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ON REGION UNKNOTTING NUMBERS

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ABSTRACT. A region crossing change at a region of a knot diagram is the crossing changes at all the crossing points on the boundary of the region. In this paper, we show that for any knot diagram and any region R, we can make any crossing change by a sequence of region crossing changes except at R. We also discuss about region unknotting numbers of 3-braids.

1. INTRODUCTION

A *region crossing change* at a region R of a link diagram D on S^2 is the crossing changes at all the crossing points on the boundary of R [3]. For example, we obtain the diagram D' from the knot diagram D by the region crossing change at the region R in Figure 1.

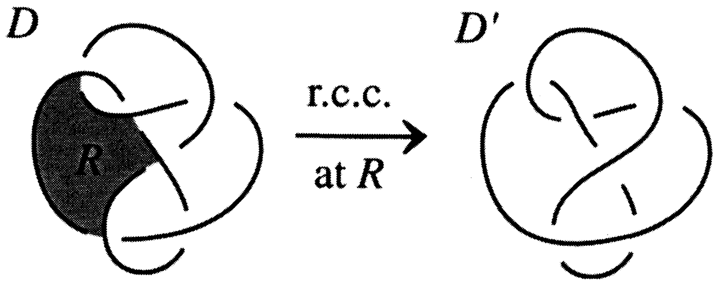

FIGURE 1

We remark that K. Kishimoto proposed a region crossing change at a seminar at Osaka City University, and asked whether a region crossing change is an unknotting operation. To give the positive answer to this question, the following theorem is shown in [3]:

Theorem 1.1 ([3]). *For any knot diagram D, we can make any crossing change on D by a sequence of region crossing changes.*

Since a crossing change is an unknotting operation, a region crossing change on a knot diagram is also an unknotting operation. Moreover, we have the following theorems:

Theorem 1.2. *Let D be a knot diagram and let R be a region of D. We can make any crossing change on D by a sequence of region crossing changes at regions of D except R.*

Theorem 1.3. *Let D be a reduced knot diagram. For any region R of D, there exists a region $S \neq R$ of D such that we can make any crossing change on D by a sequence of region crossing changes at regions of D except R and S.*

The proofs are given in Section 2. For example, for the diagram D and the region R in Figure 2, the region S satisfies the above condition: We can change the crossing at c_1 (resp. c_2) by region crossing changes at T_1 and T_3 (resp. T_1, T_2 and T_3).

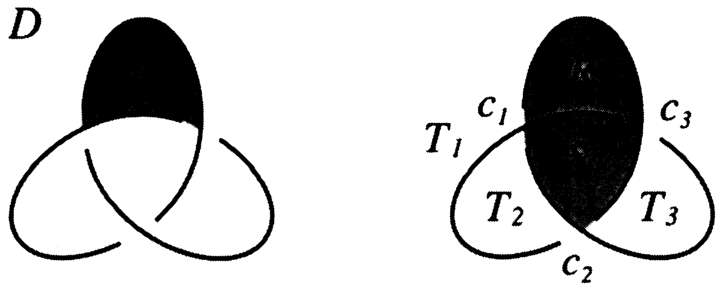

FIGURE 2

The *region unknotting number* $u_R(D)$ of a knot diagram D is the minimal number of region crossing changes which are needed to obtain a diagram of the trivial knot (without Reidemeister moves) [3]. For example, the diagram D in Figure 1 has the region unknotting number one. The *region unknotting number* $u_R(K)$ of a knot K is the minimal $u_R(D)$ for all minimal crossing diagrams D of K [3]. We have $u_R(D) \leq c(D)/2+1$ for any reduced knot diagram D, and hence we have $u_R(K) \leq c(K)/2+1$ for any knot K, and we have $u_R(K) = m$ for the $(2, 4m\pm1)$-torus knot K $(m = 1, 2, \dots)$ [3].

We will discuss about region unknotting numbers of the standard diagrams of $(3, n)$-torus knots in Section 3.

