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The optimistic limit of the colored Jones polynomial and the volume calculation

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Abstract

We discuss how to calculate the complex volume of a hyperbolic knot using the optimistic limit of the colored Jones polynomial. This method is based on the colored Jones version of Yokota theory. This is a joint-work with Jun Murakami.

Let K be a hyperbolic knot, $\mathrm{vol}(K)$ be the hyperbolic volume of K and $\mathrm{cs}(K)$ the Chern-Simons invariant of the knot complement $\mathbb{S}^3\backslash K$ modulo π^2. We call $\mathrm{vol}(K) + i\,\mathrm{cs}(K)$ *the complex volume* of K. The purpose of this article is to survey the method of calculating the complex volume using the optimistic limit of the colored Jones polynomial in [3]. Note that this method relies on Yokota's theory of the optimistic limit of the Kashaev invariant in [11].

The calculation consists of the following four steps.

1 First step : drawing oriented (1,1) tangle

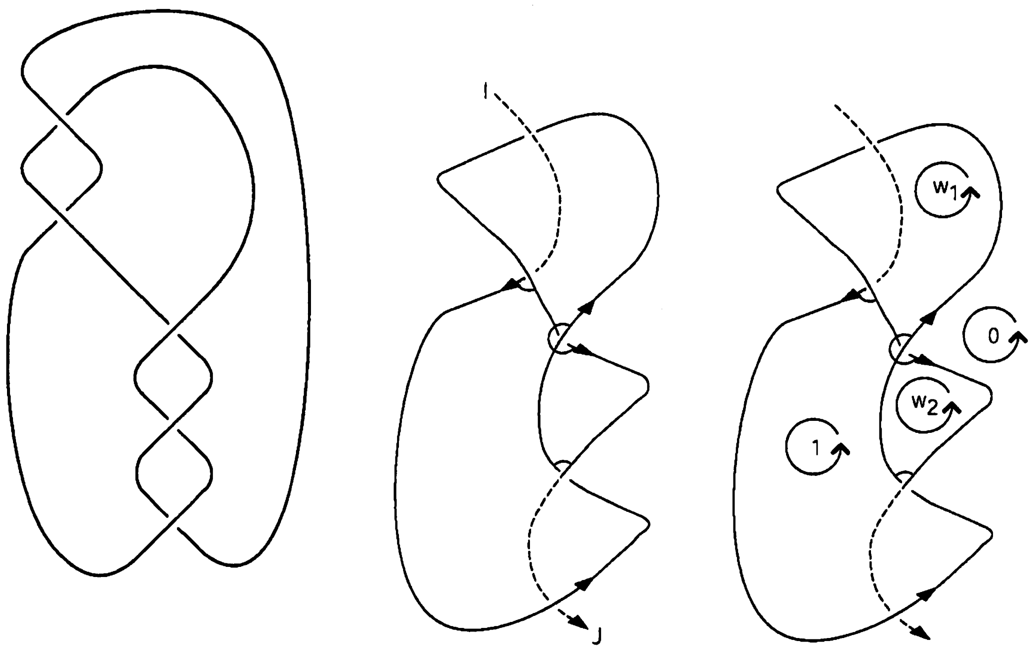
We fix a diagram D of K. We define *sides* of D as arcs connecting two adjacent crossing points. For example, Figure 1(a) has 10 sides.

Now we split a side of D open to make a (1,1)-tangle diagram. (See Figure 1(b).) We assume that the diagram satisfies Yokota's Assumptions 1–6 in [11]. This roughly means the diagram has no crossing points that can be reduced trivially. We can always deform any knot diagram to satisfy these assumptions.

Let the two open sides be I and J. Assume I and J are in an over-bridge and in an under-bridge respectively. Then extend I and J so that non-boundary endpoints of I and J become the first under-crossing point and the last over-crossing point respectively, as in Figure 1(b). Assume the two non-boundary endpoints of I and J do not coincide. Yokota proved in [11] that we can always choose I and J with these conditions because, if not, the knot should be the trefoil knot, which is not hyperbolic. Finally, we remove I and J from the tangle diagram and put the result G. We always assume G is oriented by the direction from I to J.

