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# The matrix coefficients of the large discrete series of $SU(3,1)$

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## 1. Introduction

This is an announcement of the forthcoming paper [HKO3]. In this paper we present the explicit formula of matrix coefficients of the large discrete series representations of $SU(3,1)$, the unitary group of signature $(3+,1-)$ without proofs.

In the theory of automorphic forms, the dimension formula is of a great concern. In the situation of hermitian symmetric domains of type I, there is a result of Suehiro Kato [K1, K2] when the group is the special unitary group of signature $(p+,1-)$ which treats a dimension formula of holomorphic automorphic forms. This form stems from so-called (anti-)holomorphic discrete series representations of this $\mathbb{Q}$-rank 1 semi-simple Lie group. With representation theoretical view, there is a non-holomorphic discrete series, or "large" discrete series representation. The Selberg-Godement's formula [Go], which computes the dimension of bounded automorphic forms, requires no assumption for discrete series except integrability. However, the computation of kernel function at the "large" case seems still open. This is because we think the combinatorics of the weight basis of $U(3)$ would remain hard. On the other hand, in the papers [HiO, HiO2] there is a nice way to treat these basis in very concrete fashion. By using this, we calculate the matrix coefficients of $SU(3,1)$ in this paper.

The content of this paper is as follows. In Section 2 we treat the matrix coefficients of the discrete series in general. Then the Dirac-Schmid equations are introduced. In Section 3, we introduce the unitary group $SU(3,1)$ and its Lie algebra. Then the discrete series of $SU(3,1)$ is introduced concretely. We need the Harish-Chandra parameter, the Blattner parameter and the associated non-compact roots.

In Section 4, the canonical basis of $GL(3)$ is introduced. It is described using Gel'fand-Tsetlin pattern, and also called Gel'fand-Zelevinsky basis. Since the matrix coefficients reflect detailed geometric nature of weight basis, we investigate the weight diagram very closely.

In Section 5, we specify the Dirac-Schmid equality by the $SU(3,1)$-data. We compute the injectors in a very concrete way. In Section 6 the main results of this paper are given. We select the $\mathbb{Q}$-generating set of matrix coefficients, which we call *standard functions* (Theorem 6.3). Then the problem to compute matrix coefficients is reduced to that of standard functions. The standard functions are described by the hypergeometric functions ${}_2F_1$ (Proposition 6.8 and Lemma 6.12). We state these main results without proofs. The detailed ingredients are discussed in forthcoming paper [HKO3].

*Notation.* $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the ring of integers, the fields of rational numbers, real numbers and complex numbers. Let $M_n(\mathbb{C})$ be the space of complex square matrices of

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degree $n$. Then $E_{ij}$ denotes the matrix units with 1 at the $(i,j)$-th entry and zeros at the other entries.

## 2. Generalities

**2.1. Spherical functions or matrix coefficients belonging to the discrete series.** Let $G$ be a real semi-simple Lie group of finite center. Let $K$ be a maximal compact subgroup of $G$. We assume that $G$ has a compact Cartan subgroup $T$. Let $L^2(G)$ be the $L^2$-space of measurable functions on $G$ with respect to the Haar-Hurwitz measure, which is a $G \times G$ bi-module with the action

$$\mathcal{L}(g_1)\mathcal{R}(g_2)\varphi(x) := \varphi(g_1^{-1}xg_2) \quad (\varphi \in L^2(G),\ x \in G,\ (g_1, g_2) \in G \times G).$$

The discrete series representations are, by definition, in the sum of the closed invariant subspace under this action of $G \times G$. With this definition and from the results of Harish-Chandra, we have the discrete series

$$L^2(G)_d := \bigoplus_{\lambda \in \Xi} \pi_\lambda^* \boxtimes \pi_\lambda$$

as $G \times G$ bi-module, with parameters $\lambda$ in the dominant regular integral weights $\Xi$ with respect to a compact Cartan subalgebra $\mathfrak{t} = \operatorname{Lie} T$ (Harish-Chandra parameters).

Let $(\pi_\lambda, H_\lambda)$ be a discrete series representation with Harish-Chandra parameter $\lambda$. Let $\mathcal{T} : \pi_\lambda^* \boxtimes \pi_\lambda \to L^2(G)$ be the unique $G \times G$ homomorphism up to constant multiple. If we denote by

$$\langle\ ,\ \rangle \colon H_\lambda^* \times H_\lambda \to \mathbb{C}$$

be the $G$-equivariant canonical coupling, then $\mathcal{T}$ is given as

$$\mathcal{T}(v^* \boxtimes v)(x) = \langle v^*, \pi(x)v\rangle \quad \text{for } v \in H_\lambda \text{ and } v^* \in H_\lambda^*,$$

because we can check the intertwining property

$$\mathcal{L}(g_1)\mathcal{R}(g_2)\mathcal{T}(v^* \boxtimes v)(x) = \mathcal{T}((\pi_\lambda^*(g_1)v^* \boxtimes \pi_\lambda(g_2)v)(x)$$

immediately.

Let $\iota \colon W_\tau \to H_\lambda$ and $\iota^* \colon W_{\tau^*} \to H_\lambda^*$ be $K$ modules and $K$-injections. Then we define *the matrix coefficients of $\pi_\lambda$ with $K$-type $\tau$ at vector $f \boxtimes f'$* by

$$c(f \boxtimes f'; x) := \mathcal{T}(\iota^*(f') \boxtimes \iota(f))(x)$$

Since $G$ has a Cartan decomposition $G = KAK$, where $A$ is the connected component of split $\mathbb{R}$-torus of $G$, the matrix coefficients $c(f \boxtimes f'; x)$ can be determined by the value at $a_r \in A$, which we call the radial component.

