数理解析研究所講究録
第 1767 巻 2011 年 29-43

p-ADIC SIEGEL-EISENSTEIN SERIES OF DEGREE TWO

SHO TAKEMORI (竹森 翔)
DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY

1. Introduction

In this note, we introduce an explicit formula for Fourier coefficients of Siegel-Eisenstein series of degree two with a primitive character of any conductor. Moreover, we introduce that there exists the p-adic analytic family which consists of Siegel-Eisenstein series of degree two and a certain p-adic limit of Siegel-Eisenstein series of degree two is actually a Siegel-Eisenstein series of degree two.

2. Statement of the main theorems

For a field K and positive integer g, we put

$$\mathrm{Sp}_g(K) = \{\alpha \in M_{2g}(K) \mid {}^t\alpha\eta\alpha = \eta\},$$
$$P(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(K) \mid a, b, c, d \in M_g(K), c = 0 \right\},$$

where $\eta = \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix}$. We denote by $\mathfrak{H}_g$ the Siegel upper half space of degree g. Let N be a positive integer. We define $\mathrm{Sp}_g(\mathbb{Z})$ and $\Gamma_0(N)$ by

$$\mathrm{Sp}_g(\mathbb{Z}) = \mathrm{Sp}_g(\mathbb{Q}) \cap \mathrm{GL}_{2g}(\mathbb{Z}),$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z}) \mid a, b, c, d \in M_g(\mathbb{Z}), c \equiv 0 \mod N \right\}.$$

Let ψ be a Dirichlet character mod N and k be an integer such that $\psi(-1) = (-1)^k$. We define Siegel-Eisenstein series $E^{(g)}_{k,\psi}(z)$ of degree g, weight k, character ψ and level N by

$$E^{(g)}_{k,\psi}(z) = \sum_{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in P(\mathbb{Q}) \cap \Gamma_0(N) \backslash \Gamma_0(N)} \overline{\psi}(\det(d)) \det(cz + d)^{-k}, \quad z \in \mathfrak{H}_g.$$

The right hand side is absolutely convergent when $k > g + 1$. Let

$$E^{(g)}_{k,\psi}(z) = \sum_{0 \le h \in \mathrm{Sym}^*_g(\mathbb{Z})} a(h, E^{(g)}_{k,\psi}) \exp(2\pi i \mathrm{Tr}(hz))$$

be the Fourier expansion of $E^{(g)}_{k,\psi}(z)$. Here we denote $\mathrm{Sym}^*_g(\mathbb{Z})$ by the set of half integral symmetric matrices of size g and denote $h \ge 0$ if h is positive semi-definite.

First, we state the theorem about an explicit formula of $a(h, E^{(2)}_{k,\psi})$.

Theorem 2.1. *Let ψ be a primitive Dirichlet character mod N and $h \in \mathrm{Sym}_2^*(\mathbb{Z})$ be a half integral positive-definite symmetric matrix. We denote the h-th Fourier coefficient of Siegel-Eisenstein series of degree two by $a(h, E_{k,\psi}^{(2)})$. Suppose $k > 3$. Then we have*

$$a(h, E_{k,\psi}^{(2)}) = 2\frac{L^{(N)}(2-k, \chi_h\psi)}{L(1-k,\psi)L^{(N)}(3-2k,\psi^2)} \times \prod_{\substack{q:prime \\ q\nmid N}} F_q^{(2)}(h; \psi(q)q^{k-3}) \prod_{\substack{q:prime \\ q|N}} c_q(h,\psi; q^{k-3}).$$

The notations are as follows. For a Dirichlet L-function $L(s,\chi)$ and a positive integer M, we put $L^{(M)}(s,\chi) = \prod_{q|M}(1-\chi(q)q^{-s})L(s,\chi)$. The conductor of a Dirichlet character χ is denoted by $\mathfrak{f}(\chi)$. $F_q^{(2)}(h;T)$ is a polynomial of (4.4), which is explicitly calculated by Kaufhold [4], and χ_h is the primitive Dirichlet character associated with $\mathbb{Q}(\sqrt{-\det(2h)})/\mathbb{Q}$. For a prime $q|N$, we define $c_q(h,\psi;T) \in \mathbb{Q}(\psi)(T)$ as follows.
(1) If (q, ψ_q, h) satisfies the condition (i) or (ii) below, then we define $c_q(h,\psi;T) =$ 0.
(2) If (q,ψ_q,h) satisfies the condition neither (i) nor (ii), and $\psi_p^2 \neq 1$, then we define $c_q(h,\psi;T) = 1$.
(3) If (q,ψ_q,h) satisfies the condition neither (i) nor (ii), and $\psi_p^2 = 1$, then we define $c_q(h,\psi;T)$ by

$$c_q(h,\psi;T) =$$
$$1 + q^{-1}(1-q)\frac{1-\chi_h\overline{\psi}(q)q^{-2}T^{-1}}{(1-\overline{\psi}^2(q)q^{-4}T^{-2})(1-\chi_h\psi(q)qT)}(q^3\psi^2(q)T^2)^{\beta_q - n_q + 1}.$$

where n_q and $\beta_q = \beta_q(h)$ are given by

$$n_q = \mathrm{ord}_q(\mathfrak{f}(\psi)),$$
$$2\beta_q = 2\beta_q(h) = \mathrm{ord}_q\left(\frac{\mathfrak{f}(\psi)\mathfrak{f}(\psi^2)^2}{\mathfrak{f}(\psi\chi_h)}\right) + \mathrm{ord}_q(\det 2h).$$

The conditions (i) and (ii) are as follows.

(i) $q = 2$, $\mathfrak{f}(\psi_q) \geq 4$ and $\mathfrak{f}(\psi_q) \neq 8$, and $h \in \mathrm{Sym}_2^*(\mathbb{Z}) \setminus \mathrm{Sym}_2(\mathbb{Z})$.
(ii) $q = 2$, $\mathfrak{f}(\psi_q) = 8$ and h is $\mathrm{GL}_2(\mathbb{Z}_2)$-equivalent to a matrix of the form;

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad 2^m\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad or \quad 2^m\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{Z}_2^\times$ and $m \in \{0,1\}$.

Remark 2.1. Mizuno [5] calculated $a(h, E_{k,\psi}^{(2)})$ explicitly when N is square-free and odd. Gunji [1] calculated the p-Euler factor of $a(h, E_{k,\psi}^{(2)})$ explicitly when p is an odd prime and $p \mid N$.

Next, we state the theorems about a p-adic limit of Eisenstein series and a p-adic analytic family of Eisenstein series. From now on, we fix a prime p and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$.

For p-adic interpolation of Siegel-Eisenstein series, we need to define a Eisenstein series whose p-Euler factor is equal to one.

