(\mathbf{z}, \bar{\mathbf{z}})$ is *k-convenient* if $f^I \not\equiv 0$ for any $I \subset \{1, 2, \dots, n\}$ with $|I| \geq n - k$. The following is obvious by the definition.

Proposition 20. [23] *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simple polar weighted homogeneous polynomial with s monomials and assume that f is k-convenient. Then $k \leq s - 1$.*

Now we have the following result about the connectivity of F.

Theorem 21. [23] *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simple polar weighted homogeneous polynomial with s monomials and assume that f is k-convenient. Then F is* $\min(\mathrm{k}, \mathrm{n} - 2)$*-connected.*

2.4.7. *Join type polynomials.* Another special type of mixed functions are mixed polynomials of join type. Consider the rad-polar weighted homogeneous polynomials $g(\mathbf{z},\bar{\mathbf{z}})$, $h(\mathbf{w},\bar{\mathbf{w}})$ with $\mathbf{z}=(z_1,\dots,z_n)$ and $\mathbf{w}=(w_1,\dots,w_m)$. Consider $f(\mathbf{z},\mathbf{w},\bar{\mathbf{z}},\bar{\mathbf{w}})=g(\mathbf{z},\bar{\mathbf{z}})+h(\mathbf{w},\bar{\mathbf{w}})$. Then

Theorem 22. (Cisneros-Molina [5]) *The Milnor fiber of f is homotopic to the join of $g^{-1}(1)*h^{-1}(1)$ and the monodromy is also the join of the respective monodromy.*

Proof. Let $(p_1,\dots,p_n)$ and $(r_1,\dots,r_m)$ be the normalized polar weights. Then f has the nomalized polar weight $(p_1,\dots,p_n,r_1,\dots,r_m)$. The polar weight is given by multiplying the least common multiple of the denominator. The proof is divided into three steps. Let $F_f=f^{-1}(1)\subset\mathbb{C}^{n+m}$, $F_g=g^{-1}(g)$, $F_h=h^{-1}(1)$. Let

$$F_1=\{(\mathbf{z},\mathbf{w})|g(\mathbf{z},\bar{\mathbf{z}})\in\mathbb{R}\},\ F_2=\{(\mathbf{z},\mathbf{w})\in F_1|0\le g(\mathbf{z},\bar{\mathbf{z}})\le 1\}.$$

Step 1. $F_1\subset F$ is a defoemation retract which is compatible with the monodromy.

Step 2. $F_2\subset F_1$ is also a deformation retract.

Step 3. F_2 is homotopic to the join F_g*F_h and the joined monodromy.

□

Corollary 23. *Suppose that F_1 and F_2has the homotopy types of bouques of spheres of dimension $n-1$ and $m-1$. Then F is $n+m-2$ connected and*

$$\begin{array}{ccc} H_{n+m-1}(F) & \xrightarrow{h_*} & H_{n+m-1}(F) \\ \downarrow\cong & & \downarrow\cong \\ H_{n-1}(F_1)\otimes H_{m-1}(F_2) & \xrightarrow{h_{1*}\otimes h_{2*}} & H_{n-1}(F_1)\otimes H_{m-1}(F_2) \end{array}$$

2.5. **General mixed functions.** This section is completely included in [24].

2.5.1. *Newton boundary of a mixed function.* Suppose that we are given a mixed analytic function $f(\mathbf{z},\bar{\mathbf{z}})=\sum_{\nu,\mu}c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$. We always assume that $c_{0,0}=0$ so that $O\in f^{-1}(0)$. We call the variety $V=f^{-1}(0)$ *the mixed hypersurface.* The *radial Newton polygon* $\Gamma_+(f;\mathbf{z},\bar{\mathbf{z}})$ (at the origin) of a mixed function $f(\mathbf{z},\bar{\mathbf{z}})$ is defined by the convex hull of

$$\bigcup_{c_{\nu,\mu}\neq 0}(\nu+\mu)+\mathbb{R}^{+n}.$$

Hereafter we call $\Gamma_+(f;\mathbf{z},\bar{\mathbf{z}})$ simply the Newton polygon of $f(\mathbf{z},\bar{\mathbf{z}})$. *The Newton boundary* $\Gamma(f;\mathbf{z},\bar{\mathbf{z}})$ is defined by the union of compact faces of $\Gamma_+(f)$. Observe that $\Gamma(f)$ is nothing but the ordinary Newton boundary if f is a complex analytic function. For a given positive integer vector $P=(p_1,\dots,p_n)$, we associate a linear function ℓ_P on $\Gamma(f)$ defined by $\ell_P(\nu)=\sum_{j=1}^n p_j\nu_j$ for $\nu\in\Gamma(f)$ and let $\Delta(P,f)=\Delta(P)$ be the face where ℓ_P takes its minimal value. In other words, P gives radial weights for variables $z_1,\dots,z_n$ by $\mathrm{rdeg}_P\, z_j=\mathrm{rdeg}_P\bar{z}_j=p_j$ and $\mathrm{rdeg}_P\,\mathbf{z}^\nu\bar{\mathbf{z}}^\mu=\sum_{j=1}^n p_j(\nu_j+\mu_j)$. To distinguish the points on the Newton boundary and weight vectors, we denote by N the set of integer weight vectors and denote a vector $P\in N$ by a column vectors. We denote by N^+, N^{++} the subset of positive or strictly positive weight vectors respectively. Thus $P={}^t(p_1,\dots,p_n)\in N^{++}$ (respectively $P\in N^+$) if and only if $p_i>0$ (resp. $p_i\ge 0$) for any $i=1,\dots,n$. We denote the minimal value of ℓ_P by $d(P;f)$ or simply $d(P)$. Note that

$$d(P;f)=\min\{\mathrm{rdeg}_P\,\mathbf{z}^\nu\bar{\mathbf{z}}^\mu\,|\,c_{\nu,\mu}\neq 0\}.$$

For a positive weight P, we define *the face function* $f_P(\mathbf{z}, \bar{\mathbf{z}})$ by

$$f_P(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu+\mu\in\Delta(P)} c_{\nu,\mu}\, \mathbf{z}^\nu \bar{\mathbf{z}}^\mu.$$

