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Differential equations satisfied by principal series Whittaker functions on $SU(2,2)$

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Abstract

In this talk, we discuss about a holonomic system of differential equations for Whittaker functions associated with the principal series representation of $SU(2;2)$ with higher dimensional minimal K-type.

1 Introduction

Throughout, let G be the simple real Lie group $SU(2,2)$ of rank two, and let

$K=S(U(2)\times U(2))$: the maximal compact subgroup of G

π : an irreducible representation of G which is K-admissible.

For the representation π, there are two types of Whittaker model with respect to a character η of N (a spherical subgroup of G). One is the smooth model, and the other is the algebraic models induced by the space of algebraic Whittaker vectors:

$$W(\pi,\eta):=\mathrm{Hom}_{(\mathfrak{g},K)}(\pi\mid_K, C^{\infty}\text{-}\mathrm{Ind}_N^G(\eta)),$$

Here, $\mathfrak{g}$ is the Lie algebra of G, $\pi\mid_K$ is the subspace of K-finite vectors in π and C^{∞}-$\mathrm{Ind}_N^G(\eta)$ is the right G-module smoothly induced from η.

Our aim is a characterization of the space $W(\pi,\eta)$ for the principal series representation π of G associated with a minimal parabolic subgroup, which leads to a description of the following challenging question associated to π.

Question. For each intertwiner Φ in $W(\pi,\eta)$, what is the image of Φ ? Equivalently, for each K-type τ occurring in π, one can ask the image of the τ-isotypic component in π. The latter image is called the space of Whittaker functions of π with respect to τ.

The natural and classical approach. Let τ be a K-type belonging to π, and $f_1,\dots,f_n$ be its a basis in π. Denote by $\phi_j(g)$ the image of f_j under a fixed intertwiner Φ. Then, for each j and k in K, the function $(k\phi_j)(g)=\phi_j(gk)$ is a linear combination of the functions $\phi_1(g),\dots,\phi_n(g)$. Thus, it is enough to consider the functions ϕ_j on A for our purpose.

Assume that $\mathcal{C}$ is a square matrix of size $\dim(\tau)$, with entries in the universal enveloping algebra of $\mathfrak{g}$, so that

$$\pi(\mathcal{C})\circ\begin{pmatrix} f_1\\ f_2\\ \vdots\\ f_n\end{pmatrix} = \gamma\cdot\begin{pmatrix} f_1\\ f_2\\ \vdots\\ f_n\end{pmatrix}, \tag{1}$$

for some constant $\gamma\in\mathbb{C}$.

By applying Φ to the identity (1) we get the following system of differential equations (the A-radial part)

$$\mathcal{R}(\mathcal{C})\circ\begin{pmatrix} \phi_1(a)\\ \phi_2(a)\\ \vdots\\ \phi_n(a)\end{pmatrix} = \gamma\cdot\begin{pmatrix} \phi_1(a)\\ \phi_2(a)\\ \vdots\\ \phi_n(a)\end{pmatrix}, \quad a\in A$$

where $\mathcal{R}$ denotes the infinitesimal action of G on $C^\infty\text{-}\mathrm{Ind}_N^G(\eta)$. Thus, one can regard the space $W(\pi,\eta)$ as a subset of the solution space $Sol(\mathcal{R}(\mathcal{C}))$ of the system by sending Φ to the functions $\{\phi_j(a)\}$.

Remark. Recall that Whittaker functions satisfy differential equations with regular singularities at "0". The most important requirements for choosing a basis for τ are the simplicity and symmetricity of the series expansion of these functions $\phi_j(a)(a\in A)$ around 0 and of the system of differential equations arising from (1).