Let $M_k^{(g)}(\Gamma_0(N),\psi)$ be the space of Siegel modular forms of degree g, weight k, level N and character ψ. Suppose $f \in M_k^{(g)}(\Gamma_0(N),\psi)$. Then f has the following Fourier expansion.

$$f(z) = \sum_{0 \le h \in \mathrm{Sym}_g^*(\mathbb{Z})} a(h,f)\mathbf{e}(hz).$$

We define a Hecke operator $U(p)$ as follows.

$$(f \mid U(p))(z) = \sum_{0 \le h \in \mathrm{Sym}_g^*(\mathbb{Z})} a(ph,f)\mathbf{e}(hz).$$

By the definition of $U(p)$, we have

$$f \mid U(p) \in \begin{cases} M_k^{(g)}(\Gamma_0(pN),\psi) & \text{if } p \nmid N, \\ M_k^{(g)}(\Gamma_0(N),\psi) & \text{if } p \mid N. \end{cases}$$

We define Hecke operators $V(p)$ and $W(p)$ as follows.

$$V(p) = \begin{cases} \dfrac{1-\overline{\psi}(p)^2 p^{3-2k} U(p)}{1-\overline{\psi}^2(p)p^{3-2k}} & \text{if } p \neq 2, \\ \dfrac{U(p)^2-\overline{\psi}(p)^2 p^{3-2k} U(p)^3}{1-\overline{\psi}^2(p)p^{3-2k}} & \text{if } p = 2, \end{cases}$$

$$W(p) = \frac{\left(U(p)-\psi(p)p^{k-1}\right)\left(U(p)-\psi(p)p^{k-3}\right)\left(U(p)-\psi^2(p)p^{2(k-3)}\right)}{\left(1-\psi(p)p^{k-1}\right)\left(1-\psi(p)p^{k-3}\right)\left(1-\psi^2(p)p^{2(k-3)}\right)}.$$

Let N be a positive integer divisible by p and ψ be a Dirichlet character mod N. Put $N = N_0 p^r$ with $p \nmid N_0$ and $r \ge 1$. Suppose that ψ_q is primitive for all $q|N_0$ and ψ_p is primitive if $r > 1$. We put

$$E'_{k,\psi} = E^{(2)}_{k,\psi} \mid \prod_{q|N} V(q),$$

and

$$G^{(2)}_{k,\psi} = \begin{cases} \frac{1}{2}L(1-k,\psi)L^{(N)}(3-2k,\psi^2)E'_{k,\psi} & \text{if } \psi_p \text{ is primitive}, \\ \frac{1}{2}L(1-k,\psi)L^{(N)}(3-2k,\psi^2)E'_{k,\xi} \mid W(p) & \text{if } \psi_p \text{ is the trivial character mod } p. \end{cases}$$

Here $\xi = \prod_{q|N_0} \psi_q$. Let $0 \le h \in \mathrm{Sym}_2^*(\mathbb{Z})$ be a half integral positive semi-definite symmetric matrix and suppose that $k > 3$. Then we can prove the following assertions by Theorem 2.1.

(i) If rank $h = 0$,

$$a(h, G^{(2)}_{k,\psi}) = \frac{1}{2}L(1-k,\psi)L^{(N)}(3-2k,\psi^2).$$

(ii) If rank $h = 1$,

$$a(h, G^{(2)}_{k,\psi}) = L^{(N)}(3-2k,\psi^2) \prod_{\substack{q:\text{prime} \\ q \nmid Np}} F_q^{(1)}(\varepsilon(h);\psi(q)q^{k-2}).$$

Here $F_q^{(1)}(m;T)$ is $1+qT+\cdots+(qT)^{\mathrm{ord}_q(m)}$ and $\varepsilon(h)$ is defined as follows.

$$\varepsilon(h) = \max\left\{m \in \mathbb{Z}_{\ge 0} \mid m^{-1}h \in \mathrm{Sym}_2^*(\mathbb{Z})\right\}.$$

(iii) If rank $h=2$,

$$a(h,G_{k,\psi}^{(2)})=L^{(N)}(2-k,\chi_h\psi)\prod_{\substack{q:\text{prime}\\ q\nmid Np}}F_q^{(2)}(h;\psi(q)q^{k-3})\prod_{\substack{q:\text{prime}\\ q|N_0}}c_q(h,\psi;q^{k-3}).$$

Theorem 2.2. *Let N be a positive integer divisible by p and ψ be a Dirichlet character mod N. Put $N=N_0p^r$ with $p\nmid N_0$ and $r\geq 1$. Suppose that ψ_q is primitive for all $q|N_0$ and ψ_p is primitive if $r>1$. We fix a topological generator u of $1+\mathbf{p}\mathbb{Z}_p$. Here, $\mathbf{p}$ is given by $\mathbf{p}=\begin{cases}p & \text{if } p\neq 2,\\ 4 & \text{if } p=2.\end{cases}$ We denote by ω the Teichmüller character. For a half integral positive semi-definite symmetric matrix $h\in \mathrm{Sym}_2^*(\mathbb{Z})$, there exists $\mathbf{a}(h,\psi;T)\in \mathrm{Frac}(\mathbb{Z}_p[\psi][\![T]\!])$ which satisfies the following interpolation property.*

$$\mathbf{a}(h,\psi;\varepsilon(u)u^k-1)=a(h,G_{k,\varepsilon\psi\omega^{-k}}^{(2)}),$$

for any finite order character ε of $1+\mathbf{p}\mathbb{Z}_p$ and integer k such that $k\geq 3$.

We define X and X_ψ by

$$X=\mathbb{Z}_p\times\mathbb{Z}/\phi(\mathbf{p})\mathbb{Z}\cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times),$$
$$X_\psi=\left\{(s,a)\in X\mid (-1)^a=\psi(-1)\right\}.$$

Here ϕ is Euler's phi function , $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)$ is the set of continuous group homomorphisms from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p^\times$. X is equipped with the p-adic topology. We embed $\mathbb{Z}$ in X by $\mathbb{Z}\ni m\to(m \bmod \phi(\mathbf{p}),m)\in X$. Let

$$\mathbb{C}_p[\![\mathbf{q}]\!]=\Big\{f=\sum_{0\leq h\in\mathrm{Sym}_2^*(\mathbb{Z})}a(h,f)\mathbf{e}(hz)\mid a(h)\in\mathbb{C}_p\Big\},$$

be the space of formal Fourier expansions, where $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$. We put $|f|_p=\sup_{0\leq h\in\mathrm{Sym}_2^{(*)}(\mathbb{Z})}|a(h,f)|_p$.

Theorem 2.3. *Let N be a positive integer such that $p\nmid N$ and ψ be a primitive Dirichlet character mod N. Suppose $(k,a)\in X_\psi$ and let k be an integer such that $k>3$. For any sequence $\{l_m\}_m\subset X_\psi$ such that $l_m>3$, $\lim_{m\to\infty}l_m=+\infty\in\mathbb{R}$ and $\lim_{m\to\infty}l_m=(a,k)\in X_\psi$, we have*

$$\lim_{m\to\infty}\mid G_{l_m,\psi}^{(2)}-G_{k,\psi\omega^{a-k}}^{(2)}\mid_p=0.$$

Here we regard $\psi\omega^{a-k}$ as a Dirichlet character mod Np.

Remark 2.2. Katsurada and Nagaoka [3] proved the modularity of $\lim_{m\to\infty}G_{l_m,\psi}^{(2)}$ when $N=1$, p is an odd prime and ψ is the quadratic character by using the genus theta series.

Since Theorem 2.2 and Theorem 2.3 can be deduced from Theorem 2.1, we sketch the proof of Theorem 2.1 in the following sections.

3. The Fourier expansion of Siegel-Eisenstein series

Let ψ be a primitive Dirichlet character mod N and ω be idele class character of finite order corresponding to ψ.