Example 24. Consider a mixed function $f := z_1^3\bar{z}_1^2 + z_1^2 z_2^2 + z_2^3\bar{z}_2$. The Newton boundary $\Gamma(f; \mathbf{z}, \bar{\mathbf{z}})$ has two faces Δ_1, Δ_2 which have weight vectors $P := {}^t(2,3)$ and $Q := {}^t(1,1)$ respectively. The corresponding invariants are

$$f_P(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3\bar{z}_1^2 + z_1^2 z_2^2,\ d(P; f) = 10$$
$$f_Q(\mathbf{z}, \bar{\mathbf{z}}) = z_1^2 z_2^2 + z_2^3\bar{z}_2,\ d(Q; f) = 4.$$

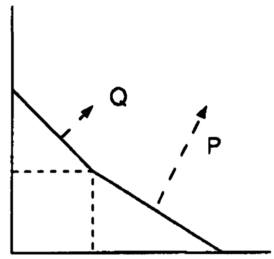

FIGURE 1. $\Gamma(f)$

2.5.2. *Non-degenerate functions.* Suppose that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a given mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$. For $P \in N^{++}$, the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial of type $(p_1, \dots, p_n; d)$ with $d = d(P; f)$.

Definition 25. Let P be a strictly positive weight vector. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *non-degenerate* for P, if the fiber $f_P^{-1}(0) \cap \mathbb{C}^{*n}$ contains no critical point of the mapping $f_P : \mathbb{C}^{*n} \to \mathbb{C}$. In particular, $f_P^{-1}(0) \cap \mathbb{C}^{*n}$ is a smooth real codimension 2 manifold or an empty set. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly non-degenerate* for P if the mapping $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical points. If dim $\Delta(P) \geq 1$, we further assume that $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ is surjective onto $\mathbb{C}$.

A mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *non-degenerate* (respectively *strongly non-degenerate*) if f is non-degenerate (resp. *strongly non-degenerate*) for any strictly positive weight vector P.

Consider the function $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. Then $V = f^{-1}(0)$ is a single point $\{O\}$. By the above definition, f is a non-degenerate mixed function. To avoid such an unpleasant situation, we say that a mixed function $g(\mathbf{z}, \bar{\mathbf{z}})$ is *a true non-degenerate function* if it satisfies further *the non-emptiness condition*:

(NE) : For any $P \in N^{++}$ with dim $\Delta(P, g) \geq 1$, the fiber $g_P^{-1}(0) \cap \mathbb{C}^{*n}$ is non-empty.

Remark 26. *Assume that $f(\mathbf{z})$ is a holomorphic function. Then $f_P(\mathbf{z})$ is a weighted homogeneous polynomial and we have the Euler equality:*

$$d(P; f) f_P(\mathbf{z}) = \sum_{i=1}^n p_i z_i \frac{\partial f_P}{\partial z_i}(\mathbf{z}).$$

Thus $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical point over $\mathbb{C}^*$. Thus f is non-degenerate for P implies f is strongly non-degenerate for P. This is also the case if $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial.

2.5.3. *Isolatedness of the singularities.* Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. As we are mainly interested in the topology of a germ of a mixed hypersurface at the origin, we always assume that f does not have the constant term so that $O \in f^{-1}(0)$. Put $V = f^{-1}(0) \subset \mathbb{C}^n$.

2.5.4. *Mixed singular points.* We say that $\mathbf{w} \in V$ is a *mixed singular point* if $\mathbf{w}$ is a critical point of the mapping $f : \mathbb{C}^n \to \mathbb{C}$. We say that V is *mixed non-singular* if it has no mixed singular points. If V is mixed non-singular, V is smooth variety of real codimension two. Note that a singular point of V (as a point of a real algebraic variety) is a mixed singular point of V but the converse is not necessarily true. For example, every point of the sphere $S = \{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = 1\}$ is a mixed singular point.

2.5.5. *Non-vanishing coordinate subspaces.* For a subset $J \subset \{1, 2, \dots, n\}$, we consider the subspace $\mathbb{C}^J$ and the restriction $f^J := f|_{\mathbb{C}^J}$. Consider the set

$$\mathcal{NV}(f) = \{I \subset \{1, \dots, n\} \mid f^I \not\equiv 0\}.$$

We call $\mathcal{NV}(f)$ *the set of non-vanishing coordinate subspaces for* f. Put

$$V^\sharp = \bigcup_{I \in \mathcal{NV}(f)} V \cap \mathbb{C}^{*I}.$$

Theorem 27. [24] *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a true non-degenerate mixed function. Then there exists a positive number r_0 such that the following properties are satisfied.*

(1) (Isolatedness of the singularity) *The mixed hypersurface $V^\sharp \cap B_{r_0}$ is mixed non-singular. In particular,* $\mathrm{codim}_{\mathbb{R}} V^\sharp = 2$.
(2) (Transversality) *The sphere S_r with $0 < r \le r_0$ intersects $V^\sharp$ transversely.*

We say that f is *k-convenient* if $J \in \mathcal{NV}(f)$ for any $J \subset \{1, \dots, n\}$ with $|J| = n-k$. We say that f is *convenient* if f is $(n-1)$-convenient. Note that $V^\sharp = V \setminus \{O\}$ if f is convenient. For a given ℓ with $0 < \ell \le n$, we put $W(\ell) = \{\mathbf{z} \in \mathbb{C}^n \mid |I(\mathbf{z})| \le \ell\}$ where $I(\mathbf{z}) = \{i | z_i = 0\}$. Thus $W(n-1) = \mathbb{C}^{*n}$. If f is ℓ-convenient, $V \cap W(\ell) \subset V^\sharp$.