The rest of this paper is organized as follows: In Section 2, we prove Theorem 1.2 and Theorem 1.3. In Section 3, we discuss about region unknotting numbers of closed 3-braid diagrams.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 after proving Theorem 1.3. The following lemmas are shown in [3]:

Lemma 2.1 ([3]). *For a reduced knot diagram D and the set B of all the black-colored regions of D with a checkerboard coloring, we obtain D from D by region crossing changes at B.*

Lemma 2.2 ([3]). *Let D be a reduced knot diagram, and let B be the set of all the black-colored regions of D with a checkerboard coloring. Let P be a subset of B. Then we obtain the same diagram from D by the region crossing changes at P and the region crossing changes at $B - P$.*

We prove Theorem 1.3.

Proof of Theorem 1.3. Let B (resp. W) be the set of all the black-colored (resp. white-colored) regions of D with a checkerboard coloring. If $R \in B$ (resp. $R \in W$), we can take any white-colored (resp. black-colored) region as S. By Lemma 2.2, the region crossing change at R is equivalent to the region crossing changes at $B - R$, and the region crossing change at S is equivalent to the region crossing changes at $W - S$. By Theorem 1.1, we can make any crossing change on D by region crossing changes at regions of D except R and S. □

From Theorem 1.3, we have the following corollaries:

Corollary 2.3. *Let D be a reduced knot diagram. For any two regions R and S of D which are adjacent to each other, we can make any crossing change on D by a sequence of region crossing changes except at R and S.*

Corollary 2.4. *Let T be a one-string tangle diagram. We can make any crossing change by a sequence of region crossing changes at regions of T except the outer region.*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. It is enough to show that for any knot diagram D on $\mathbb{R}^2$ and any crossing point c, we can make the crossing change at c by region crossing changes at regions of D except the outer region of D. If D is a knot diagram which has only one reducible crossing as c as shown in Figure 3, we can change the crossing at c by region crossing changes as follows: We splice D at c, and apply the checkerboard coloring to the knot diagram corresponding to A in Figure 3 so that the outer region of the knot diagram is colored white. Then, if we apply region crossing changes at all the regions of D corresponding to the black-colored regions, the crossing of only c is changed. This theorem also holds for reduced knot diagrams

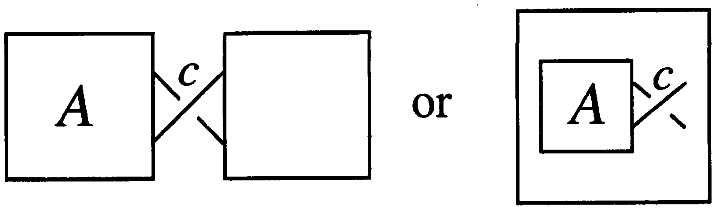

FIGURE 3

by Theorem 1.3. For other cases, we can prove by an induction on the number of reducible crossings as shown in Figure 4. □

3. REGION UNKNOTTING NUMBERS OF CLOSED 3-BRAID DIAGRAMS

In this section we discuss about region unknotting numbers of closed 3-braid diagrams. For standard diagrams of $(3, m)$-torus knots, we have the following proposition:

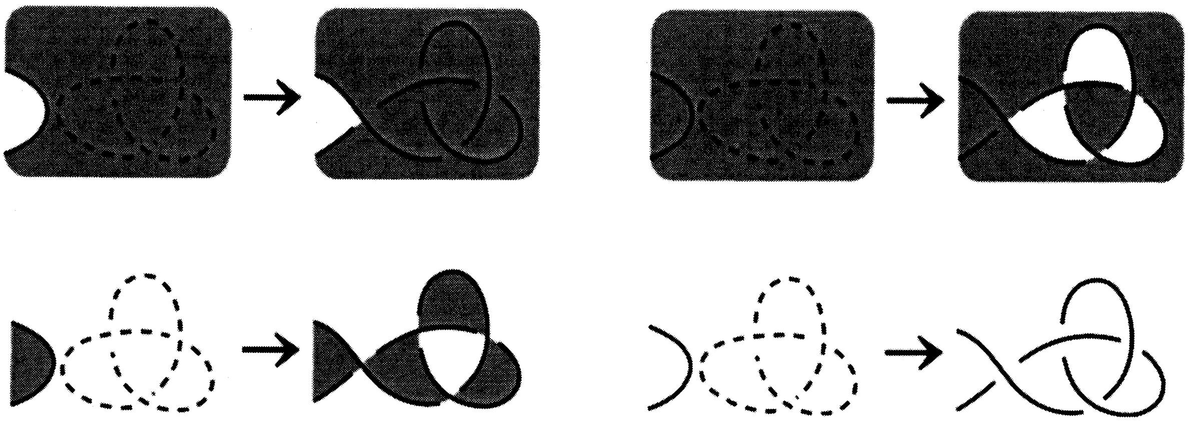

FIGURE 4

Proposition 3.1. *Let $D_{3,m}$ be the standard diagram of the $(3,m)$-torus link ($m = 1, 2, 3, \dots$). We have $u_R(D_{3,3n+1}) \leq n$ and $u_R(D_{3,3n+2}) \leq n+1$ ($n = 0, 1, 2, \dots$).*

Proof. We have $u_R(D_{3,1}) = 0$ and $u_R(D_{3,2}) = 1$. Since we can deform the braid diagram of $(\sigma_2\sigma_1)^3$ into a braid diagram which represents the trivial 3-braid by one region crossing change (see for example Figure 5), we have the inequalities. □

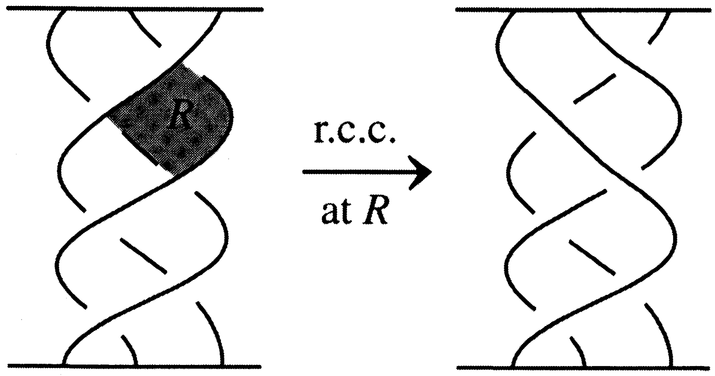

FIGURE 5

From Proposition 3.1, we have the following corollary:

Corollary 3.2. *The closed braid diagram of* $(\sigma_2^{-1}\sigma_1)^{3n+1}$ *has the region unknotting number less than or equal to* $n+1$*, and the closed braid diagram of* $(\sigma_2^{-1}\sigma_1)^{3n+2}$ *has the region unknotting number less than or equal to* $n+2$ $(n = 0, 1, 2, \dots)$.

Proof. We can obtain $D_{3,m}$ from the closed braid diagram of $(\sigma_2^{-1}\sigma_1)^m$ by one region crossing change (Figure 6). □

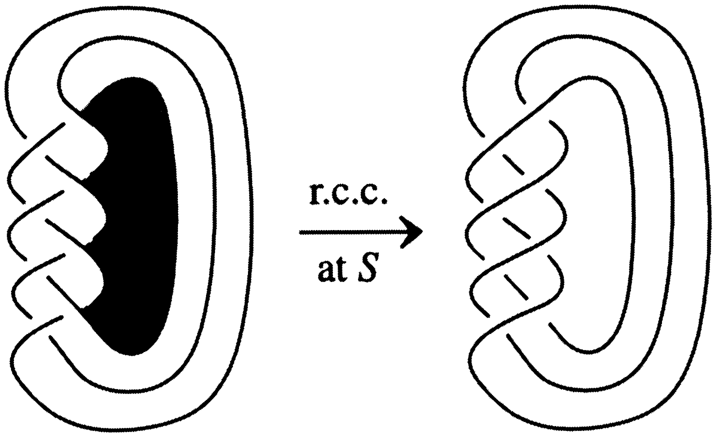

FIGURE 6

Remark. 3.3. Z. Cheng and H. Gao showed in [1] that a region crossing change on a diagram of a 3-component link such that the linking number of each two components is even is an unknotting operation. For example, a region crossing change on the closed braid diagram of $(\sigma_2^{-1}\sigma_1)^{3n}$ is an unknotting operation. As shown in Figure 5, we can obtain a trivial link diagram from $D_{3,3n}$ $(n = 0, 1, 2, \dots)$ by at most n region crossing changes, i.e., a region crossing change on $D_{3,3n}$ is also an unknotting operation.