**2.2. The Dirac-Schmid equations.** Let $\tau$ be a multiplicity-one $K$-type of $\pi_\lambda$ and let $\tau^{(e)}$ be a constituent of $\operatorname{Ad}(K) \otimes \tau$ and $I^{(e)} \colon \tau^{(e)} \to \mathfrak{p}_{\mathbb{C}} \otimes W_\tau$ be an injective $K$-homomorphism. If $\tau^{(e)}$ is not a constituent of $\pi_\lambda|K$ then for each $I^{(e)}(f) = \sum_i X_i \otimes v_i$, we have

$$\sum_i \mathcal{R}_{X_i}\mathcal{T}(\iota^*(v^*) \boxtimes \iota(v_i)) = 0 \quad (v^* \in H_{\pi_\lambda}^*)$$

since $\tau^{(e)} \to \mathfrak{p}_{\mathbb{C}} \otimes W_\tau \to H_\pi$ becomes also a $K$-homomorphism. We call this the (right) Dirac-Schmid equations.

## 3. The unitary group $SU(3,1)$, root systems and the Harish-Chandra parameters

**3.1. The Lie group and the Lie algebra.** The Lie group $G := SU(3,1)$ is realized as

$$\{g \in M_4(\mathbb{C}) \mid {}^t\bar{g} 1_{3,1} g = 1_{3,1},\ \det g = 1\}$$

with $1_{3,1} = \mathrm{diag}(1,1,1,-1)$, and its Lie algebra $\mathfrak{g} := \mathfrak{su}(3,1)$.

We choose a Cartan involution

$$\theta : g \in G \mapsto {}^t\bar{g}^{-1} \in G,$$

and the induced involution on the Lie algebra:

$$\theta : X \in \mathfrak{g} \mapsto -{}^t\bar{X} \in \mathfrak{g}.$$

Fix a compact Cartan subgroup $T$ in $K = G^\theta$ consisting of the diagonal matrices in $G$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan symmetric decomposition. The adjoint action of a central elements $\mathrm{diag}(z,z,z,z^{-3})$ with $z = \exp(\pi\sqrt{-1}/8)$ defines the canonical complex structure on $\mathfrak{p}$. Then the $(+1,-1)$ part $\mathfrak{p}_+$ and the $(-1,+1)$ part $\mathfrak{p}_-$ in $\mathfrak{p} \otimes \mathbb{C}$ is given by

$$\mathfrak{p}_+ = \mathbb{C}E_{14} \oplus \mathbb{C}E_{24} \oplus \mathbb{C}E_{34}, \quad \mathfrak{p}_- = \mathbb{C}E_{41} \oplus \mathbb{C}E_{42} \oplus \mathbb{C}E_{43}.$$

Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$ generated by $H_\mathfrak{a} = E_{14} + E_{41}$. We set $A = \exp(\mathfrak{a})$.

We put $M := Z_K(A)$ the centralizer of $A$ in $K$:

$$M = \left\{ \begin{pmatrix} u_1 & & \\ & U_2 & \\ & & u_4 \end{pmatrix} \mid u_1 \in U(1),\ U_2 \in U(2), u_4 = u_1,\ u_1^2 \det(U_2) = 1 \right\},$$

which is isomorphic to $U(1) \times U(2)$.

**3.2. The root systems.** The unitary characters of the compact Cartan subgroup

$$T := \{t := \mathrm{diag}(u_1,u_2,u_3,u_4) \mid u_i \in U(1),\ u_4 = (u_1u_2u_3)^{-1}\}$$

are expressed as

$$\chi : T \ni t \mapsto \prod_{i=1}^{3} u_i^{l_i} \in U(1),$$

with some triple of integers $(l_1,l_2,l_3) \in \mathbb{Z}^3$.

The root system $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is given by $\{\beta_{ij}\ (i \neq j)\}$ with $\beta_{ij}(t) = u_i/u_j$ $(1 \le i,j \le 3)$ which is of type $A_3$.

From now on we fix as a positive system $\Phi^+ := \{\beta_{ij}\ (i<j)\}$. Then two compact roots $\beta_{12} = (1,-1,0)$ and $\beta_{23} = (0,1,-1)$ and a non-compact root $\beta_{34} = (1,1,2)$ form the simple roots in this positive system. Note that $\rho$ the half sum of positive roots is given by $\rho = (3,2,1)$.

The root system $\Phi(\mathfrak{g}, \mathfrak{a})$ is the restricted root system of type $A_1$ with multiplicities.

3.3. **Harish-Chandra parametrization.** We recall here the Harish-Chandra parametrization of the discrete series representations for $SU(3,1)$ and their minimal $K$-types.

The set of finite-dimensional irreducible representations of $K$ is parametrized by a subset $\Xi$ of integral weights $L_T = \mathrm{Hom}(T, U(1))$ of the representations of $K$, which are dominant, i.e., its parameter is of the form $(l_1, l_2, l_3) \in \mathbb{Z}^3$ satisfying $l_1 \geq l_2 \geq l_3$.

Since $T$ is also a Cartan subgroup of $G$, the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{t}$ defines the natural inner product on $L_T$:

$$(l, l') := \sum_{i=1}^{3} l_i l'_i - \frac{1}{4}\left(\sum_{i=1}^{3} l_i\right)\left(\sum_{i=1}^{3} l'_i\right).$$

The set of equivalence classes of the discrete series representations of $G$ is parametrized by a subset of $L_T + \rho = L_T$, which is positive with respect compact positive roots $\{\beta_{12}, \beta_{23}\}$.

There are 4 different positive systems in $\Phi(\mathfrak{g}, \mathfrak{t})$ compatible with the positive system of compact roots generated by simple roots $\{\beta_{12}, \beta_{23}\}$; these are specified by 4 simple root systems $\Delta_J$ $(J \in \{I, II, III, IV\})$:

$$\begin{aligned} &(\mathrm{I}) \quad : \{\beta_{12}, \beta_{23}, \beta_{34}\}; \quad (\mathrm{II}) \quad : \{\beta_{12}, \beta_{24}, \beta_{43}\}; \\ &(\mathrm{III}) \quad : \{\beta_{14}, \beta_{42}, \beta_{23}\}; \quad (\mathrm{IV}) \quad : \{\beta_{41}, \beta_{12}, \beta_{23}\} \end{aligned}$$

with the corresponding sets of positive non-compact roots $\Phi_{J,n}$:

$$\begin{aligned} &(\mathrm{I}) \quad : \Phi^+_{I,n} = \{\beta_{14}, \beta_{24}, \beta_{34}\}; \quad (\mathrm{II}) \quad : \Phi^+_{II,n} = \{\beta_{14}, \beta_{24}, \beta_{43}\}; \\ &(\mathrm{III}) \quad : \Phi^+_{III,n} = \{\beta_{14}, \beta_{42}, \beta_{43}\}; \quad (\mathrm{IV}) \quad : \Phi^+_{IV,n} = \{\beta_{41}, \beta_{42}, \beta_{43}\}. \end{aligned}$$