Corollary 28. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient true non-degenerate mixed polynomial. Then $V = f^{-1}(0)$ has an isolated mixed singularity at the origin.*

Remark 29. *The assumption "true" is to make sure that $V^* = f^{-1}(0) \cap \mathbb{C}^{*n}$ is non-empty.*

2.6. **Milnor fibration.** In this section, we study the Milnor fibration, assuming that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. We have seen in Theorem 27 that there exists a positive number r_0 such that $V = f^{-1}(0)$ is mixed non-singular except at the origin in the ball $B_{r_0}^{2n}$ and the sphere S_r^{2n-1} intersects transversely with V for any $0 < r \le r_0$. The following is a key assertion for which we need the strong non-degeneracy.

Lemma 30. [23] *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. For any fixed positive number r_1 with $r_1 \le r_0$, there exists positive numbers $\delta_0 \ll r_1$ such that for any $\eta \ne 0$, $|\eta| \le \delta_0$ and r with $r_1 \le r \le r_0$,* (a) *the fiber*

$V_\eta := f^{-1}(\eta)$ has no mixed singularity inside the ball $B^{2n}_{r_0}$ and (b) *the intersection $V_\eta \cap S^{2n-1}_r$ is transverse and smooth.*

2.6.1. *Milnor fibration, the second description.* Put

$$D(\delta_0)^* = \{\eta \in \mathbb{C} \mid 0 < |\eta| \le \delta_0\},\ S^1_{\delta_0} = \partial D(\delta_0)^* = \{\eta \in \mathbb{C} \mid |\eta| = \delta_0\}$$
$$E(r,\delta_0)^* = f^{-1}(D(\delta_0)^*) \cap B^{2n}_r,\ \partial E(r,\delta_0)^* = f^{-1}(S^1_{\delta_0}) \cap B^{2n}_r.$$

By Lemma 30 and the theorem of Ehresman ([31]), we obtain the following description of the Milnor fibration of the second type ([10]).

Theorem 31. (The second description of the Milnor fibration) *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient, strongly non-degenerate mixed function. Take positive numbers r_0, r_1 and δ_0 such that $r \le r_0$ and $\delta_0 \ll r_1$ as in Lemma 30. Then $f : E(r,\delta_0)^* \to D(\delta_0)^*$ and $f : \partial E(r,\delta_0)^* \to S^1_{\delta_0}$ are locally trivial fibrations and the topological isomorphism class does not depend on the choice of δ_0 and r.*

2.6.2. *Milnor fibration, the first description.* We consider now the original Milnor fibration on the sphere, which is defined as follows:

$$\varphi : S^{2n-1}_r \setminus K_r \to S^1, \quad \mathbf{z} \mapsto \varphi(\mathbf{z}) = f(\mathbf{z},\bar{\mathbf{z}})/|f(\mathbf{z},\bar{\mathbf{z}})|$$

where $K_r = V \cap S^{2n-1}_r$. The fibrations of this type for mixed functions and related topics have been studied by many authors ([27, 28, 6, 29, 26, 3]). But most of the works treat rather special classes of functions. The mapping φ can be identified with $\varphi(\mathbf{z}) = -\Re(i \log f(\mathbf{z}))$, taking the argument θ as a local coordinate of the circle S^1. We use the basis $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \mid j = 1,\dots,n\}$ of the tangent space $T_{\mathbf{z}}\mathbb{C}^n \otimes \mathbb{C}$. For a mixed function $g(\mathbf{z},\bar{\mathbf{z}})$, we use two complex "gradient vectors" defined by

$$dg = (\frac{\partial g}{\partial z_1},\dots,\frac{\partial g}{\partial z_n}), \quad \bar{d}g = (\frac{\partial g}{\partial \bar{z}_1},\dots,\frac{\partial g}{\partial \bar{z}_n}).$$

Take a smooth path $\mathbf{z}(t)$, $-1 \le t \le 1$ with $\mathbf{z}(0) = \mathbf{w} \in \mathbb{C}^n \setminus V$ and put $\mathbf{v} = \frac{d\mathbf{z}}{dt}(0) \in T_{\mathbf{w}}\mathbb{C}^n$. Then we have

$$\begin{aligned}
&-\frac{d}{dt}\left(\Re(i\ \log f(\mathbf{z}(t),\bar{\mathbf{z}}(t))\right)_{t=0} \\
&= -\Re\left(\sum_{i=1}^n i\left\{\frac{\partial f}{\partial z_j}(\mathbf{w},\bar{\mathbf{w}})\frac{dz_j}{dt}(0) + \frac{\partial f}{\partial \bar{z}_j}(\mathbf{w},\bar{\mathbf{w}})\frac{d\bar{z}_j}{dt}(0)\right\}/f(\mathbf{w},\bar{\mathbf{w}})\right) \\
&= \Re(\mathbf{v}, i\,\overline{d\log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\bar{\mathbf{v}}, i\,\overline{\bar{d}\log f}(\mathbf{w},\bar{\mathbf{w}})) \\
&= \Re(\mathbf{v}, i\,\overline{d\log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\mathbf{v}, -i\,\bar{d}\log f(\mathbf{w},\bar{\mathbf{w}})) \\
&= \Re(\mathbf{v}, i\,(\overline{d\log f} - \bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})).
\end{aligned}$$

Namely we have

$$-\frac{d}{dt}\left(\Re(i\ \log f(\mathbf{z}(t),\bar{\mathbf{z}}(t))\right)_{t=0} = \Re(\mathbf{v}, i\,(\overline{d\log f} - \bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})). \tag{3}$$

Thus by the same argument as in Milnor [15], we get

Lemma 32. [24] *A point $\mathbf{z} \in S^{2n-1}_r \setminus K_r$ is a critical point of φ if and only if the two complex vectors $i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))$ and $\mathbf{z}$ are linearly dependent over $\mathbb{R}$.*

The key assertion is the following.

Lemma 33. [24] *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a strongly non-degenerate mixed function. Then there exists a positive number r_0 such that the two complex vectors $i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))$ and $\mathbf{z}\in S_r\setminus K_r$ are linearly independent over $\mathbb{R}$ for any r with $0<r\le r_0$.*

Now we are ready to prove the existence of the Milnor fibration of the first description.

Theorem 34. [24](Milnor fibration, the first description) *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a strongly non-degenerate convenient mixed function. There exists a positive number r_0 such that*

$$\varphi = f/|f| : S_r^{2n-1}\setminus K_r \to S^1$$

is a locally trivial fibration for any r with $0<r\le r_0$.