For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g$ with $a, b, c, d \in M_g$, we write $a = a_\alpha, b = b_\alpha, c = c_\alpha, d = d_\alpha$. The Siegel upper half plane of degree g is defined by

$$\mathfrak{H}_g = \left\{ z \in \mathrm{Sym}_g(\mathbb{C}) \mid \mathrm{Im}(z) > 0 \right\}.$$

For $\alpha \in \mathrm{Sp}_g(\mathbb{R}), z \in \mathfrak{H}_g$, we define

$$\alpha \cdot z = (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1}, \qquad j(\alpha, z) = \det(c_\alpha z + d_\alpha).$$

Put $\mathrm{Sp}_g(\mathbb{Z}) = \mathrm{Sp}_g(\mathbb{Q}) \cap \mathrm{GL}_{2g}(\mathbb{Z})$. Let $\Gamma \subset \mathrm{Sp}_g(\mathbb{Z})$ be a congruence subgroup and $\chi : \Gamma \to \mathbb{C}^\times$ be a character. For an integer $k \in \mathbb{Z}$ and a $\mathbb{C}$-valued function f on $\mathfrak{H}_g$, we set

$$(f|_k\gamma)(z) = f(\gamma \cdot z) j(\gamma, z)^{-k}.$$

We denote by $F_k(\Gamma, \chi)$ the space of functions on $\mathfrak{H}_g$ satisfying the following automorphic property:

(3.1) $$(f|_k\gamma)(z) = \chi(\gamma) f(z) \quad \text{for } \gamma \in \Gamma.$$

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. We denote by $\mathbf{f}$ (resp. ∞) the set of finite places of $\mathbb{Q}$ (resp. the infinite place). The adelization of $\mathrm{Sp}_g(\mathbb{Q})$ is denoted by $\mathrm{Sp}_g(\mathbb{A})$.

We put $\mathrm{Sp}_g(\mathbb{A}_\mathbf{f}) = \mathrm{Sp}_g(\mathbb{A}) \cap \prod_{v \in \mathbf{f}} \mathrm{Sp}_g(\mathbb{Q}_v)$. For $\alpha \in \mathrm{Sp}_g(\mathbb{A})$, we put

(3.2) $$\alpha = \alpha_\mathbf{f} \alpha_\infty, \qquad \alpha_\mathbf{f} \in \mathrm{Sp}_g(\mathbb{A}_\mathbf{f}), \quad \alpha_\infty \in \mathrm{Sp}_g(\mathbb{R}).$$

For a place v of $\mathbb{Q}$, a maximal compact subgroup C_v of $\mathrm{Sp}_g(\mathbb{Q}_v)$ is defined by

$$C_v = \begin{cases} \{\alpha \in \mathrm{Sp}_g(\mathbb{R}) \mid \alpha\mathbf{i} = \mathbf{i}\} & \text{if } v = \infty, \\ \mathrm{Sp}_g(\mathbb{Q}_v) \cap \mathrm{GL}_{2g}(\mathbb{Z}_v) & \text{if } v \in \mathbf{f}. \end{cases}$$

Here $\mathbf{i} = i1_g \in \mathfrak{H}_g$. Then, a maximal compact subgroup C of $\mathrm{Sp}_g(\mathbb{A})$ is defined by $C = \prod_{v \in \mathbf{f}} C_v$. We define algebraic subgroups P_g, Q_g, R_g of Sp_g by

$$P_g = \left\{ \alpha \in \mathrm{Sp}_g \mid c_\alpha = 0 \right\}, \quad Q_g = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_g \right\},$$

$$R_g = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathrm{Sym}_g \right\}.$$

Then the Iwasawa decomposition holds:

$$\mathrm{Sp}_g(\mathbb{A}) = P_g(\mathbb{A}) C C_\infty, \qquad \mathrm{Sp}_g(\mathbb{Q}) = P_g(\mathbb{Q}) \mathrm{Sp}_g(\mathbb{Z}).$$

For $0 \le i \le g$, we put

$$\eta^{(i)} = \left(\begin{array}{cc|cc} 0_i & 0 & -1_i & 0 \\ 0 & 1_j & 0 & 0_j \\ \hline 1_i & 0 & 0_i & 0 \\ 0 & 0_j & 0 & 1_j \end{array} \right), \qquad j = g - i.$$

Then the Bruhat decomposition holds:

(3.3) $$\mathrm{Sp}_g(\mathbb{Q}) = \coprod_{i=0}^{g} P_g(\mathbb{Q}) \eta^{(i)} P_g(\mathbb{Q}).$$

For an open subgroup of D of C, we put

$$\Gamma = D \cap \mathrm{Sp}_g(\mathbb{Z}).$$

Then Γ is a congruence subgroup of $\mathrm{Sp}_g(\mathbb{Z})$. Conversely, we obtain every congruence subgroup in this way.
Let $\chi : D \to \mathbb{C}^\times$ a group homomorphism. We denote the restriction to Γ of χ by the same letter. For a $\mathbb{C}$-valued function f on $\mathfrak{H}_g$ satisfying (3.1), we define a $\mathbb{C}$-valued function ϕ_f on $\mathrm{Sp}_g(\mathbb{A})$ by

$$\phi_f(\xi) = f(g_\infty \cdot \mathbf{i}) j(g_\infty, \mathbf{i})^{-k} \chi^{-1}(\delta), \quad \text{where } \xi = \alpha\delta g_\infty, \quad \alpha \in \mathrm{Sp}_g(\mathbb{Q}), \delta \in D, g_\infty \in \mathrm{Sp}_g(\mathbb{R}). \tag{3.4}$$

By strong approximation theorem, we have $\mathrm{Sp}_g(\mathbb{A}) = \mathrm{Sp}_g(\mathbb{Q}) D \mathrm{Sp}_g(\mathbb{R})$. Therefore, ϕ_f is defined on $\mathrm{Sp}_g(\mathbb{A})$ and is well-defined by (3.1). $\phi = \phi_f$ satisfies the following three conditions.

$$\phi(\alpha\xi) = \phi(\xi), \qquad \text{for } \alpha \in \mathrm{Sp}_g(\mathbb{Q}), \tag{3.5}$$

$$\phi(\xi\delta) = \chi^{-1}(\delta)\phi(\xi), \qquad \text{for } \delta \in D \tag{3.6}$$

$$\phi(\xi u) = j(u, \mathbf{i})^{-k} \phi(\xi), \qquad \text{for } u \in C_\infty. \tag{3.7}$$

We denote the space of $\mathbb{C}$-valued functions on $\mathrm{Sp}_g(\mathbb{A})$ satisfying (3.5),(3.6) and (3.7) by $\mathcal{F}_k(D, \chi)$. For $\phi \in \mathcal{F}_k(D, \chi)$, we put

$$f_\phi(z) = \phi(\xi_\infty) j(\xi_\infty, \mathbf{i})^k, \qquad \text{where } z = \xi_\infty \cdot \mathbf{i}, \quad \xi_\infty \in \mathrm{Sp}_g(\mathbb{R}). \tag{3.8}$$

Then $f \mapsto \phi_f$ is a bijection from $F_k(\Gamma, \chi)$ to $\mathcal{F}_k(D, \chi)$, and $\phi \mapsto f_\phi$ is its inverse.