And the half sum of positive roots $\rho_J$ for each $J$ is given by

$$\rho_I = \rho = (3, 2, 1); \quad \rho_{II} = (2, 1, -1); \quad \rho_{III} = (1, -1, -2); \quad \rho_{IV} = (-1, -2, -3).$$

Here we set

$$\Xi = \Xi_I \cup \Xi_{II} \cup \Xi_{III} \cup \Xi_{IV}.$$

with

$$\begin{aligned} &\Xi_I := \{l = (l_1, l_2, l_3) \in L_T |\ l_1 > l_2 > l_3 > 0\}, \quad \Xi_{II} := \{l \in L_T |\ l_1 > l_2 > 0 > l_3\}, \\ &\Xi_{III} := \{l \in L_T |\ l_1 > 0 > l_2 > l_3\}, \qquad\qquad \Xi_{IV} := \{l \in L_T |\ 0 > l_1 > l_2 > l_3\}. \end{aligned}$$

For each element $\lambda \in \Xi_J$, there is the unique discrete series representation $\pi_\lambda$, which is specified by its character formula of $T$. The minimal $K$-type $\mu$ of $\pi_\lambda$ is given by $\mu = \lambda + \rho_n(\lambda) - \rho_c = \lambda + \rho(\lambda) - 2\rho_c$. Note here that $2\rho_c = (2, 0, -2)$, i.e., $\rho_c = (1, 0, -1)$.

## 4. The Gel'fand-Zelevinsky basis for simple $GL(3)$-modules

Recall the paper [GZ] of Gel'fand-Zelevinsky and the previous papers [HiO, HiO2].

Given a highest weight $\mu$, we get a highest weight module $(\tau_\mu, W_\mu)$ of $GL(3)$. Let $W(\mu)$ be the set of weights in $(\tau_\mu, W_\mu)$. For its weight basis, we can associate the set $G(\mu)$ of the Gel'fand-Tsetlin patterns $M$:

$$M = \begin{pmatrix} \mu_1, \ \mu_2, \ \mu_3 \\ m_{12}, \ m_{22} \\ m_{11} \end{pmatrix} \qquad (\mu_1 \geq m_{12} \geq \mu_2 \geq m_{22} \geq \mu_3, \ m_{12} \geq m_{11} \geq m_{22})$$

Then according to [GZ], the simple $K$-module $V_\mu$ of highest weight $\mu$ has a basis $\{f_\mu(M) | M \in G(\mu)\}$, which is called a Gel'fand-Zelevinsky basis.

From now on, we use the offset notation for Gel'fand-Tsetlin patterns (i.e., GT-patterns):

$$M\begin{pmatrix} a,b \\ c \end{pmatrix} = \begin{pmatrix} \mu_1, \ \mu_2, \ \mu_3 \\ m_{12}+a, \ m_{22}+b \\ m_{11}+c \end{pmatrix} \ \text{for } M = \begin{pmatrix} \mu_1, \ \mu_2, \ \mu_3 \\ m_{12}, \ m_{22} \\ m_{11} \end{pmatrix} \ \text{and the offset} \begin{pmatrix} a,b \\ c \end{pmatrix}.$$

Moreover the symbol $M[a]$ means $M\begin{pmatrix} a,-a \\ 0 \end{pmatrix}$. We also denote $\mu_i = m_{i3}$ $(i = 1,2,3)$.

The weight $wt(M)$ of the GT-pattern $M$ is given as

$$(m_{11}, m_{12}+m_{22}-m_{11}, \sum_{i=1}^{3}\mu_i - (m_{12}+m_{22}))$$

Note that $wt(M[a]) = wt(M)$. We sometimes identify the basis $f_\mu(M)$ with the GT-pattern $M$ when the highest weight $\mu$ is clear from the context.

4.1. **Decomposition of the weight polygon.** Given a dominant integral weight $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbf{Z}^3$ $(\mu_1 \geq \mu_2 \geq \mu_3)$, the convex closure of all the permutations $(\mu_a, \mu_b, \mu_c)$ of three components of $\mu$ ($\{a,b,c\}$=\{ 1, 2, 3 \}) generically makes a hexagon $\mathrm{Hex}(\mu)$ in the plane $\{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid \sum_i x_i = \sum_i \mu_i\}$ of the Euclidean 3-space. If $\mu_1 = \mu_2 = \mu_3$, this is a point, and when either $\mu_1 > \mu_2 = \mu_3$ or $\mu_1 = \mu_2 > \mu_3$ it is a triangle.

Then the intersection $\mathrm{Hex}(\mu) \cap \mathbf{Z}^3$ coincides with the set of weights $W(\mu)$ belonging to the highest weight $\mu$. We divide this set into seven parts in the following manner.

$$\begin{aligned}
D(1) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \geq \mu_2, \ w_2 \leq \mu_2, \text{ and } w_3 \leq \mu_2\}, \\
D(3) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \geq \mu_2, \ w_2 \leq \mu_2, \text{ and } w_3 \geq \mu_2\}, \\
D(6) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \geq \mu_2, \ w_2 \geq \mu_2, \text{ and } w_3 \leq \mu_2\}, \\
D(4) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \leq \mu_2, \ w_2 \geq \mu_2, \text{ and } w_3 \leq \mu_2\}, \\
D(5) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \leq \mu_2, \ w_2 \leq \mu_2, \text{ and } w_3 \geq \mu_2\}, \\
D(2) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_1 \leq \mu_2, \ w_2 \geq \mu_2, \text{ and } w_3 \geq \mu_2\}, \\
D(7^+) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_i \geq \mu_2, \text{ for all } i\}, \\
D(7^-) &:= \{w = (w_1,w_2,w_3) \in W(\mu) \mid w_i \leq \mu_2, \text{ for all } i\}.
\end{aligned}$$

