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ON THE TRACE FORMULA APPROACH TO DISCRETE SERIES MULTIPLICITIES

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ABSTRACT. Let G be a connected reductive group over $\mathbb{Q}$, and Γ an arithmetic subgroup of $G(\mathbb{Q})$. In this note we survey the trace formula method of Arthur for computing (sums of) multiplicities of discrete series representations in the spectrum of $L^2(\Gamma\backslash G(\mathbb{R}))$. His formula expresses these multiplicities as a combination of p-adic orbital integrals, and asymptotic values of discrete series characters. We write down the most basic example of GL_2 in some detail. However we omit many details for general G, giving references instead. We also state a conjecture which should refine Arthur's formula to give the multiplicities directly. We conclude with an amusing application of the conjecture, together with known multiplicity formulas, towards computing p-adic orbital integrals and associated distributions.

1. NOTATION

For G an algebraic group defined over a field F and $\gamma \in G(F)$, write G_γ for the centralizer in G of γ. If T is a torus in a reductive group G, write T_r for the set of regular elements of T. Write $\mathbb{A}$ for the adeles of $\mathbb{Q}$, write $\mathbb{A}_f$ for the finite adeles of $\mathbb{Q}$, and $\hat{\mathbb{Z}}$ for the integers in $\mathbb{A}_f$.

2. ASYMPTOTICS OF DISCRETE SERIES CHARACTERS

2.1. **Case of GL_2.** Let $G = \mathrm{GL}_2$. Write A for the subgroup of diagonal matrices in G. We identify the character group $X^*(A)$ with $\mathbb{Z}^2$ so that $(m,n) \in \mathbb{Z}^2$ corresponds to the character

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^m b^n.$$

We view the root $\alpha = (1,-1)$ as positive. Given $(m,n) \in X^*(A)$, with $m \geq n$, there is a unique irreducible finite-dimensional algebraic representation E of $G(\mathbb{C})$ whose highest weight for A is (m,n). In fact,

$$E = \mathrm{Sym}^{m-n} V \otimes \det{}^n, \tag{2.1}$$

where $V = \mathbb{C}^2$ is the standard representation of $G(\mathbb{C})$. Note that E has dimension $m-n+1$.

There is a unique discrete series representation $\pi = \pi_E$ with the same central character and infinitesimal character as E. Let us specify π by its Harish-Chandra character Θ_π, viewed as a function defined on regular semisimple elements $\gamma \in G(\mathbb{R})$. Then Θ_π is given by the following formulas:

If γ has complex eigenvalues $z \neq \overline{z}$, then

$$\Theta_\pi(\gamma) = \frac{z^n \overline{z}^{m+1} - z^{m+1}\overline{z}^n}{z - \overline{z}}. \tag{2.2}$$

If γ has real eigenvalues $\lambda_1 \neq \lambda_2$, with $|\lambda_1| \geq |\lambda_2|$, then

$$\Theta_\pi(\gamma) = \begin{cases} \dfrac{2\lambda_1^n \lambda_2^{m+1}}{|\lambda_1 - \lambda_2|}, \text{ if } \lambda_1\lambda_2 > 0 \\ 0, \text{ if } \lambda_1\lambda_2 < 0 \end{cases} . \tag{2.3}$$

Let γ be regular semisimple with nonreal eigenvalues. Write T for the centralizer in G of γ, and T_r for its regular elements. The restriction of Θ_π to $T_r(\mathbb{R})$ extends continuously to all of $T(\mathbb{R})$. Moreover, it extends to the union of all conjugates of $T(\mathbb{R})$. Write $\Phi_G(\gamma, \Theta_\pi)$ for this extension. For $a \in \mathbb{R}^\times$, we have

$$\Phi_G(aI, \Theta_\pi) = -a^{m+n}(m-n+1). \tag{2.4}$$

We record some more special values of $\Phi_G(\gamma, \Theta_\pi)$ for later use. Put

$$\gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\gamma_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us define two numerical functions

$$t_4(j) = \begin{cases} 0 & \text{if } j \text{ is even} \\ 1 & \text{if } j \equiv 1 \bmod 4 \\ -1 & \text{if } j \equiv 3 \bmod 4 \end{cases}$$

and

$$t_3(j) = \begin{cases} 0 & \text{if } j \equiv 0 \bmod 3 \\ 1 & \text{if } j \equiv 1 \bmod 3 \\ -1 & \text{if } j \equiv 2 \bmod 3. \end{cases}$$

Then one computes that $\Phi_G(\gamma_4, \Theta_\pi) = -t_4(m-n+1)$ and $\Phi_G(\gamma_3, \Theta_\pi) = -t_3(m-n+1)$.