We define an open subgroup $C_0(N)$ of C by

$$C_0(N) = \prod_{p \in \mathbf{f}} C_0(N)_p, \quad C_0(N)_p = \left\{ \alpha \in \mathrm{Sp}_g(\mathbb{Z}_p) \mid c_\alpha \equiv 0 \mod N \right\}.$$

Let ω be the character of $\mathbb{A}^\times / \mathbb{Q}^\times$ corresponding to Dirichlet character ψ. For $v \in \mathbf{f} \cup \{\infty\}$, the v-component ω_v satisfies the following.

If $v = \infty, \omega_\infty(x) = \mathrm{sgn}^k(x)$.
If $v \in \mathbf{f}, v = p, p \nmid N, \quad \omega_p(p) = \psi(p), \quad \omega_p(u) = 1, \quad$ for $u \in \mathbb{Z}_p^\times$.
If $v \in \mathbf{f}, v = p, p \mid N, \quad \omega_p(p) = \psi_p^*(p), \quad \omega_p(u) = \overline{\psi}_p(u), \quad$ for $u \in \mathbb{Z}_p^\times$.

Here, ψ_p is the Dirichlet character mod p^{n_p} such that $\psi = \prod_{p|N} \psi_p$ and ψ_p^* is

$$\psi_p^* = \prod_{\substack{q|N \\ q \neq p}} \psi_q. \tag{3.9}$$

If $p \mid N$, then we consider ω_p a character of $C_0(N)_p$ by

$$\omega_p(\gamma) = \omega_p(\det d_\gamma), \quad \gamma \in C_0(N)_p.$$

Then the restriction of $\prod_{p|N} \overline{\omega}_p$ to $\Gamma_0(N) = C_0(N) \cap \mathrm{Sp}_g(\mathbb{Z})$ is equal to ψ.

For $v \in \mathbf{f} \cup \{\infty\}$, we define $\mathbb{C}$-valued function $f_v^{(k)}$ on $\mathrm{Sp}_g(\mathbb{Q}_v)$ as follows. Note that the Iwasawa decomposition holds.
(1) If $v = \infty$,

$$f_\infty^{(k)}(\xi) = j(\xi, \mathbf{i})^{-k} = |\det a_\alpha|_\infty^k \omega_\infty(\det a_\alpha) \det(u + \mathbf{i}v)^{-k},$$
$$\text{for } \xi = \alpha\gamma, \alpha \in P_g(\mathbb{Q}_\infty), \gamma = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in C_\infty.$$

(2) If $v = p \in \mathbf{f}$ and $p \nmid N$,

$$f_p^{(k)}(\xi) = |\det a_\alpha|_p^k \overline{\omega}_p(\det a_\alpha), \quad \text{for } \xi = \alpha\gamma, \alpha \in P_g(\mathbb{Q}_p), \gamma \in C_p.$$

(3) If $v = p \in \mathbf{f}$ and $p \mid N$,

$$f_p^{(k)}(\xi) = \begin{cases} 0 & \text{if } \xi \notin P_g(\mathbb{Q}_p)C_0(N)_p, \\ |\det a_\alpha|_p^k \overline{\omega}_p(\det a_\alpha)\omega_p(\gamma) & \text{if } \xi = \alpha\gamma, \alpha \in P_g(\mathbb{Q}_p), \gamma \in C_0(N)_p. \end{cases}$$

We define a function $f_{k,\psi}$ on $\mathrm{Sp}_g(\mathbb{A})$ by

$$f_{k,\psi}(\xi) = \prod_{v \in \mathbf{f} \cup \{\infty\}} f_v^{(k)}(\xi_v)$$

We define Eisenstein series $\mathcal{E}_{k,\psi}^{(g)}(\xi)$ on $\mathrm{Sp}_g(\mathbb{A})$ as follows.

$$\mathcal{E}_{k,\psi}^{(g)}(\xi) = \sum_{\alpha \in P_g(\mathbb{Q}) \backslash \mathrm{Sp}_g(\mathbb{Q})} f_{k,\psi}(\alpha\xi).$$

The right hand side is absolutely convergent when $\mathrm{Re}(s) + k > g + 1$. By definition, $\mathcal{E}_{k,\psi}^{(g)}(\xi)$ satisfies (3.5), (3.6) and (3.7) for $\Gamma = C_0(N), \chi = \prod_{p|N} \omega_p$. We can prove that $\mathcal{E}_{k,\psi}^{(g)}(\xi)$ corresponds to $E_{k,\psi}^{(g)}(z)$ by (3.4) and (3.8).

Next we consider the Fourier coefficients of Eisenstein series.
For a place v of $\mathbb{Q}$, we define a character $\mathbf{e}_v$ of $\mathbb{Q}_v$ by

$$\mathbf{e}_v(x) = \begin{cases} \mathbf{e}(-\iota_v(x)) & v \in \mathbf{f}, \\ \mathbf{e}(x) & v = \infty. \end{cases}$$

Here ι_v is the inclusion $\iota_v : \mathbb{Q}_v/\mathbb{Z}_v \hookrightarrow \bigoplus_{v \in \mathbf{f}} \mathbb{Q}_v/\mathbb{Z}_v = \mathbb{Q}/\mathbb{Z}$ when v is a finite place. By definition, $\mathbf{e}_v$ is trivial on $\mathbb{Z}_v$ when $v \in \mathbf{f}$. A character $\mathbf{e}_\mathbb{A}$ of $\mathbb{A}/\mathbb{Q}$ is defined by

$$\mathbf{e}_\mathbb{A}(x) = \prod_{v \in \mathbf{f} \cup \{\infty\}} \mathbf{e}_v(x).$$

For $X \in S_g(\mathbb{A})$ or $X \in S_g(\mathbb{Q}_v)$, we put

$$\mathbf{e}_\mathbb{A}(X) = \mathbf{e}_\mathbb{A}\big(\mathrm{Tr}(X)\big), \qquad \mathbf{e}_v(X) = \mathbf{e}_v\big(\mathrm{Tr}(X)\big).$$

Next we define Haar measure on $S_g(\mathbb{A})$. If $v \in \mathbf{f}$, we take a Haar measure dx_v on $S_g(\mathbb{Q}_v)$ such that $\int_{S_g(\mathbb{Z}_v)} dx_v = 1$. If $v = \infty$, we take a Haar measure dx_∞ on $S_g(\mathbb{R})$ such that $dx_\infty = \prod_{i \le j} dx_\infty^{(ij)}$. Here $x_\infty^{(ij)}$ is the (i,j) component of x_∞. Then we define a Haar measure dx on $S_g(\mathbb{A})$ by

$$dx = \prod_{v \in \mathbf{f} \cup \{\infty\}} dx_v.$$

This measure satisfies the following.

$$\int_{S_g(\mathbb{Q}) \backslash S_g(\mathbb{A})} dx = 1.$$

For $x \in S_g$, we put

$$\tau(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}_g.$$

For $\xi_\infty \in \mathrm{Sp}_g(\mathbb{R})$, we see that the function $\mathcal{E}^{(g)}_{k,\psi}(\tau(x)\xi_\infty)$ on $S_g(\mathbb{A})$ is $S_g(\mathbb{Q})$ invariant by (3.5). Therefore, the following equations hold.