2 Second step : defining potential function

For each region of G, we assign 1 to one bounded region, variables $w_1, \ldots, w_n$ to the other bounded regions, and 0 to the unbounded region. (See Figure 1(c).) Then we draw circles



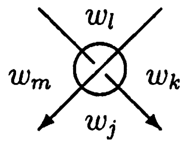
(a) 5_2 knot (b) Graph G (c) Parametrizing regions

Figure 1: Example of the 5_2 knot

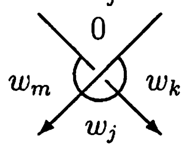
on vertices of G. If some arcs of the circles are in the unbounded region or meet $I \cup J$, then we remove these arcs.

For each vertex of G, we assign the following function according to the types of the vertex and the shape of the arcs. Note that $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt$ is the dilogarithm function.

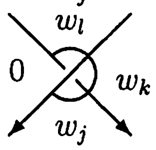
For positive crossings :



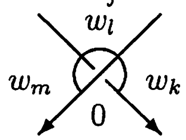
$: \mathrm{Li}_2(\frac{w_l}{w_m}) + \mathrm{Li}_2(\frac{w_l}{w_k}) - \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m}) - \mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_k}{w_j}) + \frac{\pi^2}{6} - \log\frac{w_m}{w_j}\log\frac{w_k}{w_j}$



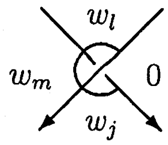
$: -\mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_k}{w_j}) + \frac{\pi^2}{6} - \log\frac{w_m}{w_j}\log\frac{w_k}{w_j}$



$: \mathrm{Li}_2(\frac{w_l}{w_k}) + \mathrm{Li}_2(\frac{w_j}{w_k}) - \frac{\pi^2}{6} + \log\frac{w_k}{w_l}\log\frac{w_k}{w_j}$

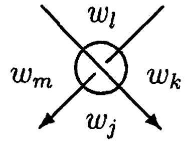


$: -\mathrm{Li}_2(\frac{w_m}{w_l}) - \mathrm{Li}_2(\frac{w_k}{w_l}) + \frac{\pi^2}{6} - \log\frac{w_m}{w_l}\log\frac{w_k}{w_l}$

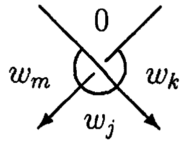


$: \mathrm{Li}_2(\frac{w_l}{w_m}) + \mathrm{Li}_2(\frac{w_j}{w_m}) - \frac{\pi^2}{6} + \log\frac{w_m}{w_l}\log\frac{w_m}{w_j}$

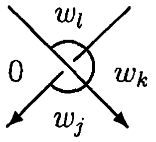
For negative crossings :



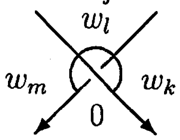
$: -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_l}{w_k}) + \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m}) + \mathrm{Li}_2(\frac{w_m}{w_j}) + \mathrm{Li}_2(\frac{w_k}{w_j}) - \frac{\pi^2}{6} + \log\frac{w_j}{w_m}\log\frac{w_j}{w_k}$



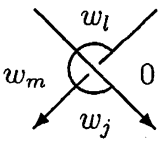
$: \mathrm{Li}_2(\frac{w_m}{w_j}) + \mathrm{Li}_2(\frac{w_k}{w_j}) - \frac{\pi^2}{6} + \log\frac{w_j}{w_m}\log\frac{w_j}{w_k}$



$: -\mathrm{Li}_2(\frac{w_l}{w_k}) - \mathrm{Li}_2(\frac{w_j}{w_k}) + \frac{\pi^2}{6} - \log\frac{w_l}{w_k}\log\frac{w_j}{w_k}$



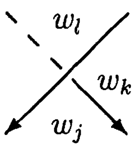
$: \mathrm{Li}_2(\frac{w_m}{w_l}) + \mathrm{Li}_2(\frac{w_k}{w_l}) - \frac{\pi^2}{6} + \log\frac{w_l}{w_m}\log\frac{w_l}{w_k}$



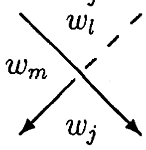
$: -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_j}{w_m}) + \frac{\pi^2}{6} - \log\frac{w_l}{w_m}\log\frac{w_j}{w_m}$

For the end points of I and J, we use the same formula whether some arcs of the circles are removed or not.