On the other hand, we see from (2.3) that the restriction of Θ_π to $A_r(\mathbb{R})$ diverges as γ approaches the scalar matrices in $A(\mathbb{R})$. Nonetheless we can understand its asymptotic behavior in this direction. For γ as in (2.3), put

$$\begin{aligned} D^G(\gamma) &= \det(\mathrm{Ad}(\gamma) - 1; \mathrm{Lie}(G)/\mathrm{Lie}(A)) \\ &= -\frac{1}{\lambda_1\lambda_2}(\lambda_1 - \lambda_2)^2. \end{aligned} \tag{2.5}$$

Here we have written $\mathrm{Lie}(G)$ for the Lie algebra of G, and similarly for A. For $\gamma \in A_r(\mathbb{R})$ define

$$\Phi_A(\gamma, \Theta_\pi) = |D^G(\gamma)|^{\frac{1}{2}} \Theta_\pi(\gamma). \tag{2.6}$$

Explicitly,

$$\Phi_A\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \Theta_\pi\right) = \begin{cases} \dfrac{2\lambda_1^n \lambda_2^{m+1}}{\sqrt{\lambda_1\lambda_2}}, \text{ if } \lambda_1\lambda_2 > 0 \\ 0, \text{ if } \lambda_1\lambda_2 < 0 \end{cases} . \tag{2.7}$$

Note that $\Phi_A(\gamma,\Theta_\pi)$ extends continuously to $A(\mathbb{R})$. We also write $\Phi_A(\gamma,\Theta_\pi)$ for this extension. For $a\in\mathbb{R}^\times$, we have

$$\Phi_A(aI,\Theta_\pi)=2\operatorname{sgn}(a)\cdot a^{m+n}. \tag{2.8}$$

2.2. **General Case.** The functions Φ_A and Φ_G of the previous section are instances of a more general theory due to Arthur and Shelstad. Let G be a connected reductive group over $\mathbb{R}$. Write Z for the center of G. Let $K_\mathbb{R}$ be a maximal compact subgroup of $G(\mathbb{R})$, and put $X=G(\mathbb{R})/K_\mathbb{R}Z(\mathbb{R})$. Write $q(G)$ for half of the dimension of X.

Definition 1. *We say that a torus $T\subset G$ is elliptic if $T(\mathbb{R})/Z(\mathbb{R})$ is compact. An element $\gamma\in G(\mathbb{R})$ is elliptic in $G(\mathbb{R})$ if it is contained in an elliptic torus. We call G cuspidal if it has a maximal torus which is elliptic.*

Let G be cuspidal, and let E be an irreducible finite-dimensional (algebraic) representation of $G(\mathbb{C})$. There corresponds to E an "L-packet" $\Pi=\Pi_E$ of representations, comprised of discrete series representations with the same central character and infinitesimal character as E. For γ a regular semisimple element in $G(\mathbb{R})$, put

$$\Theta_\Pi(\gamma)=\sum_{\pi\in\Pi}\Theta_\pi(\gamma). \tag{2.9}$$

Let T be a maximal torus of G. Write M for the centralizer of the maximal split subtorus of T. Then M is is a Levi subgroup of G, and T is an elliptic maximal torus in M. For $\gamma\in T(\mathbb{R})$, put

$$D_M^G(\gamma)=\det(\operatorname{Ad}(\gamma)-1;\operatorname{Lie}(G)/\operatorname{Lie}(M)). \tag{2.10}$$

It is an important fact (see [1], [2]) that the function

$$\gamma\mapsto\left|D_M^G(\gamma)\right|^{\frac{1}{2}}\Theta_\Pi(\gamma)$$

on $T_r(\mathbb{R})$ extends continuously to $T(\mathbb{R})$. We denote the extension by $\Phi_M(\gamma,\Theta_\Pi)$. It is called Arthur's Φ-function.

For example, it is well-known that $\Phi_G(\gamma,\Theta_\Pi)=(-1)^{q(G)}\operatorname{tr}(\gamma;E)$. Many other cases are computed in [13], for instance if A is a split torus of G, then $\Phi_A(1,\Theta_\Pi)=|W|$.

The Harish-Chandra characters may also be viewed as distributions defined on suitable functions f on $G(\mathbb{R})$. One has an operator

$$\pi(f)=\int f(g)\pi(g)dg \tag{2.11}$$

on the space of π with a trace $\operatorname{tr}\pi(f)$. For an L-packet Π, we define

$$\operatorname{tr}\Pi(f)=\sum_{\pi\in\Pi}\operatorname{tr}\pi(f). \tag{2.12}$$

For later use, let us write

$$S\Phi'_M(f)=\sum_{\Pi}\Phi_M(\gamma^{-1},\Theta_\Pi)\operatorname{tr}\Pi(f), \tag{2.13}$$

the sum being over all discrete series L-packets of $G(\mathbb{R})$. (In this paper we are glossing over various standard normalizing factors; we use the prime to distinguish this distribution from the proper one in [6] and [14].)

3. Orbital Integrals, Constant Terms, and Endoscopy

Let G be a connected reductive group over $\mathbb{Q}$. For simplicity, we assume that G is split and defined over $\mathbb{Z}$. Put $K_0 = G(\hat{\mathbb{Z}})$. Pick a Borel subgroup $B = AN$ of G, with A a split torus; then $G(\mathbb{A}_f) = B(\mathbb{A}_f)K_0$.