$$\mathcal{E}^{(g)}_{k,\psi}(\tau(x)\xi_\infty) = \sum_{h \in S_g(\mathbb{Q})} b(h,k,\xi_\infty)\mathbf{e}_{\mathbb{A}}(hx), \quad x \in S_g(\mathbb{A}), \xi_\infty \in \mathrm{Sp}_g(\mathbb{R}). \tag{3.10}$$

$$b(h,k,\xi_\infty) = \int_{S_g(\mathbb{Q})\backslash S_g(\mathbb{A})} \mathcal{E}^{(g)}_{k,\psi}(\tau(x)\xi_\infty)\mathbf{e}_{\mathbb{A}}(-hx)dx. \tag{3.11}$$

By (3.8), $\mathcal{E}^{(g)}_{k,\psi}(\xi)$ corresponds to $E^{(g)}_{k,\psi}(z)$.
Therefore if we put $\xi_\infty = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ in (3.10), we have the following.

$$E^{(g)}_{k,\psi}(z) = \sum_{h \in S_g(\mathbb{Q})} a(h,k,y)\mathbf{e}(hx), \tag{3.12}$$

$$a(h,k,y) = \det(y)^{-k/2} b(h,k,\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}). \tag{3.13}$$

We can prove the following proposition by the standard argument. We omit the proof.

Proposition 3.1. *Let $b(h,k,\xi_\infty)$ be as above. If $\det h \neq 0$, then we have*

$$b(h,k,\xi_\infty) = \int_{S_g(\mathbb{A})} f_{k,\psi}(\eta\tau(x)\xi_\infty)\mathbf{e}_{\mathbb{A}}(-hx)dx.$$

By Proposition 3.1 and the definition of $f_{k,\psi}$, we have.

(3.14) $b(h,k,\xi_\infty)$

$$= \int_{S_g(\mathbb{R})} f^{(k)}_\infty(\eta_\infty\tau(x)\xi_\infty)\mathbf{e}(-h_\infty x)dx \times \prod_{p \in \mathfrak{f}} \int_{S_g(\mathbb{Q}_p)} f^{(k)}_p(\eta_p\tau(x))\mathbf{e}_p(-h_p x)dx.$$

4. Euler factors of Fourier coefficients of Siegel-Eisenstein series

The Euler factor of $b(h,k,\xi_\infty)$ is examined by several authors. We recall some of their results.

First we introduce the result for the Euler factor at the infinite place.

For $\alpha, \beta \in \mathbb{C}$, we define a function ξ by

$$\xi(y,h;\alpha,\beta) = \int_{S_g(\mathbb{R})} \det(x+iy)^{-\alpha}\det(x-iy)^{-\beta}\mathbf{e}(-hx)dx. \tag{4.1}$$

By the definition of $f^{(k)}_\infty$, we have

$$\int_{S_g(\mathbb{R})} f^{(k)}_\infty(\eta\tau(x)\xi_\infty)\mathbf{e}(-hx)dx = \det(y)^{(k+s)/2}\xi(y,h;k+s/2,s/2) \tag{4.2}$$

Theorem 4.1 ([6] (4.34K),(4.35K); [7] (7.11),(7.12)). *Suppose $y, h \in \mathrm{Sym}_g(\mathbb{R})$ be symmetric matrices and y is positive definite. Let p and q be the number of positive and negative eigenvalue of h respectively and put $t = g - p - q$. We denote by*

$\delta_+(hy)$ (resp. $\delta_-(hy)$) the product of all positive (resp. negative) eigen values of $y^{1/2}hy^{1/2}$. For $m \in \mathbb{Z}_{\geq 0}$, we put

$$\Gamma_m(s) = \begin{cases} 1 & \text{if } m = 0, \\ \pi^{m(m-1)/4} \prod_{i=0}^{m-1} \Gamma\left(s - \frac{i}{2}\right) & \text{if } m \geq 1. \end{cases}$$

Then, there exists a function $\omega(y, h; \alpha, \beta)$ holomorphic with respect to α and β and satisfies the following equation.

$$\xi(y, h; \alpha, \beta) = i^{g(\alpha-\beta)} 2^{\tau} \pi^{\theta} \Gamma_t(\alpha + \beta - \frac{m+1}{2}) \Gamma_{g-q}(\alpha)^{-1} \Gamma_{g-p}(\beta)^{-1}$$
$$\times \det(y)^{(g+1)/2-\alpha-\beta} \delta_+(hy)^{\alpha-(g+1)/2+q/4} \delta_-(hy)^{\beta-(g+1)/2+p/4} \omega(y, h; \alpha, \beta).$$

Here τ, θ is

$$\tau = p\alpha + q\beta + t + \frac{1}{2}\{t(t-1) - pq\},$$
$$\theta = (2p - g)\alpha + (2q - g)\beta + g + \frac{t(g+1)}{2} + \frac{pq}{2}.$$

If h is positive definite, then the following equation holds.

$$\xi(y, h; \alpha, 0) = 2^{g(1-g)/2} i^{-g\alpha} (2\pi)^{g\alpha} \Gamma_g(\alpha)^{-1} \det(h)^{\alpha-(g+1)/2} \mathbf{e}(iyh).$$

Next we introduce the result for the Euler factor at unramified places, which is called the singular series.

Let p be a prime. For $x \in S_g(\mathbb{Q}_p)$, we define $\nu(x) \in \mathbb{Q}_{>0}$ by $\nu(x) = |\det c|_p^{-1}$, $x = c^{-1}d$, where $c \in \mathrm{GL}_g(\mathbb{Z}_p), d \in M_g(\mathbb{Z}_p)$ and c, d is co-prime. Here $c, d \in M_g(\mathbb{Z}_p)$ is said to be co-prime if there exist unimodular matrices $u \in \mathrm{GL}_g(\mathbb{Z}_p)$ and $v \in \mathrm{GL}_{2g}(\mathbb{Z}_p)$ such that $u(c\ d)v = (1_g\ 0_g)$. By the next lemma, ν is well-defined.

Lemma 4.1 ([8] 3.6 Proposition (3)). *Suppose $c, c' \in M_g(\mathbb{Z}_p) \cap \mathrm{GL}_g(\mathbb{Q}_p)$ and $d, d' \in M_g(\mathbb{Z}_p)$. Assume $c^{-1}d = c'^{-1}d'$ and (c, d), (c', d') are co-prime. Then there exists $u \in \mathrm{GL}_g(\mathbb{Z}_p)$ such that $c = uc', d = ud'$. Therefore, if we put*

$$\mathfrak{M} = \{(c, d) \in M_{g,2g}(\mathbb{Z}) \mid \det c \neq 0, c \cdot {}^t d = d \cdot {}^t c,\ (c, d) \text{ is co-prime}\}.$$

Then $(c, d) \to x = c^{-1}d$ is a bijection from $\mathrm{GL}_g(\mathbb{Z}_p)\backslash\mathfrak{M}$ to $S_g(\mathbb{Q}_p)$