Principal series $\pi_{s,\chi}$. Let

$$\mathfrak{a}=\{a(t_1,t_2)=\begin{pmatrix} 0&0&t_1&0\\ 0&0&0&t_2\\ t_1&0&0&0\\ 0&t_2&0&0\end{pmatrix}\mid t_1,t_2\in\mathbb{R}\}\subset\mathfrak{g},$$
$$M=\{\mathrm{diag}(e^{i\theta},e^{-i\theta},e^{i\theta},e^{-i\theta})\}\oplus\{1_4,\mathrm{diag}(1,-1,1,-1)\}.$$

Define linear functions λ_1 and λ_2 on $\mathfrak{a}$ by $\lambda_1(a(t_1,t_2))=t_1$ and $\lambda_2(a(t_1,t_2))=t_2$. Then the set $\{\pm\lambda_1\pm\lambda_2,\pm2\lambda_1,\pm2\lambda_2\}$ forms the restricted root system of type C_2 for the pair $(\mathfrak{g},\mathfrak{a})$. Define $\lambda_1\pm\lambda_2, 2\lambda_1$ and $2\lambda_2$ to be positive. Let P_{min} be the minimal parabolic subgroup of G with Langlands decomposition $P_{min}=MAN$, where N is the unipotent subgroup defined by the root spaces corresponding to positive roots. For the character $s\otimes\chi$ of M, $s\in\mathbb{Z}$, and the $\mathbb{C}$-valued real linear form $\mu=\mu_1\lambda+\mu_2\lambda_2$, one has the principal series representation

$$\pi_{s,\chi}:=\mathrm{Ind}_P^G((s\otimes\chi)_M\otimes e^{\mu+\rho}\otimes 1_N),$$

where 1_N is the trivial character of N.

The main object in the paper is the 8-dimensional space $W(\pi_{s,\chi},\eta)$ of algebraic Whittaker vectors (see Kostant [2]) for non-degenerate character η (unitary) of N. Note that it is sufficient for our purpose to assume that $s\geq 0$.

1.1 Some previous results

Let us recall some known identities as in (1) and previous results for the space $W(\pi, \eta)$. The first example is the classical **Casimir** equation: let Ω be the Casimir operator of G. Then we have the following identity

$$\pi_{s,\chi}(\Omega)v = \chi_{\pi_{s,\chi}}(\Omega)v,$$

where $\chi_{\pi_{s,\chi}}$ is the infinitesimal character and v is a differential vector. This identity gives us an injection of $W(\pi_{s,\chi}, \eta)$ into the solution space $Sol(\mathcal{R}(\Omega))$ of the above equation. Note that the space $Sol(\mathcal{R}(\Omega))$ is of infinite dimension.

Let π be a discrete series representation of G and τ be its minimal K-type. Then Yamashita [10] defined an operator $D_{\pi,\tau}$ on τ under π:

$$\pi(D_{\pi,\tau})\tau = 0.$$

This gives us an injection of $W(\pi, \eta)$ into the solution space $Sol(\mathcal{R}(D_{\pi,\tau}))$ of the operator $\mathcal{R}(D_{\pi,\tau})$. Moreover, under certain conditions, he showed that

$$W(\pi, \eta) \cong Sol(\mathcal{R}(D_{\pi,\tau}))$$

as vector spaces. This result is not just for the group G (see [10] and [11]).

Let π be the principal series representation of $G = Sp(2, \mathbb{R})$ as in [6], and τ be the minimal K-type of π. In [6], the authors obtained a matrix, of size $\dim(\tau)$, formula of the form $\pi(\mathcal{D})v = \gamma v$ which implies

$$W(\pi, \eta) \cong Sol(\mathcal{R}(\Omega), \mathcal{R}(\mathcal{D})),$$

where Ω stands for the Casimir operator of $Sp(2, \mathbb{R})$. Note that the possible value of $\dim(\tau)$ is 1 or 2. The degree of $\mathcal{D}$ is 4 if $\dim(\tau) = 1$, and 2 for the case of dimension 2.