One picks Haar measures dg_f on $G(\mathbb{A}_f)$ and dg_{f_γ} on centralizers $G_\gamma(\mathbb{A}_f)$ of semisimple elements of $G(\mathbb{A}_f)$. These and other Haar measures must be picked in a consistent fashion (using Tamagawa measures). This is explained carefully in [6] and [14], but for this survey we omit mention of these compatibility requirements. They will be specified for the example of GL_2 below.

For $h \in C_c^\infty(G(\mathbb{A}_f))$, and $\gamma \in G(\mathbb{A}_f)$, the orbital integral $O_\gamma^G(h) = O_\gamma(h)$ is defined by

$$O_\gamma(h) = \int_{G_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} h(g^{-1}\gamma g)\frac{dg_f}{dg_{f_\gamma}}.$$

Given an element $\gamma \in G(\mathbb{A}_f)$, write $SO_\gamma(h)$ for the stable orbital integral given by

$$SO_\gamma(h) = \sum_{\gamma'} O_\gamma(h). \tag{3.1}$$

Here the sum is taken over conjugacy classes of elements $\gamma' \in G(\mathbb{A}_f)$ with γ'_v stably conjugate to γ_v for each place v. (The definition of "stably conjugate" may be found in [7]. When the derived group of G is simply connected, two elements of $G(\mathbb{Q}_v)$ are stably conjugate if and only if they are conjugate in $G(\overline{\mathbb{Q}}_v)$.)

Definition 2. *Let M be a Levi component of a parabolic subgroup P of G and $h \in C_c^\infty(G(\mathbb{A}_f))$. Then the "$M$-constant term" of h is the function $h_M \in C_c^\infty(M(\mathbb{A}_f))$ defined via*

$$h_M(m) = \delta_P^{-\frac{1}{2}}(m)\int_{N(\mathbb{A}_f)}\int_{K_0} h(k^{-1}nmk)dkdn.$$

Here δ_P is the modulus function on $P(\mathbb{A}_f)$.

The constant term provides a "matching function" on $M(\mathbb{A}_f)$ in the sense that, if $\gamma \in M(\mathbb{A}_f)$ is regular semisimple, then

$$|D_M^G(\gamma)|^{\frac{1}{2}}O_\gamma^G(h) = O_\gamma^M(h_M). \tag{3.2}$$

Given an endoscopic group H for G, a local field F, and a nice function φ on $G(F)$, one has a "matching function" φ^H on $H(F)$, and "transfer factors" $\Delta(\gamma_H, \gamma)$ for regular semisimple $\gamma \in G(F)$ and $\gamma_H \in H(F)$ satisfying

$$SO'_{\gamma_H}(\varphi^H) = \sum_\gamma \Delta(\gamma_H, \gamma)O_\gamma(\varphi).$$

See, for instance ([11], [8], [10]). This is a large topic for which we have no space, but the basic idea is that one uses these identities to express an unstable distribution on $G(F)$ in terms of stable distributions on the groups $H(F)$. The function φ^H is not unique, but the difference of any two choices must be in the kernel of any stable distribution. For a function $f = \prod_v f_v$ on $G(\mathbb{A})$, we write f^H for a product $\prod_v f_v^H$ of matching functions f_v^H for f_v.

Given a measure dg_f on $G(\mathbb{A}_f)$, and a compact open subgroup K of $G(\mathbb{A}_f)$, we define the normalized characteristic function e_K of K to be the product of $\mathrm{vol}_{dg_f}(K)^{-1}$ with the characteristic function of K. Let M be a Levi subgroup of G, and put $K_M = M(\hat{\mathbb{Z}})$. We have $(e_{K_0})_M = e_{K_M}$. By the Fundamental Lemma [10], we obtain $(e_{K_0})^H = e_{K_H}$, where $K_H = H(\hat{\mathbb{Z}})$ (assuming H is split and defined over $\mathbb{Z}$).

4. Classical Modular Forms: The Case of GL_2

4.1. **Dimensions of Cusp Forms.** Let $G = \mathrm{GL}_2$, and $\Gamma \subset G(\mathbb{R})$ be an arithmetic subgroup. Write $S_k(\Gamma)$ for the usual space of cusp forms of weight k and level Γ on the upper half plane. In this section we express $\dim_{\mathbb{C}} S_k(\Gamma)$ for $k > 2$ in terms of orbital integrals and Arthur's Φ-function.

In the case where $\Gamma = \Gamma_0 = \mathrm{GL}_2(\mathbb{Z})$, it is well-known that $\dim_{\mathbb{C}} S_k(\Gamma)$ is 0 if $k = 2$, and given by the formula

$$\dim_{\mathbb{C}} S_k(\Gamma) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \left[\frac{k}{12}\right] & \text{if } k \text{ is even and } k \not\equiv 2 \mod 12 \\ \left[\frac{k}{12}\right] - 1 & \text{if } k \text{ is even and } k \equiv 2 \mod 12. \end{cases} \tag{4.1}$$

when $k \neq 2$. (Here $[x]$ denotes the greatest integer less than or equal to x.)