Let $w = (w_1,w_2,w_3)$ be a weight belonging to a highest weight $\mu$. Then the set of GT-patterns belonging to $w$ is of the form

$$G(\mu; w) := \{M[-t] \mid t \in \mathbb{Z} \cap [a,b]\}$$

with some $G$-pattern $M$ and some integers $a$, $b$. In particular, we may choose $M$ and $a$ such that $a = 0$. Then $M$ is called the *leading GT-pattern belonging to the weight* $w$. Moreover we call the difference $b - a$ *the meson number* $m(w) \geq 0$ *of the weight* $w$. Therefore the cardinality of $G(\mu; w)$ is $m(w)+1$. We have $m(w) \leq \inf\{\mu_1 - \mu_2, \mu_2 - \mu_3\}$, and the equality $m(w) = \inf\{\mu_1 - \mu_2, \mu_2 - \mu_3\}$ is valid if and only if $w \in D(7^+)$ or $w \in D(7^-)$.

Let $w = (w_1, w_2, w_3) \in W(\mu)$ be a weight belonging to $\mu$. Then we call $i := \mu_1 - w_1$ *the level* of the weight $w$. Let $L(i)$ be the set of weights of level $i$ in $W(\mu)$. Then this is a line segment of at most length $k + l + 1$.

We spell out explicitly all the weights belonging to each domain $D(J)$ $(J = 1, \cdots, 7)$. The result is different depending on the cases:

$$\text{the case } (\alpha): \quad \mu_1 - \mu_2 = k \leq \mu_2 - \mu_3 = l + 1;$$
$$\text{the case } (\beta): \quad \mu_1 - \mu_2 = k \geq \mu_2 - \mu_3 = l + 1.$$

In the former case we denote $\mu = (k+l+1, l+1, 0) \in (\alpha)$, and in the second case $\mu \in (\beta)$.

If the first component $w_1 = k + l + 1 - i$ of the weight $w \in W(\mu)$ belongs to the range such that $0 \leq i \leq \inf\{k, l+1\}$, we say that the weight belongs to the top range, and denote this by $w \in \mathcal{T}$ symbolically; if $i$ satisfies $\inf\{k, l+1\} \leq i \leq \sup\{k, l+1\}$, then the weight belongs to the middle range ($w \in \mathcal{M}$); lastly if $\sup\{k, l+1\} \leq i$, we say that the weight belongs to the bottom range ($w \in \mathcal{B}$).

4.2. **Parametrization of the leading GT-patterns.** We write the exhaustive list of the leading GT-patterns on each $D(J)$ $(1 \leq J \leq 7)$. In what follows,we define the highest weight vector $\mathfrak{m}_0 := \begin{pmatrix} \mu_1, \ \mu_2, \ \mu_3 \\ \mu_1, \ \mu_2 \\ \mu_1 \end{pmatrix}$. We also set $c(i) = |l + 1 - i|$.

On $D(6)$:

(1) If $\mu \in (\alpha)$, or $\mu \in (\beta)$ and $w \in \mathcal{T}$, the leading GT-patterns in $D(6)$ is exhausted by

$$M_{(a)} := \mathfrak{m}_0 \begin{pmatrix} 0, -i+a \\ -i \end{pmatrix} \qquad (0 \leq a \leq i \leq \inf\{k, l+1\})$$

with $wt(M_{(a)}) = (k + l + 1 - i, l + 1 + a, i - a)$, $\delta(M_{(a)}) = a$, and the meson number $m(M_{(a)}) = i - a$.

(2) The case $\mu \in (\beta)$ and $w \in \mathcal{M}$:

$$M_{(b)} := \mathfrak{m}_0 \begin{pmatrix} 0, -l-1+b \\ -i \end{pmatrix} \qquad (l + 1 \leq i \leq k, \ 0 \leq b \leq l + 1)$$

with $wt(M_{(b)}) = (k+l+1-i, b+i, l+1-b)$, $\delta(M_{(b)}) = -l-1+i+b$, and $m(M_{(b)}) = l+1-a$.

On $D(3)$:

(1) If $\mu \in (\alpha)$, or if $\mu \in (\beta)$ and $w \in \mathcal{T}$:

$$M_{(a)} := \mathfrak{m}_0 \begin{pmatrix} -a, -l-1 \\ -i \end{pmatrix} \qquad (0 \leq a \leq i \leq k)$$

with $wt(M_{(a)}) = (k+l+1-i, i-a, l+1+a)$, $\delta(M_{(a)}) = -l-1+i-a$ and $m(M_{(a)}) = i-a$.

(2) if $\mu \in (\beta)$ and $w \in \mathcal{M}$:

$$M_{(b)} := \mathfrak{m}_0 \begin{pmatrix} l+1-i-b, -l-1 \\ -i \end{pmatrix} \qquad (l + 1 \leq i \leq k, \ 0 \leq b \leq l + 1)$$

with $wt(M_{(b)}) = (k + l + 1 - i, l + 1 - b, i + b)$, $\delta(M_{(b)}) = -b$, and $m(M_{(b)}) = l + 1 - b$.

On $D(4)$:

(1) If $\mu \in (\alpha)$ and $w \in \mathcal{M}$ :

$$M_{(a)} := \mathfrak{m}_0 \begin{pmatrix} 0, -i+a \\ -i \end{pmatrix} \qquad (0 \leq a \leq k \leq i \leq l + 1)$$