For the end point of I :

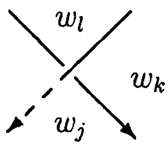


$: -\mathrm{Li}_2(\frac{w_k}{w_l}) + \mathrm{Li}_2(\frac{w_j}{w_l})$

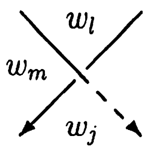


$: \mathrm{Li}_2(\frac{w_m}{w_l}) - \mathrm{Li}_2(\frac{w_j}{w_l})$

For the end point of J :



$: -\mathrm{Li}_2(\frac{w_k}{w_j}) + \mathrm{Li}_2(\frac{w_l}{w_j})$



: $\mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_l}{w_j})$

We define *the potential function* $W(w_1, \ldots, w_n)$ of the knot by the summation of all functions assigned to the vertices. (By the definition, this function depends on the diagram. Note that this potential function is the minus of the one defined in [3]. The sign is chosen to make the identity (3) hold.) For example, the potential function of the 5_2 knot from Figure 1(c) becomes

$$W(w_1, w_2) = \mathrm{Li}_2(w_1) - \mathrm{Li}_2(\frac{1}{w_1}) + 2\mathrm{Li}_2(w_2) + \log\frac{1}{w_1}\log\frac{1}{w_2} - \frac{\pi^2}{6}. \tag{1}$$

Although we do not discuss the optimistic limit of the colored Jones polynomial here, the definition of this potential function came from the R-matrix of the colored Jones polynomial in [6] and the optimistic limit. The exact relation can be found in [3].

3 Third step : finding essential solutions

For the potential function $W(w_1, \ldots, w_n)$, we define *hyperbolicity equation* $\mathcal{H}$ by

$$\mathcal{H} = \left\{ \exp\left(w_k \frac{\partial W(w_1, \ldots, w_n)}{\partial w_k} \right) = 1 \,\middle|\, k = 1, \ldots, n \right\}.$$

For example, the previous potential function (1) of the 5_2 knot determines the hyperbolicity equation

$$\mathcal{H} = \left\{ \frac{w_2}{(1-w_1)(1-\frac{1}{w1})} = 1, \frac{w_1}{(1-w_2)^2} = 1 \right\}. \tag{2}$$

We remark that the hyperbolicity equation of $\mathcal{H}$ is actually the hyperbolicity equation of the Thurston triangulation of the knot complement $\mathbb{S}^3 \backslash K$, which consists of the edge relation and the cusp condition of the triangulation. See [3] for the definition of Thurston triangulation and the relation of it with $\mathcal{H}$.

Among the solutions of $\mathcal{H}$, we need only the solutions satisfying that none of the variables inside the dilogarithm functions $\mathrm{Li}_2(*)$ of the potential function becomes 0, 1, or ∞. We call these solutions *essential solutions.* In the view of the Thurston triangulation, the essential solution means the corresponding Thurston triangulation has homotopically nontrivial edges.

For example of the previous 5_2 knot, the hyperbolicity equation has the three essential solutions

$$\begin{aligned}
(w_1^{(0)}, w_2^{(0)}) &:= (0.1226... - i\,0.7449..., 1.6624... - i\,0.5629...), \\
(\overline{w}_1^{(0)}, \overline{w}_2^{(0)}) &= (0.1226... + i\,0.7449..., 1.6624... + i\,0.5629...), \\
(w_1^{(1)}, w_2^{(1)}) &:= (1.7549..., -0.3247...).
\end{aligned}$$

If $\mathcal{H}$ has an essential solution, then we can go to the fourth step. If not, we cannot calculate the complex volume by this method.

For the actual calculation of an example, checking the existence of an essential solution is very simple by using computer. However, the existence problem in general case is not an easy problem. The rigorous result on the existence of an essential solution for the 2-bridge knots with the diagram of the Conway notation $C(a_1, \ldots, a_m)$ was recently done in [1] by proving the Sakuma-Weeks conjecture proposed in [8]. (See Section 4 of [2] and [7] for detailed discussions.)