4.2. **Multiplicity in terms of orbital integrals.** Returning to the case of general Γ, let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup with that $K \cap G(\mathbb{Q}) = \Gamma$. Let $k \geq 2$ and take $E = \mathrm{Sym}^{k-2} V$ for our representation of $\mathrm{GL}_2(\mathbb{C})$. We now define constants $a(\gamma, M)$ for each pair (γ, M), where $M = A$ or G and $\gamma \in M(\mathbb{Q})$ is elliptic in $M(\mathbb{R})$. For all $\gamma \in A$ put $a(\gamma, A) = \dfrac{1}{8}$. If γ is central in G put $a(\gamma, G) = \dfrac{1}{24}$. Finally, suppose that $\gamma \in G(\mathbb{Q})$ has eigenvalues which generate an imaginary quadratic extension F of $\mathbb{Q}$. In this case, put

$$a(\gamma, G) = |\mathrm{cl}(F)| \left|\mathcal{O}_F^\times\right|^{-1},$$

where $\mathrm{cl}(F)$ is the class group of F and $\mathcal{O}_F^\times$ is the group of units in the integers of F. For a Levi subgroup M of G and a function $h \in C_c^\infty(M(\mathbb{A}_f))$ we set

$$T_g(h, M) = \sum_\gamma a(\gamma, M) O_\gamma^M(h) \Phi_M(\gamma^{-1}, \Theta^E). \tag{4.2}$$

The sum runs over conjugacy classes of elements $\gamma \in G(\mathbb{Q})$ which are elliptic in $G(\mathbb{R})$. The orbital integrals O_γ^G are defined with the measure on $G(\mathbb{A}_f)$ giving K_0 mass one, and with the measure on $G_\gamma(\mathbb{A}_f)$ giving $K_0 \cap G_\gamma(\mathbb{A}_f)$ mass one. We let the trivial orbital integrals O_γ^A be simply given by $O_\gamma^A(h_A) = h_A(\gamma)$.

Finally, we put

$$T_g(h) = T_g(h, G) + T_g(h_A, A). \tag{4.3}$$

The following can be extracted from Arthur [1]. We will discuss his more general formula in the next section.

Proposition 1. *Let $k \geq 2$. Write e_K for the characteristic function of K. Then*

$$T_g(e_K) = \dim_{\mathbb{C}} S_k(\Gamma). \tag{4.4}$$

For $\gamma \in M(\mathbb{Q})$ as above, put

$$T_g(h, \gamma, M) = a(\gamma, M) O^M_\gamma(h) \Phi_M(\gamma^{-1}, \Theta_\pi),$$

so that

$$T_g(h) = \sum_{\gamma, M} T_g(h_M, \gamma, M).$$

As a warm-up, let's treat the case of odd $k \geq 3$.

Corollary 1. *$S_k(\Gamma) = \{0\}$ when k is odd and $-1 \in \Gamma$.*

Proof. Consider a pair (γ, M) with γ being $\mathbb{R}$-elliptic in $M(\mathbb{Q})$.

Since $-1 \in K$ we have $h_M(-m) = h_M(m)$ for all $m \in M(\mathbb{A}_f)$, and so

$$O^M_{-\gamma}(h_M) = O^M_\gamma(h_M).$$

Moreover note that $a(-\gamma, M) = a(\gamma, M)$.

But we also have

$$\Phi_M(-\gamma, \Theta_\pi) = (-1)^k \Phi_M(\gamma, \Theta_\pi).$$

It follows that $T_g(h, -\gamma, M) = (-1)^k T_g(h, \gamma, M)$, whence the result. □

4.3. **Verification.** Our next goal is to recover the dimension formula (4.1) in the case of $K = K_0$. In view of the corollary, we assume k is even. Let $n = k - 1 = \dim_{\mathbb{C}} E$. We put $h = e_{K_0}$.

The central elements of $K_0 \cap G(\mathbb{Q})$ are merely ± 1, and so we obtain

$$T_g(h, \pm 1, G) = \frac{n}{24}. \tag{4.5}$$

Let $\gamma \in G$ be $\mathbb{R}$-elliptic regular semisimple. The condition that $\gamma \in G(\mathbb{Q}) \cap K_0$ implies that E is either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

There is a unique $G(\mathbb{Q})$-conjugacy class in $G(\mathbb{Z})$ whose eigenvalues generate $\mathbb{Q}(\sqrt{-1})$, given by

$$\gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have $a(\gamma_4, G) = \frac{1}{4}$. The orbital integral $O_{\gamma_4}(h) = 1$; we refer to [5] for the computation of the orbital integrals which we need.

Thus we obtain

$$T_g(h, \gamma_4, G) = -\frac{1}{4} t_4(n).$$

There are two $G(\mathbb{Q})$-conjugacy classes in $G(\mathbb{Z})$ whose eigenvalues generate $\mathbb{Q}(\sqrt{-3})$, given by

$$\gamma_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

and its negative $-\gamma_3$.