Remark. In the case $s = 0$ and $s = 1$, the corresponding spaces $W(\pi_{s,\chi}, \eta)$ behave quite similar to the above mentioned cases for $G = Sp(2, \mathbb{R})$, and are studied in [4], .

2 Differential equations

We begin by providing some formulas for the multiplicity one K-types $\tau_{[0,s;l]}$ in the principal series $\pi_{s,\chi}$. These formulas come from the explicit $(\mathfrak{g}, K)$-module structure of $\pi_{s,\chi}$ which originally discussed by Oda [7].

Note that the space of the adjoint K-representation $(Ad, \mathfrak{p}_{\mathbb{C}})$ is generated by the matrix units E_{ij+2} and E_{i+2j} $(1 \leq i, j \leq 2)$ and denote by $\mathcal{E}_{ij+2}$ and $\mathcal{E}_{i+2j}$ their infinitesimal actions with respect to π_s. Let denote $\mathbf{F}_{[s;l]}$ the transpose of the vector $(f_0, f_1, ..., f_s)$, where $\{f_j : 0 \leq j \leq s\}$ is the "nice" basis of $\tau_{[0,s;l]}$ introduced in [1] and $\mathbf{c}_q := q/s$ for $0 \leq q \leq s$.

Formula 1. *(Casimir equation) Let Ω be the Casimir operator. Then we have*

$$\pi_{s,\chi}(\Omega) \cdot \mathbf{F}_{[s;l]} = (\mu_1^2 + \mu_2^2 + \frac{1}{2}s^2 - 10)\mathbf{F}_{[s;l]}.$$

Formula 2. *(Shift equations) Set* $\nu_1 = \frac{1}{2}(s+l)$ *and* $\nu_2 = \frac{1}{2}(s-l)$*. Then we have*

$$\pi_{s,\chi}(\bar{\mathcal{Q}}) \cdot \mathbf{F}_{[s;l]} = \frac{1}{4}(\mu_1^2 - (\nu_1+1)^2)\mathbf{F}_{[s;l]},$$

and

$$\pi_{s,\chi}(\mathcal{Q}) \cdot \mathbf{F}_{[s;l]} = \frac{1}{4}(\mu_2^2 - (\nu_2-1)^2)\mathbf{F}_{[s;l]},$$

where $\bar{\mathcal{Q}} = \{\bar{Q}_{ij}\}_{0\le i,j\le s}$ *and* $\mathcal{Q} = \{Q_{ij}\}_{0\le i,j\le s}$ *are square matrices given by*

$$\begin{aligned}
\bar{Q}_{qq-1} &= -\mathbf{c}_q(\mathcal{E}_{24}\mathcal{E}_{32} + \mathcal{E}_{14}\mathcal{E}_{31})\\
\bar{Q}_{qq+1} &= -(1-\mathbf{c}_q)(\mathcal{E}_{23}\mathcal{E}_{42} + \mathcal{E}_{13}\mathcal{E}_{41})\\
\bar{Q}_{qq} &= (1-\mathbf{c}_q)(\mathcal{E}_{23}\mathcal{E}_{32} + \mathcal{E}_{13}\mathcal{E}_{31}) + \mathbf{c}_q(\mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42})
\end{aligned}$$

and

$$\begin{aligned}
Q_{qq-1} &= \mathbf{c}_q(\mathcal{E}_{32}\mathcal{E}_{24} + \mathcal{E}_{31}\mathcal{E}_{14})\\
Q_{qq+1} &= (1-\mathbf{c}_q)(\mathcal{E}_{42}\mathcal{E}_{23} + \mathcal{E}_{41}\mathcal{E}_{13})\\
Q_{qq} &= \mathbf{c}_q(\mathcal{E}_{32}\mathcal{E}_{23} + \mathcal{E}_{31}\mathcal{E}_{13}) + (1-\mathbf{c}_q)(\mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24})
\end{aligned}$$