2.6.3. *Equivalence of two Milnor fibrations.* Take positive numbers r, δ_0 with $\delta_0 \ll r$ as in Theorem 31. We compare the two fibrations

$$f:\partial E(r,\delta_0)\to S^1_{\delta_0},\qquad \varphi : S_r^{2n-1}\setminus K_r \to S^1$$

and we will show that they are isomorphic. However the proof is much more complicated compared with the case of holomorphic functions. The reason is that we have to take care of the two vectors

$$i\,(\overline{d\log f} - \bar{d}\log f),\ \overline{d\log f} + \bar{d}\log f$$

which are not perpendicular. (In the holomorphic case, the proof is easy as the two vectors reduce to the perpendicular vectors $i\,\overline{d\log f}$, $\overline{d\log f}$.) Consider a smooth curve $\mathbf{z}(t)$, $-1\le t\le 1$, with $\mathbf{z}(0)=\mathbf{w}\in B_r^{2n}\setminus V$ and $\mathbf{v} = \frac{d\mathbf{z}(t)}{dt}(0)$. Put $\mathbf{v} = (v_1,\dots,v_n)$. First from (3), we observe that

$$\begin{aligned}\frac{\log f(\mathbf{z}(t),\,\bar{\mathbf{z}}(t))}{dt}|_{t=0} &= \sum_{j=1}^{n}\left(v_j\,\frac{\partial \log f}{\partial z_j}(\mathbf{w},\bar{\mathbf{w}}) + \bar{v}_j\,\frac{\partial\log f}{\partial \bar{z}_j}(\mathbf{w},\bar{\mathbf{w}})\right)\\ &= \Re(\mathbf{v},\,(\overline{d\log f}+\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})) + i\,\Re(\mathbf{v},\,i\,(\overline{d\log f}-\bar{d}\log f)(\mathbf{w},\bar{\mathbf{w}})).\end{aligned}$$

Define two vectors on $\mathbb{C}^n - V$:

$$\begin{aligned}\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}) &= \overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) + \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}})\\ \mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}}) &= i\,(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))\end{aligned}$$

The above equality is translated as

$$\frac{\log f(\mathbf{z}(t),\bar{\mathbf{z}}(t))}{dt}|_{t=0} = \Re(\mathbf{v},\mathbf{v}_1(\mathbf{w},\bar{\mathbf{w}})) + i\,\Re(\mathbf{v},\mathbf{v}_2(\mathbf{w},\bar{\mathbf{w}})). \tag{4}$$

The following will play the key role for the equivalence of two fibrations:

Lemma 35. [24]*labelkey lemma Under the same assumption as in Theorem 34, there exists a positive number r_0 so that for any $\mathbf{z}$ with $\|\mathbf{z}\|\le r_0$ and $f(\mathbf{z},\bar{\mathbf{z}})\ne 0$, the three vectors*

$$\mathbf{z},\quad \mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}),\quad \mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})$$

are either (i) linearly independent over $\mathbb{R}$ or (ii) they are linearly dependent over $\mathbb{R}$ and the relation can be written as

$$\mathbf{z} = a\,\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}) + b\,\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}}),\ a,b\in\mathbb{R}. \tag{5}$$

and the coefficient a is positive.

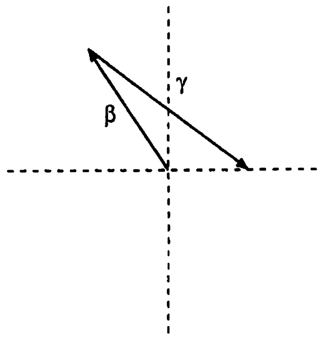

FIGURE 2. If $\lambda_0 \le 0$, $|\beta| < |\gamma|$

Now we are ready to prove the isomorphism theorem:

Theorem 36. *Under the same assumption as in Theorem 34, the two fibrations*

$$f : \partial E(r, \delta_0) \to S^1_{\delta_0}, \qquad \varphi : S^{2n-1}_r \setminus K_r \to S^1$$

are topologically isomorphic.

3. Mixed projective curves

Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a rad-polar weighted homogeneous mixed polynomial with $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. Namely there exist integers $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$ and positive integers d_r, d_p such that

$$f(t \circ \mathbf{z}, t \circ \bar{\mathbf{z}}) = t^{d_r} f(\mathbf{z}, \bar{\mathbf{z}}), \quad t \circ \mathbf{z} = (t^{q_1} z_1, \dots, t^{q_n} z_n), \ t \in \mathbb{R}^+$$
$$f(\rho \circ \mathbf{z}, \overline{\rho \circ \mathbf{z}}) = \rho^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}), \quad \rho \circ \mathbf{z} = (\rho^{p_1} z_1, \dots, \rho^{p_n} z_n), \ \rho \in \mathbb{C}, \ |\rho| = 1.$$

This gives $\mathbb{R}^+ \times S^1$ action by

$$(t, \rho) \circ \mathbf{z} = (t^{q_1} \rho^{p_1} z_1, \dots, t^{q_n} \rho^{p_n} z_n), \quad t\rho \in \mathbb{R}^+ \times S^1.$$

We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly polar weighted homogeneous* if $p_j = q_j$ for $j = 1, \dots, n$. Then the associated $\mathbb{R}^+ \times S^1$ action on $\mathbb{C}^n$ is in fact the $\mathbb{C}^*$ action which is defined by

$$(\mathbf{z}, \tau) = ((z_1, \dots, n), \tau) \mapsto \tau \circ \mathbf{z} = (z_1 \tau^{p_1}, \dots, z_n \tau^{p_n}), \ \tau \in \mathbb{C}^*.$$

We say $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly polar homogeneous* if further the weights satisfies the equalities $q_j = p_j = 1$ for any j. A strongly polar weighted homogeneous polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ satisfies the equality:

$$f((t, \rho) \circ \mathbf{z}, \overline{(t, \rho) \circ \mathbf{z}}) = t^{d_r} \rho^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}), \quad (t, \rho) \in \mathbb{R}^+ \times S^1. \tag{6}$$

Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly polar weighted homogeneous polynomial of radial degree d_r and of polar degree d_p respectively and let $P = (p_1, \dots, p_n)$ be the weight vector. Let $\tilde{V}$ be the mixed affine hypersurface

$$\tilde{V} = f^{-1}(0) = \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}.$$

Let $\varphi : S^{2n-1} \setminus K \to S^1$ be the Milnor fibration with $K = \tilde{V} \cap S^{2n-1}$ and let F be the fiber. Recall that $\varphi(\mathbf{z}) = f(\mathbf{z}, \bar{\mathbf{z}})/|f(\mathbf{z}, \bar{\mathbf{z}})|$. Thus F is defined by

$$F = \varphi^{-1}(1) = \{\mathbf{z} \in S^{2n-1} \setminus K \mid f(\mathbf{z}, \bar{\mathbf{z}}) > 0\}$$

We can equivalently consider the global fibration $f : \mathbb{C}^n - \tilde{V} \to \mathbb{C}^*$. Then the Milnor fiber is identified with the hypersurface $f^{-1}(1)$. The monodromy map $h : F \to F$ (in either case) is defined by

$$h(\mathbf{z}) = (\exp(\frac{2p_1\pi i}{d_p})z_1, \dots, \exp(\frac{2p_n\pi i}{d_p})z_n).$$

We consider also the wighted projective hypersurface V defined by

$$V = \{(z_1 : z_2 : \cdots : z_n) \in \mathbb{CP}(P)^{n-1} \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$$

where $\mathbb{CP}(P)^{n-1}$ is the weighted projective space defined by the equivalence induced by the above $\mathbb{C}^*$ action:

$$\mathbf{z} \sim \mathbf{w} \iff \exists \tau \in \mathbb{C}^*, \, \mathbf{w} = \tau \circ \mathbf{z}.$$

It is well-known that $\mathbb{CP}(P)^{n-1}$ is an orbifold with at most cyclic quotient singularities.

By (6), $\mathbf{z} \in f^{-1}(0)$ and $\mathbf{z}' \sim \mathbf{z}$, then $\mathbf{z}' \in f^{-1}(0)$. Thus the hypersurface $V = \{[\mathbf{z}] \in \mathbb{CP}^{n-1}(P) | f(\mathbf{z}) = 0\}$ is well-defined. Consider the quotient map $\pi : S^{2n-1} \to \mathbb{CP}(P)^{n-1}$ or $\pi : \mathbb{C}^n \setminus \{O\} \to \mathbb{CP}(P)^{n-1}$. For the brevity's sake, we denote the restrictions $\pi|F : F \to \mathbb{CP}^{n-1} \setminus V$ and $\pi|K : K \to V$ by the same π. We are interested in the topology of V and the relation with the Milnor fibration.

3.1. **Canonical orientation.** It is well known that a complex analytic smooth variety has a canonical orientation which comes from the complex structure (see for example p.18, [7]). Let $\tilde{V} = f^{-1}(0)$ be a mixed hypersurface. Take a point $\mathbf{a} \in \tilde{V}$. We say that $\mathbf{a}$ is a *mixed singular* point of $\tilde{V}$, if $\mathbf{a}$ is a critical point of the mapping $f : \mathbb{C}^n \to \mathbb{C}$. Otherwise, $\mathbf{a}$ is a *mixed regular point.* Note that a point $\mathbf{a} \in \tilde{V}$ to be a regular point as a point of a real analytic variety is a necessary condition but not a sufficient condition for the regularity as a point on a mixed variety. Recall that $\mathbf{a}$ is a mixed singular point if and only if $df_{\mathbf{a}} : T_{\mathbf{a}}\mathbb{C}^n \to T_{f(\mathbf{a})}\mathbb{C}$ is surjective. This is equivalent to the existence of a complex number $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$\overline{df}(\mathbf{a}, \bar{\mathbf{a}}) = \alpha \bar{d} f(\mathbf{a}, \bar{\mathbf{a}}) \quad \text{i.e.,} \quad \overline{\frac{\partial f}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}}) = \alpha \frac{\partial f}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}}), \, j = 1, \dots, n$$

([23]). We assert that

Proposition 37. *There is a canonical orientation on the smooth part of a mixed hypersurface.*

Proof. Take a regular point $\mathbf{a} \in \tilde{V}$. The normal bundle $\mathcal{N}$ of $\tilde{V} \subset \mathbb{C}^n$ has a canonical orientation so that $df_{\mathbf{a}} : \mathcal{N}_{\mathbf{a}} \to T_{f(\mathbf{a}, \bar{\mathbf{a}})}\mathbb{C}$ is an orientation preserving isomorphism. This gives a canonical orientation on $\tilde{V}$ so that the ordered union of the oriented frames $\{v_1, \dots, v_{2n-2}, n_1, n_2\}$ of $T_{\mathbf{a}}\mathbb{C}^n$ is the orientation of $\mathbb{C}^n$ if and only if $\{v_1, \dots, v_{2n-2}\}$ is an oriented frame of $T_{\mathbf{a}}\tilde{V}$ where $\{n_1, n_2\}$ is an oriented frame of normal vectors. □

Consider a mixed homogeneous hypersurface $\tilde{V}$ and let V be the corresponding mixed projective hypersurface.

Proposition 38. *Let* $\mathbf{a} \in \tilde{V} \setminus \{O\}$. *Then* $\mathbf{a} \in \tilde{V}$ *is a mixed singular point of* $\tilde{V}$ *if and only if* $\pi(\mathbf{a}) \in V$ *is a mixed singular point.*

See [24] for the proof.