Similarly, $a(\gamma_4, G) = \frac{1}{6}$ and $O_{\gamma_3}(h) = 1$, and so

$$T_g(h, \pm\gamma_3, G) = -\frac{1}{6} t_3(n).$$

Since $K_A \cap \mathrm{GL}_2(\mathbb{Q}) = \{\pm 1\}$, we have

$$T_g(h_A, \pm 1, A) = -\frac{1}{4}.$$

Thus

$$T_g(h) = \frac{n}{12} - \frac{1}{4}t_4(n) - \frac{1}{3}t_3(n) - \frac{1}{2}.$$

It is elementary to check that this is equal to (4.1). Note that for $n = 1$ one has $T_g(h) = -1$. This is the case for which E is the trivial representation; its highest weight is certainly not regular.

5. More General Multiplicity Formulas

The problem of computing dimensions of spaces of cusp forms (resp., traces of Hecke operators on them) is part of computing the discrete spectrum of $X = \Gamma\backslash G(\mathbb{R})$. To be more specific, write R for the representation defined by the action of $G(\mathbb{R})$ on $L^2(X)$. Then R decomposes as

$$R = R_{\mathrm{disc}} \oplus R_{\mathrm{cont}},$$

where R_{disc} is a direct sum of irreducible representations, and R_{cont} decomposes continuously. Given an irreducible representation π of $G(\mathbb{R})$, write $R_{\mathrm{disc}}(\pi)$ for the π-isotypic subspace of R_{disc}. Thus,

$$R_{\mathrm{disc}}(\pi) \cong \pi^{\oplus m_{\mathrm{disc}}(\pi)}$$

for some integer $m_{\mathrm{disc}}(\pi, \Gamma)$.

For $G = \mathrm{GL}_2$, and $k \geq 2$, one has $\dim_{\mathbb{C}} S_k(\Gamma) = m_{\mathrm{disc}}(\pi, \Gamma)$, where $\pi \in \Pi_E$ and $E = \mathrm{Sym}^{k-2} V$. However this case is very special. If $n > 2$ there are no discrete series representations of $\mathrm{GL}_n(\mathbb{R})$. If one considers other reductive groups G besides GL_n and its affiliates, then stably conjugate elements in $G(\mathbb{R})$ are not typically conjugate. Indeed, the partitioning of representations into packets is the spectral analogue of grouping together conjugacy classes into a stable conjugacy class. Stably invariant distributions are much simpler than unstable ones. It is significantly easier to determine the sum

$$m_{\mathrm{disc}}(\Pi, \Gamma) = \sum_{\pi \in \Pi} m_{\mathrm{disc}}(\pi, \Gamma)$$

of multiplicities, rather than the individual $m_{\mathrm{disc}}(\pi, \Gamma)$.

5.1. **Arthur's Formula.** Fix an L-packet Π. For a Levi subgroup M of G and a function $\varphi \in C_c^\infty(M(\mathbb{A}_f))$ we put

$$T_g(\varphi, M) = \sum_{\gamma} a(\gamma, M) O_\gamma^M(\varphi) \Phi_M(\gamma^{-1}, \Theta_\Pi). \tag{5.1}$$

The constants $a(\gamma, M)$ are volume-related, and straightforward but subtle. In the notation of [1], they are given by

$$a(\gamma, M) = (-1)^{\dim(A_M/A_G)} (n_M^G)^{-1} \iota^M(\gamma)^{-1} \tau(M_\gamma) v(M_\gamma)^{-1}. \tag{5.2}$$

For $h \in C_c^\infty(G(\mathbb{A}_f))$, put

$$T_g(h) = \sum_M T_g(h_M, M). \tag{5.3}$$

We say that a discrete series L-packet Π_E is regular if the highest weight of E is regular. A discrete series π is regular if it is in a regular L-packet. Suppose that Γ is an arithmetic subgroup of $G(\mathbb{Q})$, and K is an open compact subgroup of $G(\mathbb{A}_f)$ so that $K \cap G(\mathbb{Q}) = \Gamma$.

Theorem 1. *(Arthur [1]) Suppose Π is a regular L-packet. Then*

$$T_g(e_K) = m_{\text{disc}}(\Pi, \Gamma). \tag{5.4}$$

Note that this does not yield the individual multiplicities $m_{\text{disc}}(\pi, \Gamma)$, but only their average over Π_E. Since for $G = \mathrm{GL}_2$ the packets are singletons, we obtain Proposition 1. Actually, this formula generalizes to compute (stable) traces of Hecke operators.

Remark: When making the volume computations for $a(\gamma, M)O_\gamma^M(h_M)$ it is prudent to group together the product $v(M_\gamma)^{-1} \cdot \mathrm{vol}(M_\gamma(\mathbb{A}_f) \cap K_0)^{-1}$. Such product are closely related to Euler characteristics of certain symmetric space attached to $M_\gamma(\mathbb{R})$. When G is semisimple and simply connected, one may compute these using Harder's formula in [3]. We have extended his formula to reductive groups (satisfying some mild hypotheses) in [14].