For a 3-braid $\beta = \sigma_1^{n_1}\sigma_2^{n_2}\sigma_1^{n_3}\dots\sigma_2^{n_m}$, let β_1 and β_2 be the 3-braids defined to be $\beta_1 = \sigma_2^{-n_m}\dots\sigma_1^{-n_3}\sigma_2^{-n_2}\sigma_1^{-n_1}$ and $\beta_2 = \sigma_1^{-n_m}\dots\sigma_2^{-n_3}\sigma_1^{-n_2}\sigma_2^{-n_1}$ $(n_1, n_2, \dots n_m \in \mathbb{Z})$. K. Kishimoto pointed out that each closed 3-braid diagram of the following $A_1, A_2, \dots$ or B_3 can be deformed into a diagram of a trivial link by one region crossing change:

$$\begin{aligned}
A_1 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^3, \\
A_2 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^3\sigma_2^{-1}, \\
A_3 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^4, \\
B_1 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2\sigma_2^{-1}\sigma_1\sigma_2^{-1}, \\
B_2 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2(\sigma_2^{-1}\sigma_1)^2, \\
B_3 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2(\sigma_2^{-1}\sigma_1)^2\sigma_2^{-1},
\end{aligned}$$

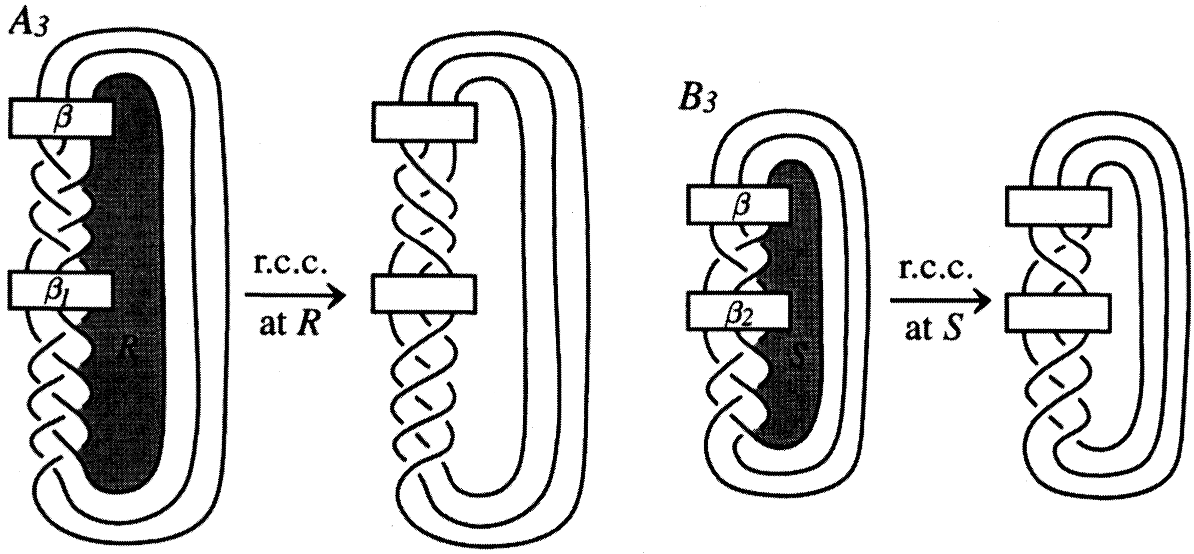

FIGURE 7

where β is a 3-braid, and A_3 and B_3 are illustrated in Figure 7.

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REFERENCES

[1] Z. Cheng and H. Gao: *On region crossing change and incidence matrix*, arXiv:1101.1129v2 (2011).
[2] A. Kawauchi: *A survey of knot theory*, Birkhauser, (1996).
[3] A. Shimizu: *Region crossing change is an unknotting operation*, preprint.

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