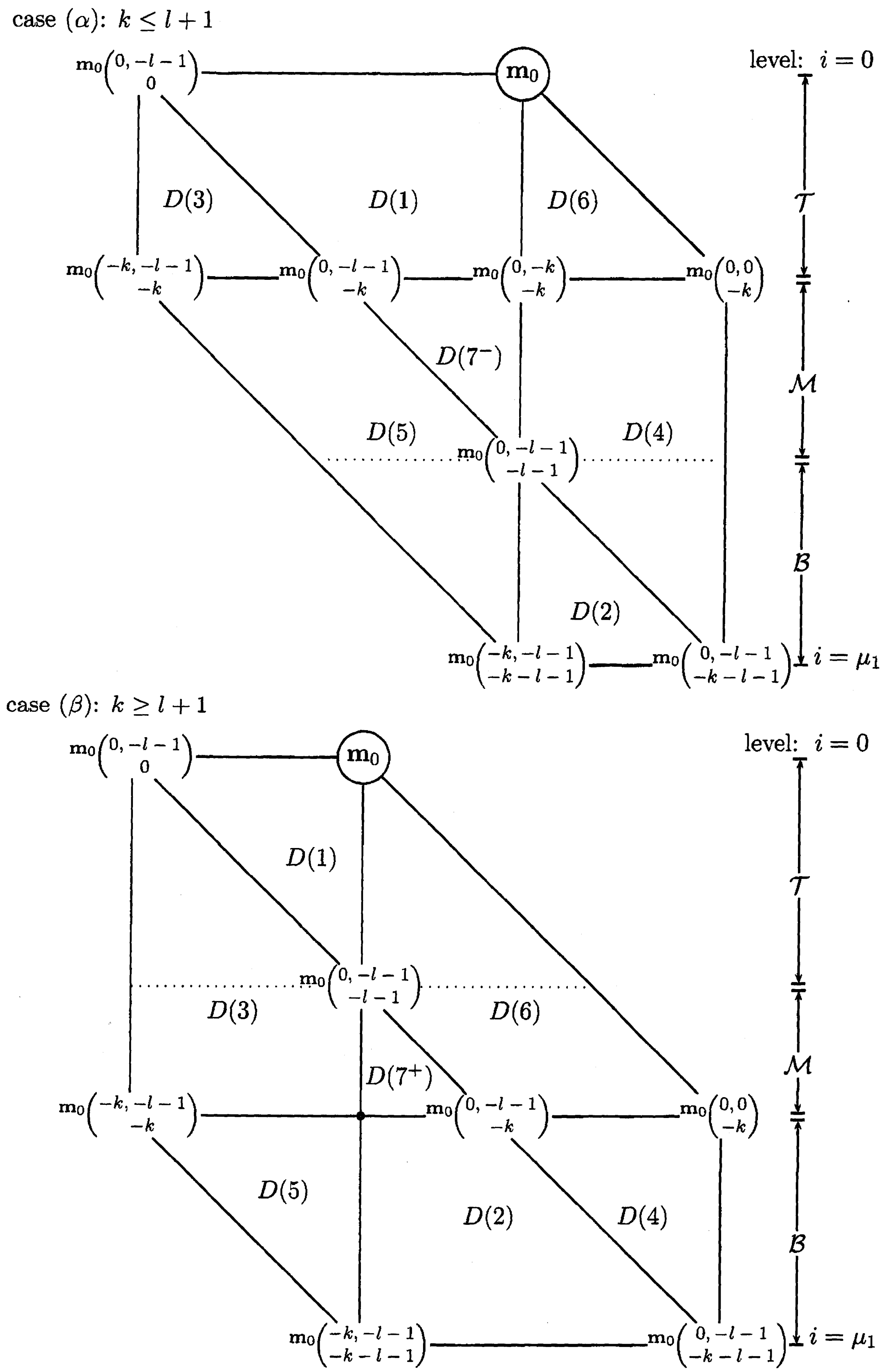


FIGURE 1. the weight polygons and leading $GT$-patterns

with $wt(M_{(a)}) = (k+l+1-i, l+1+a, i-a)$, $\delta(M_{(a)}) = a$, $m(M_{(a)}) = k-a$.
(2) If $\mu \in (\alpha)$ and $w \in \mathcal{B}$, or if $\mu \in (\beta)$:

$$M_{(b)} := \mathfrak{m}_0 \begin{pmatrix} 0, -l-1+b \\ -i \end{pmatrix} \qquad (k \leq i \leq k+l+1,\ 0 \leq b \leq k+l+1-i)$$

with $wt(M_{(b)}) = (k+l+1-i, b+i, l+1-b)$, $\delta(M_{(b)}) = -l-1+i+b$, and $m(M_{(b)}) = k+l+1-i-b$.

On $D(5)$:
(1) $(\alpha)$ and $w \in \mathcal{M}$:

$$M_{(a)} := \mathfrak{m}_0 \begin{pmatrix} -a, -l-1 \\ -i \end{pmatrix} \qquad (0 \leq a \leq k \leq i \leq l+1)$$

with $wt(M_{(a)}) = (k+l+1-i, i-a, l+1+a)$ and $\delta(M_{(a)}) = -l-1+i-a$, $m(M_{(a)}) = k-a$.
(2) If $(\alpha)$ and $w \in \mathcal{B}$, or if $\mu \in (\beta)$:

$$M_{(b)} := \mathfrak{m}_0 \begin{pmatrix} l+1-i-b, -l-1 \\ -i \end{pmatrix} \qquad (0 \leq i-(l+1) \leq k,\ 0 \leq b \leq k+l+1-i)$$

with $wt(M_{(b)}) = (k+l+1-i, l+1-b, i+b)$, $\delta(M_{(b)}) = -b$, and $m(M_{(b)}) = k+l+1-i-b$.

On $D(1)$, the forms of the leading GT-patterns are the same in the both cases $(\alpha)$, $(\beta)$. But the range of the parameters are different.

$$M_{[a]} := \mathfrak{m}_0 \begin{pmatrix} 0, -i-a \\ -i \end{pmatrix} \qquad (0 \leq i \leq \inf\{k, l+1\},\ 0 \leq a \leq l+1-i)$$

with $wt(M_{[a]}) = (k+l+1-i, l+1-a, i+a)$, $\delta(M_{[a]}) = -a$, $m(M_{[a]}) = i$.

On $D(2)$ we have the same form of GT-patterns with different ranges of parameter depending on the $(\alpha)$ case or $(\beta)$ case.

$$M^{[b]} := \mathfrak{m}_0 \begin{pmatrix} -i+(l+1)+b, -l-1 \\ -i \end{pmatrix} \qquad (\sup\{k, l+1\} \leq i \leq k+l+1,\ 0 \leq b \leq i-(l+1))$$

with $wt(M^{[b]}) = (k+l+1-i, l+1+b, i-b)$, $\delta(M^{[b]}) = b$, $m(M^{[b]}) = k+l+1-i$.