for $0 \le q \le s$*, but all other entries are 0.*

Formula 3. *(Annihilation equations) We have*

$$\pi_{s,\chi}(\mathcal{A}) \cdot \mathbf{F}_{[s;l]} = 0,$$

and

$$\pi_{s,\chi}(\bar{\mathcal{A}}) \cdot \mathbf{F}_{[s;l]} = 0,$$

where $\mathcal{A} = \{A_{ij}\}$ *and* $\bar{\mathcal{A}} = \{\bar{A}_{ij}\}$ *are square matrix whose non-zero entries are given by*

$$\begin{aligned}
A_{jj-1} &= -\mathcal{E}_{31}\mathcal{E}_{14} - \mathcal{E}_{32}\mathcal{E}_{24},\\
A_{jj} &= \mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24} - \mathcal{E}_{31}\mathcal{E}_{13} - \mathcal{E}_{32}\mathcal{E}_{23},\\
A_{jj+1} &= \mathcal{E}_{41}\mathcal{E}_{13} + \mathcal{E}_{42}\mathcal{E}_{23},
\end{aligned}$$

and

$$\begin{aligned}
\bar{A}_{jj-1} &= -\mathcal{E}_{14}\mathcal{E}_{31} - \mathcal{E}_{24}\mathcal{E}_{32},\\
\bar{A}_{jj} &= \mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42} - \mathcal{E}_{13}\mathcal{E}_{31} - \mathcal{E}_{23}\mathcal{E}_{32},\\
\bar{A}_{jj+1} &= \mathcal{E}_{13}\mathcal{E}_{41} + \mathcal{E}_{23}\mathcal{E}_{42},
\end{aligned}$$

for $1 \le j \le s-1$.

Proposition 2.1. *On the* K*-type* $\tau_{[0,s;l]}$ *with respect to the action* $\pi_{s,\chi}$ *we have*

$$\mathcal{Q} + \bar{\mathcal{Q}} = \Omega/4.$$

2.1 A holonomic system of rank 8

Coordinate system. Since the $\mathbb{R}$-split torus A for our case is two dimensional, one may choose the coordinate system (y_1, y_2). Denote the Euler operators $y_1 \frac{\partial}{\partial y_1}$ and $y_2 \frac{\partial}{\partial y_2}$ with respect to this system by ∂_1 and ∂_2, respectively.

We now define the matrix differential operator $\bar{\mathcal{D}}$ by

$$\begin{pmatrix} \bar{d}_{00} & \bar{d}_{01} & 0 & & \cdots & & 0 \\ 0 & \bar{d}_{11} & \bar{d}_{12} & & \cdots & & 0 \\ 0 & 0 & \bar{d}_{22} & & \cdots & & 0 \\ \vdots & \vdots & & & \vdots & & \\ 0 & 0 & 0 & \cdots & \bar{d}_{s-2s-2} & \bar{d}_{s-2s-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_{s-1s-1} & \bar{d}_{s-1s} \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_{ss-1} & \bar{d}_{ss} \end{pmatrix}$$

where

$$d_{qq} = \frac{1}{4}((\partial_1 - q)^2 - \mu_1^2) - \xi\bar{\xi}\, y_1^2, \qquad d_{qq+1} = \bar{\xi}\, y_1(\partial_2 + \frac{1}{2}s - q) + \bar{\xi}\, y_1 y_2$$

for $q = 0, \dots, s-1$ and

$$d_{ss} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_1^2) - \xi\bar{\xi}\, y_1^2 - y_2^2 - \nu_1 y_2$$

$$d_{ss-1} = -\xi\, y_1(\partial_2 + \frac{1}{2}s) + \xi\, y_1 y_2.$$

We also define the matrix differential operator $\mathcal{D}$ by

$$\begin{pmatrix} d_{00} & d_{01} & 0 & & \cdots & & 0 \\ d_{10} & d_{11} & 0 & & \cdots & & 0 \\ 0 & a_{32} & d_{33} & & \cdots & & 0 \\ \vdots & \vdots & & & \vdots & & \\ 0 & 0 & 0 & \cdots & d_{s-2s-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & d_{s-1s-2} & d_{s-1s-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_{ss-1} & d_{ss} \end{pmatrix}$$