3.2. **Milnor fiberation and Hopf fiberation.** Consider the Hopf fibration $\pi : S^{2n-1} \to \mathbb{CP}^{n-1}$ and its restriction to the Milnor fiber $F = \{\mathbf{z} \in S^{2n-1} \mid f(\mathbf{z}, \bar{\mathbf{z}}) > 0\}$. Put $K = f^{-1}(0) \cap S^{2n-1}$ be the link. As f is polar weighted, it is easy to see that $\pi : F \to \mathbb{CP}^{n-1} \setminus V$ is a cyclic covering of order d_p and the covering transformation is generated by the monodromy map

$$h : F \to F, \quad \mathbf{z} \mapsto \exp(\frac{2\pi i}{d_p}) \cdot \mathbf{z}.$$

Thus we have

Proposition 39. (1) $\chi(F) = d_p\chi(\mathbb{CP}^{n-1} \setminus V)$.
(2) $\chi(\mathbb{CP}^{n-1} \setminus V) = n - \chi(V)$ *and* $\chi(V) = n - \chi(F)/d_p$.
(3) *We have the following exact sequence.*

$$1 \to \pi_1(F) \xrightarrow{\pi_\sharp} \pi_1(\mathbb{CP}^{n-1} \setminus V) \to \mathbb{Z}/d_p\mathbb{Z} \to 1.$$

The following special cases are used later.

Corollary 40. (1) *Suppose $n = 2$. Then $K \subset S^3$ is a link. Put r be the number of the components. Then V is r points and r and χ are related by*

$$\frac{\chi(F)}{d_p} = 2 - r.$$

(2) *Suppose $n = 3$. Then $V \subset \mathbb{P}^2$ is a curve of genus g and*

$$1 + 2g = \frac{\chi(F)}{d_p}.$$

Remark 41. *Let f be a mixed polar weighted polynomial of two variables and let r be the number of link components S^3. Let s be the number of irreducible components of f. Then $r \geq s$. For example, For example, let $f(z_1, z_2, \bar{z}_1, \bar{z}_2) = -2z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + tz_1^2\bar{z}_2$. Then for $t = 0$, $\mathrm{lkn}(f) = 1 = s$ and for $t = 2$, $s = 1$ and $\mathrm{lkn}(f) = 3$.*

Corollary 42. *If $d_p = 1$, the projection $\pi : F \to \mathbb{CP}^{n-1} \setminus V$ is a diffeomorphism.*

The monodromy map $h : F \to F$ gives free $\mathbb{Z}/d_p\mathbb{Z}$ action on F. Thus using the periodic monodromy argument in [15], we get

Proposition 43. *The zeta function of $h : F \to F$ is given by*

$$\zeta(t) = (1 - t^{d_p})^{-\chi(F)/d_p}.$$

In particular, if $d_p = 1$, $h = \mathrm{id}_F$ and $\zeta(t) = (1 - t)^{-\chi(F)}$.

3.3. **Degree of mixed projective hypersurfaces.** Suppose that $f(\mathbf{z}, \bar{\mathbf{z}}) \in \mathcal{M}(q+2r, q; n)$ be a strongly polar homogeneous polynomial and let

$$V = \{\mathbf{z} \in \mathbb{CP}^{n-1} \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}.$$

We assume that the singular locus ΣV of V is either empty or $\mathrm{codim}_{\mathbb{R}} \Sigma V \geq 2$. We have observed that $V \setminus \Sigma V \subset \mathbb{CP}^{n-1}$ is canonically oriented so that the union of the oriented frame of $T_P V$, say $\{v_1, \dots, v_{2(n-2)}\}$ and the frame of normal bundle $\{w_1, w_2\}$ which is compatible with the local defining complex function g_j on the affine chart $U_j = \{z_j \neq 0\}$ is the oriented frame of $\mathbb{CP}^{n-1}$. (Recall that g_j is a mixed function of the variables $u_i = z_i/z_j$, $i \neq j$ defined by $g_j(\mathbf{u}, \bar{\mathbf{u}}) = f(\mathbf{z}, \bar{\mathbf{z}})/z_j^{q+r}/\bar{z}_j^r$.)

Thus it has a fundamental class $[V] \in H_{2n-4}(V;\mathbb{Z})$ by Borel-Haefliger [4]. The topological degree of V is the integer d so that $\iota_*[V] = d[\mathbb{CP}^{n-2}]$ where $\iota : V \to \mathbb{CP}^{n-1}$ is the inclusion map and $[\mathbb{CP}^{n-2}]$ is the homology class of a canonical hyperplane $\mathbb{CP}^{n-2}$.

Theorem 44. [22] *The topological degree of V is equal to the polar degree q. Namely the fundamental class $[V]$ corresponds to $q[\mathbb{CP}^{n-2}] \in H_{2(n-2)}(\mathbb{CP}^{n-1})$ by the inclusion mapping ι_*.*

3.3.1. *Residue formula for a monic mixed polynomial.* Let $g(w,\bar{w}) = \sum_{a,b} c_{a,b} w^a \bar{w}^b$ be a mixed polynomial. Put $d = \max\{a+b \mid c_{a,b} \neq 0\}$ and we call d the radial degree of g. We say that g is *a monic mixed polynomial of degree d* if g has a unique monomial of radial degree d.

Lemma 45. *Assume that $g(w,\bar{w})$ is a monic mixed polynomial of degree d which is written as*

$$\begin{aligned} g(w,\bar{w}) &= c_0(\bar{w})w^{q+r} + c_1(\bar{w})w^{q+r-1} + \cdots + c_{q+r} \\ c_j(\bar{w}) &\in \mathbb{C}[\bar{w}],\ \mathrm{rdeg}_{\bar{w}} c_j \leq r,\ j = 0,\dots,q+r \\ c_0(\bar{w}) &= c_{0r}\bar{w}^r + \cdots + c_{00},\ c_{0r} \neq 0 \end{aligned}$$

with $d = q + 2r$. Then

$$\frac{1}{2\pi}\int_{|w|=R} \mathrm{Gauss}(g)d\theta = q.$$

3.3.2. *Mixed projective curves.* In this section, we study basic examples in the projective surface $\mathbb{CP}^2$. Thus we assume that $n = 3$. We consider projective curves of degree q:

$$C = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 \mid f(z_1,z_2,z_3) = 0\}$$

where f is a strongly polar homogeneous polynomial with $\mathrm{pdeg}\, f = q$. We have seen that the topological degree of C is q by Theorem 44. The genus g of C is not an invariant of q. Recall that for a differentiable curve C of genus g, embedded in $\mathbb{CP}^2$, with the topological degree q, we have the following Thom's inequality, which was conjectured by Thom and proved by for example Kronheimer-Mrowka [13]:

$$g \geq \frac{(q-1)(q-2)}{2}$$

where the right side number is the genus of algebraic curves of degree q, given by the Plücker formula. Recall that for a mixed strongly polar homogeneous polynomial, the genus and the Euler characteristic of the Milnor fiber are related as follows (Corollary 40):

$$g = \frac{1}{2}\left(\frac{\chi(F)}{q} - 1\right)$$

where

$$F = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \mid f(z_1,z_2,z_3,\bar{z}_1,\bar{z}_2,\bar{z}_3) = 1\}.$$

Now we will see some examples which shows that $\chi(F)$ is not an invariant of q.