On $D(7)$ we have the following selections:

$$(\alpha): \quad M_{[a]} = \mathfrak{m}_0 \begin{pmatrix} 0, -i-a \\ -i \end{pmatrix} \qquad (k \leq i \leq l+1,\ 0 \leq a \leq l+1-i),$$

$$(\beta): \quad M^{[b]} = \mathfrak{m}_0 \begin{pmatrix} -i+(l+1)+b, -l-1 \\ -i \end{pmatrix} \qquad (l+1 \leq i \leq k,\ 0 \leq b \leq i-(l+1)).$$

The weights and $\delta$ are given in the formulas in the domains $D(1)$ and D(2). Moreover the meson number is $\inf\{k,\ l+1\}$.

## 5. Dirac-Schmid equations on $SU(3,1)$

Let $(\tau_\mu, W_\mu)$ be the minimal $K$-type of $(\pi_\lambda, H_\lambda)$. The action of the basis $E_{ij}$ $(i,j = 1,2,3)$ of $\mathfrak{k}_{\mathbb{C}}$ can be computed as the following Proposition.

**Proposition 5.1** (Gel'fand-Zelevinsky). *Let $f_\mu(M)$ be the basis with GT-pattern $M \in G(\mu)$. The action of the six weight vectors $E_{ij}$ $(i \neq j)$ is given as follows:*

$$E_{12}f_\mu(M) = (m_{12} - m_{11})f_\mu(M\begin{pmatrix}0,0\\1\end{pmatrix}) + (m_{23} - m_{22})\chi_+(M)f_\mu(M\begin{pmatrix}0,0\\1\end{pmatrix}[-1]),$$
$$E_{21}f_\mu(M) = (m_{11} - m_{22})f_\mu(M\begin{pmatrix}0,0\\-1\end{pmatrix}) + (m_{12} - m_{23})\chi_-(M)f_\mu(M\begin{pmatrix}0,0\\-1\end{pmatrix}[-1]),$$
$$E_{23}f_\mu(M) = (m_{13} - m_{12})f_\mu(M\begin{pmatrix}1,0\\0\end{pmatrix}) + (m_{13} - m_{12} - \delta(M))\chi_-(M)f_\mu(M\begin{pmatrix}1,0\\0\end{pmatrix}[-1]),$$
$$E_{32}f_\mu(M) = (m_{22} - m_{33})f_\mu(M\begin{pmatrix}0,-1\\0\end{pmatrix}) + (m_{22} - m_{33} + \delta(M))\chi_+(M)f_\mu(M\begin{pmatrix}0,-1\\0\end{pmatrix}[-1]),$$
$$E_{13}f_\mu(M) = (m_{13} - m_{12})f_\mu(M\begin{pmatrix}1,0\\1\end{pmatrix}) - \bar{c}_1(M)f_\mu(M\begin{pmatrix}1,0\\1\end{pmatrix}[-1]),$$
$$E_{31}f_\mu(M) = -(m_{22} - m_{33})f_\mu(M\begin{pmatrix}0,-1\\-1\end{pmatrix}) + c_1(M)f_\mu(M\begin{pmatrix}0,-1\\-1\end{pmatrix}[-1]).$$

*Here we set*

$$\delta(M) := m_{12} + m_{22} - m_{11} - m_{23},$$

*and*

$$\chi_+(M) = \begin{cases} 1, & \text{if } \delta(M) > 0 \\ 0, & \text{otherwise} \end{cases} \quad ; \quad \chi_-(M) = \begin{cases} 1, & \text{if } \delta(M) < 0 \\ 0, & \text{otherwise} \end{cases} .$$

*Moreover*

$$c_1(M) = \inf\{m_{11} - m_{22},\ m_{12} - m_{23}\},\ \bar{c}_1(M) = \inf\{m_{23} - m_{22},\ m_{12} - m_{11}\}.$$

*The actions of $E_{11}, E_{22}$ and $E_{33}$ are given by*

$$E_{11}f_\mu(M) = m_{11}f_\mu(M), \quad E_{22}f_\mu(M) = (m_{12} + m_{22} - m_{11})f_\mu(M),$$
$$E_{33}f_\mu(M) = (\sum_{i=1}^{3} m_{i3} - m_{12} - m_{22})f_\mu(M).$$

For our later purpose, we introduce more piecewise linear functions:

$$D(M) = m_{12} - m_{13} - \delta(M), \quad \bar{D}(M) = m_{33} - m_{22} + \delta(M), \quad c_2(M) = c_1(M)\bar{c}_1(M).$$

**Proposition 5.2.** *Let $(\tau_\mu, V_\mu)$ be the simple $K$-module with a dominant integral weight $\mu = (m_{13}, m_{23}, m_{33}) \in \mathbb{Z}^3$, which is equipped with an Gel'fand-Zelevinsky basis $\{f_\mu(M) | M \in G(\mu)\}$. Set $\mu^{(i)} = \mu + \mathbf{e}_i$ and $\mu^{(-i)} = \mu - \mathbf{e}_i$ $(i = 1, 2, 3)$, and let $\{f^{(\pm i)}(M) | M \in G(\mu^{(\pm i)})\}$ be a Gel'fand-Zelevinsky basis of $V_{\mu^{(\pm i)}}$.*

(1) *Up to a scalar multiple, the injector $V_{\mu+\mathbf{e}_3} \hookrightarrow V_{\mathbf{e}_1} \otimes V_\mu$ is given by*

$$f_{\mu+\mathbf{e}_3}(M') = \begin{pmatrix}1,0\\1\end{pmatrix} \otimes f_\mu\left(\mathbf{m}\begin{pmatrix}0,-1\\-1\end{pmatrix}\right) - \begin{pmatrix}1,0\\0\end{pmatrix} \otimes \left\{f_\mu\left(\mathbf{m}\begin{pmatrix}0,-1\\0\end{pmatrix}\right) + \chi_+(M')f_\mu\left(\mathbf{m}\begin{pmatrix}-1,0\\0\end{pmatrix}\right)\right\}$$
$$+ \begin{pmatrix}0,0\\0\end{pmatrix} \otimes f_\mu\left(\mathbf{m}\begin{pmatrix}0,0\\0\end{pmatrix}\right).$$