Arthur's trace formula gives an equality of distributions, one geometric and one spectral. In this application, the spectral side becomes the (stable) multiplicities of the discrete series representations that we are considering, and the geometric side gives $T_g(h)$. These expressions are obtained by evaluating the trace formula distributions at functions f on $G(\mathbb{A})$ of the form $f = h \cdot f_\infty$. Here f_∞ a function on $G(\mathbb{R})$ which is stable cuspidal, meaning that f_∞ is a linear combination of pseudocoefficients of an L-packet. In fact, Arthur uses

$$f_\Pi = \sum_{\pi \in \Pi_E} e_\pi, \tag{5.5}$$

where e_π is the product of $(-1)^{q(G)}$ with a pseudocoefficient for the contragredient of the discrete series representation π.

5.2. **Kottwitz's Formula.** In [6], Kottwitz stabilized Arthur's formula, at least for functions which were stable cuspidal at the real place. In other words, he rewrote Arthur's geometric expression as a sum of stable distributions on endoscopic groups for G.

For $f = f^\infty f_\infty$ with $f^\infty \in C_c^\infty(G(\mathbb{A}_f))$ and f_∞ stable cuspidal on $G(\mathbb{R})$, put

$$\mathcal{K}(f) = \sum_H \iota(G, H) ST_g^H(f^{\infty H} f_\infty^H), \tag{5.6}$$

with the sum running over the (elliptic) endoscopic groups for G. The function $f^{\infty H} \in C_c^\infty(H(\mathbb{A}_f))$ is an endoscopic transfer of f^∞. The constant $\iota(G, H)$ is given by

$$\iota(G, H) = \tau(G)\tau(H)^{-1} |\operatorname{Out}(H)|^{-1}, \tag{5.7}$$

where τ denotes the Tamagawa number, and $\operatorname{Out}(H)$ is the group of outer automorphisms of H (or more properly, of the endoscopic data accompanying H).

Remark: Because of the tradition of writing H for endoscopic groups of G, we now favor the notation f^∞ rather than h.

Let us now discuss the distributions ST_g^H. For $H = G$ one has

$$ST_g^G(f^\infty f_\infty) = \sum_{\gamma, M} k(\gamma, M) SO_\gamma(f^\infty) S\Phi'_M(f_\infty). \tag{5.8}$$

Here the sum runs over cuspidal Levi subgroups M of G, and stable conjugacy classes of elliptic elements $\gamma \in M(\mathbb{Q})$.

The constants $k(\gamma, M)$ are again volume-related and nonzero, being a stable variant of $a(\gamma, M)$. We omit the formula; $k(\gamma, M)$ only depends on γ and M. As with the constants $a(\gamma, M)$, one makes volume computations using Harder's formula and its generalization in [14]. For other H, the formula is the same, replacing G with H throughout. Here is the main result of [6]:

Theorem 2. *If $f = f^\infty f_\Pi$ as above, then*

$$T_g(f^\infty) = \mathcal{K}(f^\infty f_\Pi). \tag{5.9}$$

We have studied the distribution $\mathcal{K}$ evaluated instead on functions of the form $f = f^\infty e_\pi$. It is much easier to substitute these functions into Kottwitz's formula than into Arthur's formula, because only (unweighted) semisimple orbital integals are involved.

Conjecture 1. *Let π be a regular discrete series representation of $G(\mathbb{R})$, and Γ an arithmetic subgroup of $G(\mathbb{Q})$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ so that $K \cap G(\mathbb{Q}) = \Gamma$. Write e_K for its normalized characteristic function. Then*

$$\mathcal{K}(e_K e_\pi) = m_{\text{disc}}(\pi, \Gamma). \tag{5.10}$$

Note that, by Theorems 1 and 2, we already have

$$\sum_{\pi \in \Pi} \mathcal{K}(e_K e_\pi) = \sum_{\pi \in \Pi} m_{\text{disc}}(\pi, \Gamma) \tag{5.11}$$

for any L-packet Π.

6. Case of GSp_4

In this section we would like to show evidence which led us to the conjecture. Details may be found in [14]. Earlier we had evaluated the identity terms of $\mathcal{K}(e_K e_\pi)$ for the group $G = \mathrm{SO}_5$ and $\Gamma = G(\mathbb{Z})$ as part of [12]. Then Wakatsuki noted that the resulting expressions matched up with the terms in his formulas for $m_{\text{disc}}(\pi, \Gamma)$ which corresponded to unipotent elements (although his computations were for the 4×4 symplectic group). Moreover, as we will see, the contribution to $\mathcal{K}(e_{K_0} e_\pi)$ from the endoscopic group accounted for the difference in these multiplicity formulas, while the stable part corresponded to the sum. We then redid the computations, which we now discuss, for $G = \mathrm{GSp}_4$ and $\Gamma = G(\mathbb{Z})$. Write A for the group of diagonal matrices in G. As usual, let $K_0 = G(\hat{\mathbb{Z}})$ and $f^\infty = e_{K_0}$, so that $f = f^\infty e_\pi$.

There are two elliptic endoscopic groups for G. One has G itself, and the group H defined by

$$1 \to \mathbb{G}_m \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to H \to 1.$$

Here $t \mapsto tI \times t^{-1}I$.