where

$$d_{00} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi\bar{\xi}\, y_1^2 - y_2^2 - \nu_2 y_2$$

$$d_{01} = -\bar{\xi}\, y_1(\partial_2 - \frac{1}{2}s) - \bar{\xi}\, y_1 y_2$$

and

$$d_{qq} = \frac{1}{4}((\partial_1 - s + q)^2 - \mu_2^2) - \xi\bar{\xi}\, y_1^2, \quad d_{qq-1} = \xi\, y_1(\partial_2 + q - \frac{1}{2}s) - \xi\, y_1 y_2$$

for $q = 1, \dots, s$. Here, the parameters ξ and $\bar{\xi}$ are associated to the character η.

By using Formulas 2 and 3, one can see that the Whittaker functions of $\pi_{s,\chi}$ with respect to $\tau_{[0,s;l]}$ satisfy the system of differential equations $\mathcal{D} = 0$ and $\bar{\mathcal{D}} = 0$. Moreover, we have the following result which characterizes the Whittaker functions of $\pi_{s,\chi}$ with respect to $\tau_{[0,s;l]}$.

Theorem 2.2. *For $s \geq 2$, the natural map from $W(\pi_{s,\chi}, \eta)$ into $Ker(\bar{\mathcal{D}}, \mathcal{D})$ is bijection if $\pi_{s,\chi}$ is irreducible and η is a nondegenerate unitary character of N.*

Here, we also have the following formula in the case $s = 0$, which is analogue to the class one case for $Sp(2, \mathbb{R})$ in [5]. Write W for the little Weyl group for $(\mathfrak{g}, \mathfrak{a})$, and (ρ_1, ρ_2) for the pair $(3, 2)$ related to the half sum.

Theorem 2.3. *Let $\pi_{0,\chi}$ be an irreducible principal series with parameter $\mu = (\mu_1, \mu_2) \in \mathfrak{a}_{\mathbb{C}}^*$, and set $\varepsilon = \frac{1-\chi(-1)}{2}$. Then the function ϕ_μ on A defined by*

$$\phi_\mu(y_1, y_2) = y_1^{\rho_1} y_2^{\rho_2} \sum_{m,n \geq 0} \frac{\mathbf{U}^0_{m,n}}{2^{2n}(\frac{\mu_1-\varepsilon}{2}+1)_m(\frac{\mu_2-\varepsilon}{2}+1)_n} \times y_1^{\mu_1+2m} y_2^{\frac{\mu_1+\mu_2}{2}+2n}$$
$$+ \frac{\varepsilon \mathbf{U}^1_{m,n}}{2^{2n+1}(\frac{\mu_1-\varepsilon}{2}+1)_m(\frac{\mu_2-\varepsilon}{2}+1)_{n+1}} \times y_1^{\mu_1+2m} y_2^{\frac{\mu_1+\mu_2}{2}+2n+1},$$

is a Whittaker function, on A, of $\pi_{0,\chi}$ with the K-type $\tau_{[0,0;2\varepsilon]}$. Moreover, the intertwiners $\Phi_{\omega(\mu)}$ attached to the function $\phi_{\omega(\mu)}(y_1, y_2)$ form a basis of the 8-dimensional space $W(\pi_{0,\chi}, \eta)$. Here,

$$\mathbf{U}^t_{m,n} := \sum_{j=0}^{min(m,n)} \frac{(\frac{\mu_1-\varepsilon}{2}+n+1+t)_{m-j}}{(m-j)!(n-j)!j!(\frac{\mu_1+\mu_2}{2}+1)_j(\frac{\mu_1-\mu_2}{2}+1)_{m-j}}$$

for $t = 0, 1$.

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