*for each $M' = (\mu + \mathbf{e}_3; \mathbf{m}) \in G(\mu + \mathbf{e}_3)$.*

(2) *Up to a scalar multiple, the injector* $V_{\mu+\mathbf{e}_2} \hookrightarrow V_{\mathbf{e}_1} \otimes V_\mu$ *is given by*

$$\begin{aligned}(d_2+1)f_{\mu+\mathbf{e}_2}(M') &= \binom{1,0}{1} \otimes \{-(m'_{22}-m'_{33})f_\mu(\mathbf{m}\tbinom{0,-1}{-1}) + \chi_-(M')\bar{D}(M')f_\mu(\mathbf{m}\tbinom{-1,0}{-1})\} \\ &+ \binom{1,0}{0} \otimes \{(m'_{22}-m'_{33})f_\mu(\mathbf{m}\tbinom{0,-1}{0}) - \bar{c}_1(M')f_\mu(\mathbf{m}\tbinom{-1,0}{0})\} \\ &+ \binom{0,0}{0} \otimes \{(m'_{23}-m'_{22})f_\mu(\mathbf{m}\tbinom{0,0}{0}) + \chi_-(M')\bar{c}_1(M')f_\mu(\mathbf{m}\tbinom{-1,1}{0})\}.\end{aligned}$$

*for each* $M' = (\mu+\mathbf{e}_2;\mathbf{m}) \in G(\mu+\mathbf{e}_2)$. *Here* $\bar{D}(M') = -(m'_{22}-m'_{33}) + \delta(M')$.

(3) *The injector* $V_{\mu+\mathbf{e}_1} \hookrightarrow V_{\mathbf{e}_1} \otimes V_\mu$ *is given by*

$$\begin{aligned}&(d_1+1)(d_1+d_2+1)f_{\mu+\mathbf{e}_1}(M') \\ &= \binom{1,0}{1} \otimes \{-(m'_{13}-m'_{12})(m'_{22}-m'_{33})f_\mu(\mathbf{m}\tbinom{0,-1}{-1}) + \bar{E}(M')f_\mu(\mathbf{m}\tbinom{-1,0}{-1})\} \\ &+ \binom{1,0}{0} \otimes \{(m'_{13}-m'_{12})(m'_{22}-m'_{33})f_\mu(\mathbf{m}\tbinom{0,-1}{0}) - \bar{F}(M')f_\mu(\mathbf{m}\tbinom{-1,0}{0}) \\ &\qquad + c_2(M')f_\mu(\mathbf{m}\tbinom{-2,1}{0})\} \\ &+ \binom{0,0}{0} \otimes \{(m'_{13}-m'_{12})(m'_{13}-m'_{12}+1)f_\mu(\mathbf{m}) - c_2(M')f_\mu(\mathbf{m}\tbinom{-1,1}{0})\}.\end{aligned}$$

(4) *The injector* $V_{\mu-\mathbf{e}_1} \hookrightarrow V_{-\mathbf{e}_3} \otimes V_\mu$ *is given by*

$$\begin{aligned}f_{\mu-\mathbf{e}_1}(M') &= \binom{0,0}{0} \otimes f_\mu\left(M\tbinom{0,0}{0}\right) - \binom{0,-1}{0} \otimes \left\{f_\mu\left(M\tbinom{1,0}{0}\right) + \chi_-(M')f_\mu\left(M\tbinom{0,1}{0}\right)\right\} \\ &+ \binom{0,-1}{-1} \otimes f_\mu\left(M\tbinom{1,0}{1}\right).\end{aligned}$$

(5) *The injector* $V_{\mu-\mathbf{e}_2} \hookrightarrow V_{-\mathbf{e}_3} \otimes V_\mu$ *is given by*

$$\begin{aligned}(d_1+1)f_{\mu-\mathbf{e}_2}(M') &= \binom{0,0}{0} \otimes \{(m'_{12}-m'_{23})f_\mu(M\tbinom{0,0}{0}) + \chi'_+(M')c_1(M')f_\mu(M\tbinom{-1,1}{0})\} \\ &+ \binom{0,-1}{0} \otimes \{(m'_{13}-m'_{12})f_\mu(M\tbinom{1,0}{0}) - c_1(M')f_\mu(M\tbinom{0,1}{0})\} \\ &+ \binom{0,-1}{-1} \otimes \{-(m'_{13}-m'_{12})f_\mu(M\tbinom{1,0}{1}) + \chi'_+(M')D(M')f_\mu(M\tbinom{0,1}{1})\}.\end{aligned}$$

(6) *The injector* $V_{\mu-\mathbf{e}_3} \hookrightarrow V_{\mu-\mathbf{e}_3} \otimes V_\mu$ *is given by*

$$\begin{aligned}&(d_1+1)(d_1+d_2+1)f_{\mu-\mathbf{e}_3}(M') \\ &= \binom{0,0}{0} \otimes \{(m'_{12}-m'_{33}+1)(m'_{22}-m'_{33})f_\mu(M\tbinom{0,0}{0}) - c_2(M')f_\mu(M[-1])\} \\ &- \binom{0,-1}{0} \otimes \{-(m'_{13}-m'_{12})(m'_{22}-m'_{33})f_\mu(M\tbinom{1,0}{0}) + F(M')f_\mu(M\tbinom{0,1}{0}) \\ &\qquad - \chi_-(M')c_2(M')f_\mu(M\tbinom{1,0}{0}[-2])\} \\ &+ \binom{0,-1}{-1} \otimes \{-(m'_{13}-m'_{12})(m'_{22}-m'_{33})f_\mu(M\tbinom{1,0}{1}) + E(M')f_\mu(M\tbinom{0,1}{1})\}.\end{aligned}$$

5.1. **The annihilators of the minimal $K$-types.** In what follows, we restrict ourselves to the case when the discrete series is from $\Xi_{II}$.

Let $\mu$ be the Blattner parameter of $\pi_\lambda \in \Xi_{II}$. Then by the Blattner formula (proved generally by Hecht-Schmid [HS]), the $K$-types $\mu-\beta$ ($\beta \in \Phi^+_{J,n}$) do not occur.