6.1. **Discrete Series for G.** We need to specify our discrete series packets. There are three approaches here. One could use finite-dimensional representations E, which are themselves parametrized by their highest weights. Alternatively, one could specify the discrete series themselves by their minimal $K_{\mathbb{R}}$-types, via Harish-Chandra parameters. Instead of these, we prefer to use Langlands parameters, because then the endoscopic transfer from $H(\mathbb{R})$ to $G(\mathbb{R})$ is defined in terms of these.

Write $W_{\mathbb{R}}$ for the Weil group of $\mathbb{R}$, and $W_{\mathbb{C}}$ for the canonical image of $\mathbb{C}^\times$ in $W_{\mathbb{R}}$. There is an exact sequence

$$1 \to W_{\mathbb{C}} \to W_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \to 1.$$

The Weil group $W_{\mathbb{R}}$ is generated by $W_{\mathbb{C}}$ and a fixed element τ satisfying $\tau^2 = -1$ and $\tau z \tau^{-1} = \overline{z}$ for $z \in W_{\mathbb{C}}$. Write ${}^LG$ for the L-group of G, which in this case is $\mathrm{GSp}_4(\mathbb{C}) \times W_{\mathbb{R}}$.

Let a, b be odd integers with $a > b > 0$. Let t be an even integer. For $z \in \mathbb{C}^\times$, let $\theta(z) = \dfrac{z}{|z|}$. Our Langlands parameter $\varphi_G : W_{\mathbb{R}} \to {}^LG$ is given by

$$\varphi_G(z) = |z|^t \begin{pmatrix} \theta(z)^a & & & \\ & \theta(z)^b & & \\ & & \theta(z)^{-b} & \\ & & & \theta(z)^{-a} \end{pmatrix} \times z,$$

and $\varphi_G(\tau) = J \times \tau$.

The L-packet defined by this parameter corresponds to the finite-dimensional representation E of $G(\mathbb{C})$ whose highest weight is the character on A given by

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \mapsto \lambda_1^{\frac{1}{2}(a+b-4)} \lambda_2^{\frac{1}{2}(t-a+3)} \lambda_4^{\frac{1}{2}(t-b+1)}. \tag{6.1}$$

Then $\Pi_E = \{\pi^{\mathrm{Hol}}, \pi^{\mathrm{Lar}}\}$. Here π^{Hol} is the "holomorphic discrete series", and π^{Lar} is the "large discrete series".

6.2. **Transfer from H to G.** Next we set up the Langlands parameters for discrete series representations of $H(\mathbb{R})$, and describe how they transfer to L-packets for $G(\mathbb{R})$. Here the dual group to H is $\hat{H} = \{(g, h) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \mid \det(g) = \det(h)\}$, and ${}^LH = \hat{H} \times W_{\mathbb{R}}$.

Define the Langlands parameter $\varphi_H : W_{\mathbb{R}} \to {}^LH$ by

$$\varphi_H(z) = |z|^t \begin{pmatrix} \theta(z)^a & \\ & \theta(z)^{-a} \end{pmatrix} \times |z|^t \begin{pmatrix} \theta(z)^b & \\ & \theta(z)^{-b} \end{pmatrix} \times z$$

for $z \in W_{\mathbb{C}}$, and

$$\varphi_H(\tau) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \times \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \times \tau.$$

The L-packet associated to φ_H is a singleton $\{\pi_H\}$.

There is another Langlands parameter φ'_H given by switching the first two factors, i.e.,

$$\varphi'_H(z) = |z|^t \begin{pmatrix} \theta(z)^b & \\ & \theta(z)^{-b} \end{pmatrix} \times |z|^t \begin{pmatrix} \theta(z)^a & \\ & \theta(z)^{-a} \end{pmatrix} \times z,$$

and with $\varphi'_H(\tau) = \varphi_H(\tau)$ as above. The new L-packet is another singleton $\{\pi'_H\}$. In fact both L-packets $\{\pi_H\}$ and $\{\pi'_H\}$ transfer to Π_E, and we may take $e^H_{\pi^{\mathrm{Hol}}} = e_{\pi_H} + e_{\pi'_H}$ and $e^H_{\pi^{\mathrm{Lar}}} = -e^H_{\pi^{\mathrm{Hol}}}$.

Remark: Actually, the theory of transfer depends on some intricate calibrations which we do not display here. In any case, the first equality is true up to a sign, and the second is true regardless of the convention.

6.3. **Unipotent terms.** Let us write $H^{\mathrm{Hol}}_{1,\Gamma}$ and $H^{\mathrm{Lar}}_{1,\Gamma}$ for the contribution to each multiplicity from unipotent γ. These were computed by Wakatsuki [18] to be

$$H^{\mathrm{Hol}}_1 = 2^{-9}3^{-3}5^{-1}ab(a-b)(a+b) - 2^{-5}3^{-2}ab + 2^{-4}3^{-1}b$$

and

$$H^{\mathrm{Lar}}_1 = 2^{-9}3^{-3}5^{-1}ab(a-b)(a+b) + 2^{-5}3^{-2}ab - 2^{-3}3^{-1}b + 2^{-2}.$$

(To translate from his notation to ours, use $j = b-1$ and $k = \frac{1}{2}(a-b)+2$.)