Definition 4.1. Let p be a prime and ψ be a Dirichlet character mod p^n. Suppose $h \in S_g^*(\mathbb{Z}_p)$. We define a Dirichlet series $S_p(h, \psi, s)$ by

$$S_p(h, \psi, s) = \begin{cases} \displaystyle\sum_{x \in S_g(\mathbb{Q}_p)/S_g(\mathbb{Z}_p)} \nu(x)^{-s} \mathbf{e}_p(-hx) & \text{if } n = 0, \\ \displaystyle\sum_{x \in S_g(\mathbb{Q}_p)'/S_g(\mathbb{Z}_p)} \overline{\psi}_p(\nu(x)\det(x))\nu(x)^{-s} \mathbf{e}_p(-hx) & \text{if } n \geq 1. \end{cases}$$

Here $S_g(\mathbb{Q}_p)' = \{c^{-1}d \mid (c, d) \in \mathfrak{M},\ c \equiv 0 \bmod N\}$ and $\mathfrak{M}$ is as in Lemma 4.1. This Dirichlet series is called the singular series. If $n = 0$, then $S_p(h, \psi, s)$ does not depend on ψ. Therefore we denote $S_p(h, \psi, s)$ by $S_p(h, s)$ when $n = 0$. We define the formal power series $A_p(h, \psi; T)$ corresponding to $S_p(h, \psi, s)$ by

$$A_p(h, \psi; p^{-s}) = S_p(h, \psi, s).$$

We denote $A_p(h, \psi; T)$ by $A_p(h; T)$ when $n = 0$.

Suppose $a \in \mathbb{Z}_p^\times$ and $u \in \mathrm{GL}_g(\mathbb{Z}_p)$. By the definition of the singular series, we have

$$S_p(ah,s,\psi) = \overline{\psi}_p(a)^g S_p(h,\psi,s), \qquad S_p(h[u],s,\psi) = \overline{\psi}_p((\det u)^2) S_p(h,\psi,s).$$

$$\int_{S_g(\mathbb{Q}_p)} f_p^{(k)}(\eta\tau(x))\mathrm{e}_p(-hx)dx = A_p(h,\psi_p;\overline{\omega}_p(p)p^{-k-s}). \tag{4.3}$$

Proposition 4.1 ([8] 14.9.Proposition). *Suppose $h \in \mathrm{Sym}_g^*(\mathbb{Z}_p)$ and $\det h \neq 0$.*

Then $A_p(h;T)$ is $\mathbb{Z}$-coefficient polynomial with constant term 1 and divisible by a polynomial $\gamma_p(h;T)$ defined as follows.

$$\gamma_p(h;T) = \begin{cases} \dfrac{1-T}{1-\lambda(h)p^{g/2}T} \displaystyle\prod_{i=1}^{g/2}(1-p^{2i}T^2) & \text{if } g \text{ is even,} \\ (1-T) \displaystyle\prod_{i=1}^{(g-1)/2}(1-p^{2i}T^2) & \text{if } g \text{ is odd.} \end{cases}$$

Here, when g is even, we define $d = (-1)^{g/2}\det(h)$, $K_h = \mathbb{Q}_p(\sqrt{d})$ and

$$\lambda(h) = \begin{cases} 1 & K_h = \mathbb{Q}_p, \\ -1 & K_h/Q_p \text{ is unramified quadratic extension.} \\ 0 & K_h/Q_p \text{ is ramified extension.} \end{cases}$$

Moreover if h satisfies the following condition,

$$\begin{cases} \det(2h) \in \mathbb{Z}_p^\times & \text{if } g \text{ is even,} \\ \det(2h) \in 2\mathbb{Z}_p^\times & \text{if } g \text{ is odd,} \end{cases}$$

then

$$A_p(h;T) = \gamma_p(h;T).$$

We define $F_p^{(g)}(h;T)$ by $F_p^{(g)}(h;T) = A_p(h;T)/\gamma_p(h;T)$. By Proposition 4.1, if $h \in \mathrm{Sym}_g^*(\mathbb{Z}_p)$ and $\det h \neq 0$, then $F_p^{(g)}(h;T)$ is $\mathbb{Z}$-coefficient polynomial with constant term 1 and if $h \in \mathrm{Sym}_g^*(\mathbb{Z})$, then $F_p^{(g)}(h;T) = 1$ for all but a finite prime.

We introduce the result for $F_p^{(2)}(h;T)$.

Proposition 4.2 ([4] Hilfssatz 10). *Let $h \in \mathrm{Sym}_2^*(\mathbb{Z})$ be a half-integral positive definite symmetric matrix. We put $D(h) = -\det(2h), K(h) = \mathbb{Q}(\sqrt{D(h)})$. We denote the discriminant of $K(h)$ by $D_0(h)$. Then there exists a positive integer $f(h) \in \mathbb{Z}_{>0}$ such that $D(h) = D_0(h)f(h)^2$. We denote by χ_h the primitive quadratic character associated with $K(h)/\mathbb{Q}$. We put*

$$\varepsilon(h) = \max\left\{m \in \mathbb{N} \mid m^{-1}h \in \mathrm{Sym}_2^*(\mathbb{Z})\right\},$$

and

$$\alpha_1 = \mathrm{ord}_p(\varepsilon(h)), \qquad \alpha = \mathrm{ord}_p(f(h)).$$

Then the explicit form of $F_p^{(2)}(h;T)$ is as follows.

$$F_p^{(2)}(h;T) = \sum_{i=0}^{\alpha_1}(p^2T)^i\left\{\sum_{j=0}^{\alpha-i}(p^3T^2)^j - \chi_h(p)(pT)\sum_{j=0}^{\alpha-i-1}(p^3T^2)^j\right\}. \tag{4.4}$$

Moreover, $F_p^{(2)}(h;T)$ satisfies the following functional equation.

$$F_p^{(2)}(h;T) = (p^{3/2}T)^{2\alpha} F_p^{(2)}(h;p^{-3}T^{-1}).$$

Remark 4.1. Katsurada [2] proved an explicit formula and a functional equation for $F_p^{(g)}(h;T)$ for all degree g.

By proposition 4.1, we have the following theorem.

Theorem 4.2. *Suppose $k > g+1$ and $\det h \neq 0$. For a Dirichlet L-function $L(s,\chi)$ and a positive integer N, we put $L^{(N)}(s,\chi) = \prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1}$. Then $a(h, E_{k,\psi}^{(g)})$ is given as follows.*
(1) *If g is even,*

$$\xi(y,h;k,0)\mathbf{e}(-iyh)\frac{L^{(N)}(k-g/2,\chi_h\overline{\psi})}{L(k,\overline{\psi})L^{(N)}(2k-g,\overline{\psi}^2)}\prod_{i=1}^{(g-2)/2} L^{(N)}(2k-2i,\overline{\psi}^2)^{-1}$$
$$\times \prod_{p\nmid N} F_p^{(2)}(h;\overline{\psi}(p)p^{-k}) \prod_{p\mid N} A_p(h,\psi_p;\overline{\psi}_p^{\,*}(p)p^{-k}).$$

(2) *If g is odd,*

$$\xi(y,h;k,0)\mathbf{e}(-iyh)L^{(N)}(k,\overline{\psi})^{-1}\prod_{i=1}^{(g-1)/2} L^{(N)}(2k-2i,\overline{\psi}^2)^{-1}$$
$$\times \prod_{p\nmid N} F_p^{(2)}(h;\overline{\psi}(p)p^{-k}) \prod_{p\mid N} A_p(h,\psi_p;\overline{\psi}_p^{\,*}(p)p^{-k}).$$

5. Euler factors at ramified places

In this section, we introduce the result for the singular series at ramified places. For simplicity, we state the result only for singular series at an odd prime.