I. Simplicial polynomials. We consider the following simplicial polar homogeneous polynomials of polar degree q.

$$f_{s_1}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{q+r}\bar{z}_1^r + z_2^{q+r}\bar{z}_2^r + z_3^{q+r}\bar{z}_3^r$$
$$f_{s_2}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r}\bar{z}_3^r$$
$$f_{s_3}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r-1}\bar{z}_3^r z_1$$
$$f_{s_4}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{q+r+1}\bar{z}_1^r \bar{z}_2 + z_2^{q+r+1}\bar{z}_2^r \bar{z}_3 + z_3^{q+r}\bar{z}_3^r$$
$$f_{s_5}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{q+r+1}\bar{z}_1^r \bar{z}_2 + z_2^{q+r+1}\bar{z}_2^r \bar{z}_3 + z_3^{q+r+1}\bar{z}_3^r \bar{z}_1$$

Let F_{s_i} be the Milnor fiber of f_{s_i} and let C_{s_i} be the corresponding projective curves for $i = 1,\dots,5$. First, the Euler characteristic of the Milnor fibers and the genera are given as follows.

$$\chi(F_{s_i}) = q^3 - 3q^2 + 3q, \quad g(C_{s_i}) = \frac{(q-1)(q-2)}{2}, \; i = 1,2,3$$
$$\chi(F_{s_4}) = q(q^2+q+1), \quad g(C_{s_4}) = \frac{q(q+1)}{2}$$
$$\chi(F_{s_5}) = q(q^2+3q+3), \quad g(C_{s_4}) = \frac{(q+2)(q+1)}{2}$$

In [21], we have shown that C_{s_1} and C_{s_2} are isomorphic to algebraic plane curves defined by the associated homogeneous polynomials of degree q:

$$g_{s_1}(\mathbf{z}) = z_1^{q-1}z_2 + z_2^{q-1}z_3 + z_3^q$$
$$g_{s_2}(\mathbf{z}) = z_1^{q-1}z_2 + z_2^{q-1}z_3 + z_3^q.$$

We also expect that C_{s_3} is isotopic to the algebraic curve

$$z_1^{q-1}z_2 + z_2^{q-1}z_3 + z_3^{q-1}z_1 = 0,$$

as the genus of C_{s_3} suggests it (see also [21]).

II. We consider the following join type polar homogeneous polynomial.

$$h_j(\mathbf{z},\bar{\mathbf{z}}) = g_j(\mathbf{w},\bar{\mathbf{w}}) + z_3^{q+r}\bar{z}_3^r,$$
$$g_j(\mathbf{w},\bar{\mathbf{w}}) = (w_1^{q+j}\bar{w}_1^j + w_2^{q+j}\bar{w}_2^j)(w_1^{r-j} - \alpha w_2^{r-j})(\bar{w}_1^{r-j} - \beta\bar{w}_2^{r-j}),$$
$$0 \le j \le r.$$

($\alpha,\beta \in \mathbb{C}^*$ are generic.) The the Milnor fiber F_{g_j} of g_j is connected. The link component number of $g = 0$ is $\mathrm{lkn}(g) = q+2(r-j)$. Thus $\chi(F_{g_j}) = q(q-2+2(r-j))$ by Corollary 40 and $g = \frac{(q-1)(q-2+2(r-j))}{2}$.

In particular, taking $q = 2$, we obtain $g = r - j$ and thus

Corollary 46. *For any smooth surface S of genus g, there is an embedding $S \subset \mathbb{CP}^2$ so that the degree of S is 2.*

We observe that the case $q = 1$ gives only the trivial case $g = 0$ in this family.

3.4. **Twisted join type polynomial.** In this section, we introduce a new class of mixed polar weighted polynomials which we use to construct curves with embedded degree 1. For further detail, see [22]. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial of n-variables $\mathbf{z} = (z_1,\dots,z_n)$. Let $Q = {}^t(q_1,\dots,q_n)$, $P = {}^t(p_1,\dots,p_n)$ be the radial and polar weight respectively and let d, q be the radial and polar degree respectively. For simplicity, we call that $Q' = {}^t(q_1/d,\dots,q_n/d)$ and $P' =$

${}^t(p_1/q, \dots, p_n/q)$ *the normalized radial weights* and *the normalized polar weights* respectively. Consider the mixed polynomial of $(n+1)$-variables:

$$g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w}) = f(\mathbf{z}, \bar{\mathbf{z}}) + \bar{z}_n w^a \bar{w}^b, \quad a > b.$$

Consider the rational numbers $\bar{q}_{n+1}$, $\bar{p}_{n+1}$ satisfying

$$\frac{q_n}{d} + (a+b)\,\bar{q}_{n+1} = 1, \quad -\frac{p_n}{q} + (a-b)\,\bar{p}_{n+1} = 1.$$

We assume that $q_n < d$ so that $\bar{q}_{n+1}, \bar{p}_{n+1}$ are positive rational numbers. The polynomial g is a polar weighted homogeneous polynomial with the normalized radial and polar weights $\widetilde{Q'} = {}^t(q_1/d, \dots, q_n/d, \bar{q}_{n+1})$ and $\widetilde{P'} = {}^t(p_1/q, \dots, p_n/q, \bar{p}_{n+1})$ respectively. The radial and polar degree of g are given by $\mathrm{lcm}(d, \mathit{denom}(\bar{q}_{n+1}))$ and $\mathrm{lcm}(q, \mathit{denom}(\bar{p}_{n+1}))$ where $\mathit{denom}(x)$ is the denominator of $x \in \mathbb{Q}$. We call g *a twisted join of* $f(\mathbf{z}, \bar{\mathbf{z}})$ *and* $\bar{z}_n w^a \bar{w}^b$. We say that g is a polar weighted homogeneous polynomial of *twisted join type.* A twisted join type polynomial behaves differently than the simple join type, as we will see below.

We recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is called to be *1-convenient* if the restriction of f to each coordinate hyperplane $f_i := f|_{\{z_i=0\}}$ is non-trivial for $i = 1, \dots, n$ ([23])

Lemma 47. *Assume that $n \geq 2$ and f is 1-convenient. Then*

$$\phi_\sharp : \pi_1((\mathbb{C}^*)^n \setminus F_f^*) \cong \mathbb{Z}^n \times \mathbb{Z}$$

is an isomorphism where ϕ is the canonical mapping

$$\phi : (\mathbb{C}^*)^n \setminus F_f^* \to (\mathbb{C}^*)^n \times (\mathbb{C} \setminus \{1\})$$

defined by $\phi(\mathbf{z}) = (\mathbf{z}, f(\mathbf{z}, \bar{\mathbf{z}}))$ and $F_f^* := f^{-1}(1) \cap (\mathbb{C}^*)^n$.

Put $F_{f_n} := f_n^{-1}(1) = F_f \cap \{z_n = 0\} \subset \mathbb{C}^{n-1}$ with $f_n := f|_{\mathbb{C}^n \cap \{z_n=0\}}$.

Theorem 48. [22] *Assume that $n \geq 2$ and f is 1-convenient and $g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w})$ is a twisted join polynomial as above. Then*

(1) *the Milnor fiber of g, $F_g = g^{-1}(1)$, is simply connected.*
(2) *The Euler characteristic of F_g is given by the formula:*

$$\chi(F_g) = -(a - b - 1)\chi(F_f) + (a-b)\chi(F_{f_n}).$$

3.4.1. *Construction of a family of mixed curves with polar degree q.* Now we are ready to construct a key family of mixed curves with embedding degree q. Recall the polynomial:

$$h_{q,r,j}(\mathbf{w}, \bar{\mathbf{w}}) := (z_1^{q+j}\bar{z}_1^j + z_2^{q+j}\bar{z}_2^j)(z_1^{r-j} - \alpha z_2^{r-j})(\bar{z}_1^{r-j} - \beta \bar{z}_2^{r-j}), \quad \mathbf{w} = (z_1, z_2).$$

$h_{q,r,j}(\mathbf{w}, \bar{\mathbf{w}})$ is 1-convenient strongly polar homogeneous polynomial with the radial degree $q + r$ and the polar degree q respectively. The constants α, β are generic. For this, it suffices to assume that $|\alpha|, |\beta| \neq 0, 1$ and $|\alpha| \neq |\beta|$. Consider the twisted join polynomial of 3 variables z_1, z_2, z_3:

$$s_{q,r,j}(\mathbf{z}, \bar{\mathbf{z}}) = h_{q,r,j}(\mathbf{w}, \bar{\mathbf{w}}) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{r-1}, \quad \mathbf{z} = (z_1, z_2, z_3).$$

Let $F_{q,r,j} = s_{q,r,j}^{-1}(1) \subset \mathbb{C}^3$ be the Milnor fiber and let $S_{q,r,j} \subset \mathbb{P}^2$ be the corresponding mixed projective curve:

$$S_{q,r,j} = \{[\mathbf{z}] \in \mathbb{P}^2 \mid s_{q,r,j}(\mathbf{z}, \bar{\mathbf{z}}) = 0\}.$$

Note that $S_{q,r,j}$ is a smooth mixed curve. The following describes the topology of $F_{q,r,j}$ and $S_{q,r,j}$.

Theorem 49. (1) *The Euler characteristic of the Milnor fiber $F_{q,r,j}$ is given by:*

$$\chi(F_{q,r,j}) = q(q^2 - q + 1 + 2(r - j)).$$

(2) *The genus of $S_{q,r,j}$ is given by:*

$$g(S_{q,r,j}) = \frac{q(q-1)}{2} + (r - j)$$

Proof. Let $H_{q,r,j} = h_{q,r,j}^{-1}(1)$. Then by Corollary 40,

$$\chi(H_{q,r,j}) = -q(q - 2 + 2(r - j))$$
$$\chi(H_{q,r,j} \cap \{z_2 = 0\}) = q$$

and the assertion follows from Theorem 48. □

3.4.2. *Mixed curves with polar degree 1.* We consider the case $q = 1, j = 0$:

$$\begin{cases} h(\mathbf{w}, \bar{\mathbf{w}}) & := (z_1 + z_2)(z_1^r - \alpha z_2^r)(\bar{z}_1^r - \beta \bar{z}_2^r) \\ f_r(\mathbf{z}, \bar{\mathbf{z}}) & := h(\mathbf{w}, \bar{\mathbf{w}}) + \bar{z}_2 z_3^{r+1} \bar{z}_3^{r-1} \\ S_r & := \{[\mathbf{z}] \in \mathbb{P}^2 \mid f_r(\mathbf{z}, \bar{\mathbf{z}}) = 0\}. \end{cases}$$

Corollary 50. *Let S_r be the mixed curve as above. Then the embedding degree of S_r is 1 and the genus of S_r is r.*

Proof. Let $F_r = f_r^{-1}(1)$ be the Milnor fiber of f_r. By Theorem 48, we have $\chi(F_r) = 2r + 1$. Thus by Corollary 40, the assertion follows immediately. □

Remark 51. *$h(\mathbf{w}, \bar{\mathbf{w}})$ can be replaced by $(z_1^{r+1} - z_2^{r+1})(\bar{z}_1 - \beta \bar{z}_2^r)$ without changing the topology.*

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