It was proved in [9] that each essential solution $\mathbf{w}$ of the hyperbolicity equation $\mathcal{H}$ induces a parabolic representation $\rho_{\mathbf{w}} : \pi_1(K) \to \mathrm{PSL}(2, \mathbb{C})$. Furthermore, each representaion has the complex volume and it can be computed using the extended Bloch group theory in [12].

If the parabolic representation $\rho : \pi_1(K) \to \mathrm{PSL}(2, \mathbb{C})$ induces the complete hyperbolic structure of the knot complement $\mathbb{S}^3 \backslash K$, we call ρ *the geometric representation.* By the definition, the complex volume of the geometric representation is the complex volume of the hyperbolic knot K.

If K is the hyperbolic knot, there exists the geometric representation. However, we cannot guarantee this can be induced from an essential solution of $\mathcal{H}$. It was proved in Section 2.8 of [10] that if $\mathcal{H}$ has an essential solution, there exists unique solution of $\mathcal{H}$ that induces the geometric representation. We call the unique solution *geometric solution.* Therefore, if we assume the existence of an essential solution of $\mathcal{H}$, we can denote the geometric representation $\rho_{\mathbf{w}}$ for the geometric solution $\mathbf{w}$ of $\mathcal{H}$.

For the example of the 5_2 knot with the hyperbolicity equation $\mathcal{H}$ in (2), we will see the geometric solution becomes $(w_1^{(0)}, w_2^{(0)}) = (0.1226... - i\,0.7449..., 1.6624... - i\,0.5629...)$ in next section.

4 Fourth step : obtaining the complex volume by $i\,W_0$

For the potential function $W(w_1, \ldots, w_n)$, we define

$$W_0(w_1, \ldots, w_n) := W(w_1, \ldots, w_n) - \sum_{k=1}^{n} w_k \frac{\partial W(w_1, \ldots, w_n)}{\partial w_k}.$$

Then we evaluate all essential solutions of $\mathcal{H}$ to $i\,W_0$. According to Yokota's result in [11] and the result in [3], the evaluation becomes

$$i\,W_0(\mathbf{w}) \equiv \mathrm{vol}(\rho_{\mathbf{w}}) + i\,\mathrm{cs}(\rho_{\mathbf{w}}) \pmod{i\pi^2}. \tag{3}$$

(See also [2] for reference.)

On the other hand, it was proved in Corollary 5.10 of [5] that, for any representation $\rho : \pi_1(K) \to \mathrm{PSL}(2, \mathbb{C})$,

$$|\mathrm{vol}(\rho)| \le \mathrm{vol}(K). \tag{4}$$

Gromov-Thurston-Goldman rigidity in [4] implies that the equality of (4) holds if and only if ρ is the geometric representation. Therefore, after confirming the existence of an essential solution of $\mathcal{H}$, if we pick the solution $\mathbf{w}$ of $\mathcal{H}$ which gives the maximum value of the real part of $i\,W_0(\mathbf{w})$, the solution becomes the geometric solution and $i\,W_0(\mathbf{w})$ becomes the complex volume of the knot K.

For the example of the 5_2 knot with the potential function (1), the evaluations of the essential solutions become

$$\begin{aligned}
i\,W_0(w_1^{(0)}, w_2^{(0)}) &= 2.8281... \quad +i\,3.0241...,\\
i\,W_0(\overline{w}_1^{(0)}, \overline{w}_2^{(0)}) &= -2.8281... \quad +i\,3.0241...,\\
i\,W_0(w_1^{(1)}, w_2^{(1)}) &= 0 \qquad\quad -i\,1.1135....
\end{aligned}$$

The value with the maximun real part is $i\,W_0(w_1^{(0)}, w_2^{(0)}) = 2.8281... + i\,3.0241...$, so it is the complex volume of the 5_2 knot.

All these steps are combinatorial (except finding solutions and evaluation) and very easy to calculate. We believe this method can be useful for proving problems related to volumes and Chern-Simons invariant.

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