By the definition of the singular series, it is sufficient to calculate $S_p(\psi,h)$ for each representative h of $\mathrm{GL}_2(\mathbb{Z}_p a)$-equivalent class of $\mathrm{Sym}_2(\mathbb{Z}_p)$. Let m be the maximum integer which satisfies $p^{-m}h \in \mathrm{Sym}_2(\mathbb{Z}_p)^*$ and put $h' = p^{-m}h$.
If $p \neq 2$, then h' is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to the matrix of the form;

$$\begin{pmatrix} \alpha & 0 \\ 0 & p^t\beta \end{pmatrix}, \quad \alpha,\beta \in \mathbb{Z}_p^\times.$$

The explicit forms of $S_p(\psi,h)$ are as follows.

Proposition 5.1. *Let ψ be a primitive Dirichlet character mod p^n. Suppose $p \neq 2$ and put $h = p^m\begin{pmatrix} \alpha & 0 \\ 0 & p^t\beta \end{pmatrix}$, with $m \geq 0$ and α, $\beta \in \mathbb{Z}_p^\times$. We denote the quadratic character mod p by χ_{p^*}. Then the following assertions hold.*

(1) *Suppose* $\psi = \chi_{p^*}$, *then*

$$S_p(\psi,h) = \begin{cases} \psi(-1)\Big\{(p-1)\sum_{i=1}^{m+t/2} p^{(3-2s)i-2} - p^{(3-2s)(m+t/2+1)-2}\Big\} & \text{if } t \text{ is even}, \\ \psi(-1)\Big\{(p-1)\sum_{i=1}^{m+t/2+1/2} p^{(3-2s)i-2} \\ \qquad\qquad + \chi_{p^*}(\alpha\beta)p^{(3-2s)(m+t/2+1)-3/2}\Big\} & \text{if } t \text{ is odd}. \end{cases}$$

(2) *Suppose* $\psi \neq \chi_{p^*}$, *then*

$$S_p(\psi,h) = \begin{cases} \psi(\alpha\beta)G\left(\overline{\psi}\right)^2 p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } n-t \text{ is even}, \\ \varepsilon_p(\psi\chi_{p^*})(\alpha\beta)G(\overline{\psi})G(\overline{\psi}\chi_{p^*})p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } n-t \text{ is odd}. \end{cases}$$

Here, ε_p *is given by*

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4, \\ i & \text{if } p \equiv 3 \mod 4. \end{cases}$$

By the same computation in [1], we can prove this proposition. We omit the proof.

6. Sketch of the proof of Theorem 2.1

For a place v of $\mathbb{Q}$ and a quasi character $\chi : \mathbb{Q}_v^\times \to \mathbb{C}^\times$, we denote ε-factor and γ-factor from the local functional equation in Tate's thesis by $\varepsilon_v(\chi,s)$ and $\gamma_v(\chi,s)$ for fixed additive character $\mathbf{e}_v$.

By the explicit form for $\xi(y,h;k,0)$ in Theorem 4.1, and the duplication formula of the Gamma function : $\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\pi^{1/2}\Gamma(2s)$, we can prove the following Proposition.

Proposition 6.1. *Suppose that* $h \in \mathrm{Sym}_g(\mathbb{R})$ *is positive definite and put* $\chi = \mathrm{sgn}^k$, $\rho = \mathrm{sgn}$. *Then* $\xi(y,h,k,0)\mathbf{e}(-iyh)$ *is given as follows.*
(1) If g *is even,*

$$2^{g/2}i^{g^2/4}(\det 2h)^{k-(g+1)/2}\frac{\gamma_\infty(\chi,k)}{\gamma_\infty(\rho\chi,k-g/2)}\prod_{i=1}^{g/2}\gamma_\infty(\chi,2k-2i).$$

(2) If g *is odd,*

$$2^{(g+1)/2}(-1)^{(g^2-1)/8}(2^{-1}\det 2h)^{k-(g+1)/2}\gamma_\infty(\chi,k)\prod_{i=1}^{(g-1)/2}\gamma_\infty(\chi,2k-2i).$$

Next let us consider the Euler factor at a finite place. For a Dirichlet character χ mod p^n, we define $A'_p(h,\chi;T)$ as follows.

$$A'_p(h,\chi;T) = \begin{cases} A_p(h,\chi;T) & \text{if } \chi^2 \neq 1, \\ A_p(h,\chi;T) - \chi(-1)\dfrac{(p-1)p^{-n-1}(p^3T^2)^n}{1-p^3T^2} & \text{if } \chi^2 = 1. \end{cases} \tag{6.1}$$

Proposition 6.2. *Let* ψ *be a primitive Dirichlet character mod* N. *Let* ω *denote the character of* $\mathbb{A}^\times/\mathbb{Q}^\times$ *corresponding to* ψ *and* ρ_h *denote the character of* $\mathbb{Q}_p^\times$

corresponding to $\mathbb{Q}_p(\sqrt{-\det(h)})/\mathbb{Q}_p$ *by local class field theory. Suppose* $p \mid N$, $h \in \mathrm{Sym}_2^*(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ *and* $A_p(h,\omega;T) \neq 0$. *Then the following equation holds.*

$$A'_p(h,\psi_p;\overline{\psi}_p^*(p)p^{-s}) = \overline{\omega}_p(\det(2h))\frac{\gamma_p(\overline{\omega}_p,s)\gamma_p(\overline{\omega}_p^2,2s-2)}{\gamma_p(\rho_h\overline{\omega}_p,s-1)}\varepsilon_p(\rho_h,s-1)p^{(3-2s)\alpha_p}.$$

Furthermore if $\psi_p^2 = 1$,

$$A'_p(h,\psi_p;\overline{\psi}_p^*(p)p^{-s}) = \omega_p(-1)p^{-n_p}\left(\overline{\omega}_p(p^2)p^{3-2s}\right)^{\beta_p}\frac{L(\omega_p^2,3-2s)L(\rho_h\overline{\omega}_p,s-1)}{L(\overline{\omega}_p^2,2s-2)L(\rho_h\omega_p,2-s)}.$$

Here $n_p = \mathrm{ord}_p(\mathfrak{f}(\omega_p))$ *and* α_p, β_p *is given by*

$$\alpha_p = \frac{1}{2}\mathrm{ord}_p(\det(2h)/\mathfrak{f}(\rho_h)),\quad \beta_p = \frac{1}{2}\mathrm{ord}_p\left(\mathfrak{f}(\omega_p)\mathfrak{f}(\omega_p^2)^2/\mathfrak{f}(\omega_p\rho_h)\right) + \frac{1}{2}\mathrm{ord}_p\det(2h).$$

We can prove Proposition 6.2 by Proposition 5.1 and the next lemma.