Now consider the contribution to $\mathcal{K}(e_{K_0}e_\pi)$ from the central terms (STAB) of $G(\mathbb{Q})$ and the central terms (UNSTAB) of $H(\mathbb{Q})$. Here $\pi = \pi^{\mathrm{Hol}}$. We have

$$\begin{aligned}
(\mathrm{STAB}) &= \sum_{z,M} ST_g(f,z,M) \\
&= 2^{-9}3^{-3}5^{-1}ab(a+b)(a-b) - 2^{-4}3^{-1}(a-b) - 2^{-4}3^{-1}b + 2^{-3}, \\
(\mathrm{UNSTAB}) &= \iota(G,H) \sum_{z,M_H} ST_g(f^H,z,M_H) \\
&= -2^{-5}3^{-2}ab + 2^{-4}3^{-1}(a+b) - 2^{-3}.
\end{aligned}$$

We find that

$$(\mathrm{STAB}) + (\mathrm{UNSTAB}) = H^{\mathrm{Hol}}_{1,\Gamma},$$

and

$$(\mathrm{STAB}) - (\mathrm{UNSTAB}) = H^{\mathrm{Lar}}_{1,\Gamma}.$$

Thus the central-unipotent terms agree, as predicted by the conjecture, for $\pi = \pi^{\mathrm{Hol}}$. In view of the relation $e^H_{\pi^{\mathrm{Lar}}} = -e^H_{\pi^{\mathrm{Hol}}}$, the central-unipotent terms agree for $\pi = \pi^{\mathrm{Lar}}$ as well.

7. Application to Harmonic Analysis

Of the ingredients in the formula, the most troublesome to compute are the orbital integrals of the various $f^{\infty H}$ and their constant terms. The constants $k(\gamma, M)$ and $\Phi(\gamma, \Theta_\Pi)$ are comparatively straightforward. When $\Gamma = G(\hat{\mathbb{Z}})$, one may use the Fundamental Lemma to find $f^{\infty H}$ explicitly. For parahoric Γ one may use [4], but beyond that not much is known.

On the other hand, the multiplicity $m_{\mathrm{disc}}(\pi^{\mathrm{Hol}}, \Gamma)$ is equal to the dimension of a space of (vector-valued) Siegel cusp forms, and these have been extensively tabulated. For instance, when $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$ and $G = \mathrm{Sp}_4$, the dimensions of these spaces of cusp forms were calculated in Tsushima [15], [16] using the Riemann-Roch-Hirzebruch formula, and later in Wakatsuki [17] using the Selberg trace formula and the theory of prehomogeneous vector spaces. Many cases of Γ are treated. Wakatsuki then evaluated Arthur's formula in [18] to compute

$$m_{\text{disc}}(\pi^{\text{Hol}}, \Gamma) + m_{\text{disc}}(\pi^{\text{Lar}}, \Gamma)$$

in each case, thereby deducing formulas for $m_{\text{disc}}(\pi^{\text{Lar}}, \Gamma)$ as well.

This preponderance of formulas for the 4×4 symplectic groups suggests another point of view on the matter: to approach the conjecture as a method to calculate these orbital integrals and related quantities.

To illustrate this, we return to $G = \mathrm{GL}_2$ and the discrete series π from Section 4. This time fix a prime p and let $\Gamma = \Gamma_p$ be the matrices in $\mathrm{GL}_2(\mathbb{Z})$ whose $(2,1)$-entry is a multiple of p. Let K_p be the group of matrices in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ whose $(2,1)$-entry is a multiple of p, and write $h = e_{K_p}$. Recall that Arthur's formula gives

$$T_g(h) = \frac{n}{12} O_1(h) + \frac{1}{4} t_4(n) O_{\gamma_4}(h) + \frac{1}{3} t_3(n) O_{\gamma_3}(h) - \frac{1}{2} h_A(1). \tag{7.1}$$

On the other hand it is known (see [9]) that, when $k > 1$,

$$m_{\text{disc}}(\pi, \Gamma_p) = \frac{n}{12}(p+1) + \frac{1}{4}\left(1 + \left(\frac{-1}{p}\right)\right) \cdot t_4(n) + \frac{1}{3}\left(1 + \left(\frac{-3}{p}\right)\right) \cdot t_3(n) - 1. \tag{7.2}$$

Since the functions $\{1, k, t_3(k), t_4(k)\}$ of k are linearly independent it follows that

$$O_1(h) = p + 1, O_{\gamma_4}(h) = 1 + \left(\frac{-1}{p}\right), O_{\gamma_3}(h) = 1 + \left(\frac{-3}{p}\right), \text{ and } h_A(1) = 2. \tag{7.3}$$

We thus regard the many intricate multiplicity formulas as encoding values of orbital integrals of normalized characteristic functions, their matching functions, and their constant terms. We believe it is a worthwhile project to express these multiplicity formulas directly as such.

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