Lemma 6.1. *Suppose* $p \neq 2$. *Let* ψ *be a primitive Dirichlet character mod* p^n. *For* $d \in \mathbb{Q}^\times$, *we denote the primitive Dirichlet character associated with* $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ *by* χ_d. *Then the following assertions holds.*

If n *is even,*

$$G(\psi)^2 = \overline{\psi}(4)G(\psi^2)p^{n/2}.$$

If n *is odd and* $\psi^2 \neq 1$,

$$G(\psi)G(\psi\chi_{p^*}) = \varepsilon_p\overline{\psi}(4)G(\psi^2)p^{n/2}.$$

Remark 6.1. If $n = 1$, then this lemma is the special case of Davenport-Hasse's product formula.

Proof. For simplicity, we assume n is odd and $\psi^2 \neq 1$. In the first place, we shall show

$$\sum_{x \bmod p^n}\psi(\alpha+\beta x^2) = \chi_{p^*}(\beta)\varepsilon_p^3(\psi\chi_{p^*})(\alpha)G(\psi)G(\psi\chi_{p^*})^{-1}p^{\frac{n}{2}}, \tag{6.2}$$

for $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}$ with $(\beta,p)=1$. Let $a \in \mathbb{Z}/p^n\mathbb{Z}$, $(a,p)=1$. Then by induction on n, we can prove

$$\sum_{x \bmod p^n}\mathbf{e}\left(\frac{ax^2}{p^n}\right) = \varepsilon_p\chi_{p^*}(a)p^{\frac{n}{2}} \tag{6.3}$$

where $\varepsilon_p = 1$ if $p \equiv 1 \bmod 4$ and $\varepsilon_p = i$ if $p \equiv 3 \bmod 4$. Since ψ is a primitive character, we have

$$\psi(x) = \frac{1}{p^n}G(\psi)\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\overline{\psi}(a)\mathbf{e}\left(-\frac{ax}{p^n}\right).$$

By this, we have

$$\sum_{x \bmod p^n}\psi(x^2+\alpha) = p^{-n}G(\psi)\sum_{\substack{x,a \bmod p^n\\(a,p)=1}}\overline{\psi}(a)\mathbf{e}\left(-\frac{a(x^2+\alpha)}{p^n}\right)$$

$$= p^{-n}G(\psi)\sum_{\substack{x,a \bmod p^n\\(a,p)=1}}\overline{\psi}(a)\mathbf{e}\left(-\frac{ax^2}{p^n}\right)\mathbf{e}\left(\frac{-a\alpha}{p^n}\right).$$

By (6.3), we have

$$\sum_{x \bmod p^n} \psi(x^2+\alpha) = p^{-n}G(\psi) \sum_{a \bmod p^n} \varepsilon_p \chi_{p^*}(-a) p^{\frac{n}{2}} \overline{\psi}(a) \mathbf{e}(-\frac{a\alpha}{p^n})$$
$$= \varepsilon_p^3 G(\psi) p^{-\frac{n}{2}} \sum_{\substack{a \bmod p^n \\ (a,p)=1}} (\overline{\psi}\chi_{p^*})(a) \mathbf{e}(-\frac{a\alpha}{p^n}).$$
$$= \varepsilon_p^3 (\psi\chi_{p^*})(\alpha) G(\psi) G(\psi\chi_{p^*})^{-1} p^{\frac{n}{2}}.$$

Thus we obtain (6.2). For primitive Dirichlet characters $\chi, \psi \bmod p^n$, we denote by $J(\chi, \psi)$ the Jacobi sum. By (6.2), we have

$$J(\psi,\psi) = \sum_{x \bmod p^n} \psi(1-x)\psi(x) = \sum_{x \bmod p^n} \psi(1/4 - x^2)$$
$$= \chi_{p^*}(-1) \varepsilon_p^3 \overline{\psi}(4) G(\psi) G(\psi\chi_{p^*})^{-1} p^{n/2}.$$

Since $J(\psi,\psi) = G(\psi)^2/G(\psi^2)$, we obtain the assertion of the lemma. □

Proof of Theorem 2.1. For simplicity, we assume N is odd and $\psi_p^2 \neq 1$ for all $p \mid N$. Let ω be the character of $\mathbb{A}^\times/\mathbb{Q}^\times$ corresponding to ψ. By Theorem 4.2, $a(h, E_{k,\psi}^{(2)})$ is given by

$$\xi(y,h;k,0)\mathbf{e}(-iyh) \frac{L^{(N)}(k-1, \chi_h \overline{\psi})}{L(k,\overline{\psi}) L^{(N)}(2k-2, \overline{\psi}^2)}$$
$$\times \prod_{p \nmid N} F_p^{(2)}(h; \overline{\psi}(p) p^{-k}) \prod_{p \mid N} A_p(h, \psi_p; \overline{\psi}_p^*(p) p^{-k}).$$

By Proposition 6.1, Proposition 6.2 and the functional equations of the Dirichlet L-function and $F_p^{(2)}(h;T)$,

$$a(h, E_{k,\psi}^{(2)}) = 2i \det(2h)^{k-3/2} \prod_{p \mid N} \overline{\omega}_p(\det 2h) \varepsilon_p(\rho_{h,p}, k-1) p^{(3-2k)\alpha_p}$$
$$\times \prod_{p \nmid N} \varepsilon_p(\rho_{h,p}\overline{\omega}_p, k-1) \overline{\omega}_p(p^{2\alpha_p}) p^{(3-2k)\alpha_p}$$
$$\times \frac{L^{(N)}(2-k, \chi_h, \psi)}{L(1-k,\psi) L^{(N)}(3-2k, \psi^2)} \prod_{p \nmid N} F_p^{(2)}(h; \psi(p) p^{k-3}).$$

Here $\rho_{h,v}$ is v-component of the character corresponding to χ_h. From this and the following equations, we obtain the assertion of the theorem.

$$\varepsilon_p(\rho_{h,p}\overline{\omega}_p, k-1) = \overline{\omega}_p(\mathfrak{f}(\rho_{h,p})) \varepsilon_p(\rho_{h,p}, k-1),$$
$$\prod_{p:\text{prime}} \varepsilon_p(\rho_{h,p}, k-1) = -\mathfrak{f}(\chi_h)^{1-k} G(\chi_h) = -i\mathfrak{f}(\chi_h)^{3/2-k},$$
$$\det 2h = \mathfrak{f}(\chi_h) \prod_{p:\text{prime}} p^{2\alpha_p}.$$

□

References

1. K. Gunji, *On the Siegel Eisenstein series of degree two for low weights*, preprint.
2. H. Katsurada, *An explicit formula for Siegel series*, American Journal of Mathematics **121** (1999), no. 2, 415–452.
3. H. Katsurada and S. Nagaoka, *On some p-adic properties of Siegel-Eisenstein series*, Journal of Number Theory **104** (2004), no. 1, 100–117.
4. G. Kaufhold, *Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades*, Mathematische Annalen **137** (1959), no. 5, 454–476.
5. Y. Mizuno, *An explicit arithmetic formula for the Fourier coefficients of Siegel–Eisenstein series of degree two and square-free odd levels*, Mathematische Zeitschrift **263** (2009), no. 4, 837–860.
6. G. Shimura, *Confluent hypergeometric functions on tube domains*, Mathematische Annalen **260** (1982), no. 3, 269–302.
7. ———, *On Eisenstein series*, Duke Mathematical Journal **50** (1983), no. 2, 417–476.
8. ———, *Euler products and Eisenstein series*, Amer Mathematical Society, 1997.