Here we consider the action of $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ on the $K$-finite elements in the represenation space of $\pi_\lambda$, more specifically on the minimal $K$-type $(\tau_\mu, W_\mu) \hookrightarrow \pi_\lambda, H_{\pi,K}$. Then the image $\mathfrak{p}_{\mathbb{C}} W_\mu$ is the canonical image of the $K$-module $\mathfrak{p}_{\mathbb{C}} \otimes W_\mu$.

We regard $E_{i4}$ $(i = 1,2,3)$ are elements in $\mathfrak{p}_+$, and $E_{4i}$ $(i = 1,2,3)$ in $\mathfrak{p}_-$.

**Proposition 5.3.** *We have the following Dirac-Schmid equations:*

(1) $V_{\mu+\mathbf{e}_3} \otimes \det$ *does not occur in* $\pi_\lambda$, *i.e., we have a set of relations:*

$$E_{14} f_\mu(\mathbf{m}\begin{pmatrix} 0,-1 \\ -1 \end{pmatrix}) - E_{24}\left\{ f_\mu(\mathbf{m}\begin{pmatrix} 0,-1 \\ 0 \end{pmatrix}) + \chi_+(M') f_\mu(\mathbf{m}\begin{pmatrix} -1,0 \\ 0 \end{pmatrix})\right\} + E_{34} f_\mu(\mathbf{m}\begin{pmatrix} 0,0 \\ 0 \end{pmatrix}) = 0.$$

(2) $V_{\mu-\mathbf{e}_1} \otimes \det^{-1}$ *does not occur in* $\pi_\lambda$, *i.e., we have relations:*

$$E_{43} f_\mu(\mathbf{m}\begin{pmatrix} 0,0 \\ 0 \end{pmatrix}) + E_{42}\left\{ f_\mu(\mathbf{m}\begin{pmatrix} 1,0 \\ 0 \end{pmatrix}) + \chi_-(M') f_\mu(\mathbf{m}\begin{pmatrix} 0,1 \\ 0 \end{pmatrix})\right\} + E_{41} f_\mu(\mathbf{m}\begin{pmatrix} 1,0 \\ 1 \end{pmatrix}) = 0.$$

(3) $V_{\mu-\mathbf{e}_2} \otimes \det^{-1}$ *does not occur in* $\pi_\lambda$, *i.e., we have relations:*

$$\begin{aligned} &E_{43}\left\{ (m'_{12} - m'_{23}) f_\mu(\mathbf{m}\begin{pmatrix} 0,0 \\ 0 \end{pmatrix}) + \chi_+(M') c_1(M') f_\mu(\mathbf{m}\begin{pmatrix} -1,1 \\ 0 \end{pmatrix})\right\} \\ &+E_{42}\left\{ -(m'_{13} - m'_{12}) f_\mu(\mathbf{m}\begin{pmatrix} 1,0 \\ 0 \end{pmatrix}) + c_1(M') f_\mu(\mathbf{m}\begin{pmatrix} 0,1 \\ 0 \end{pmatrix})\right\} \\ &+E_{41}\left\{ -(m'_{13} - m'_{12}) f_\mu(\mathbf{m}\begin{pmatrix} 1,0 \\ 1 \end{pmatrix}) + \chi_+(M') D(M') f_\mu(\mathbf{m}\begin{pmatrix} 0,1 \\ 1 \end{pmatrix})\right\} = 0. \end{aligned}$$

*Remark.* Note that $m'_{12} - m'_{23} = k + 1 - (m'_{13} - m'_{12})$ and $D(M') = -(k+1) + c_1(M')$.

## 6. Main results

In the following we announce our main results of this paper. The proofs are to be described in detail in [HKO3].

Firstly the following Proposition asserts that the nontrivial matrix coefficients happens only around the "diagonal" entries.

**Proposition 6.1.** *Let $w, w' \in W(\mu)$ be two distinct weights of the highest weight module $\tau_\mu$. Then the radial component of matrix coefficients becomes trivial, namely*

$$c(M \boxtimes \widehat{M'})|A = 0 \quad \text{for} \quad M \in G(\mu, w),\ M' \in G(\mu, w').$$

This is the direct consequence of "$M$-compatibility":

$$\mathrm{Ad}(X) c(M \boxtimes \widehat{M'}; a_r) = c(\tau_\mu(X) M \boxtimes \widehat{M'}; a_r) + c(M \boxtimes \tau_{\hat{\mu}}(X) \widehat{M'}; a_r) = 0.$$

for $X \in \mathfrak{t} \cap \mathfrak{m}$.

6.1. **Standard functions.** In order to show the explicit formulas of matrix coefficients, we firstly try to fix a $\mathbb{Q}$-generating subset of the vector space generated by the matrix coefficients as is indicated in Theorem 6.3.

**Notation 6.2.** *We define the standard functions $\mathcal{S}_{i,a}(r)$ of level $i$, offset $a$, $b$ by*

- *If* $M_{(a)} \in (D(6) \cap \mathcal{T}) \cup (D(4) \cap \mathcal{M}) \cup (D(3) \cap \mathcal{T}) \cup (D(5) \cup \mathcal{M})$,
$$\mathcal{S}_{i,a}(r) = c(M_{(a)} \boxtimes \widehat{M}_{(a)}; a_r)$$
- *If* $M_{(b)} \in (D(6) \cap \mathcal{M}) \cup (D(4) \cap \mathcal{B}) \cup (D(3) \cap \mathcal{M}) \cup (D(5) \cup \mathcal{B})$,
$$\mathcal{S}_{i,b}(r) = c(M_{(b)} \boxtimes \widehat{M}_{(b)}; a_r)$$

**Theorem 6.3.** *Let $M_w$, $M'_w \in G(\mu, w)$ be GT-patterns of weight $w$. Then any matrix element $c(M_w \boxtimes \widehat{M'_w}; a_r)$ on $A$ is a $\mathbb{Q}$-linear combination of the standard functions, with explicitly determined these coefficients.*

We sketch the procedure to show this by the following: first the case where $w'$ belongs to $D(6) \cup D(4)$ is handled. Also we have a similar result for $w' \in D(5) \cup D(3)$ in terms of the co-standard functions; next we consider the case where $w' \in D(1) \cup D(7) \cup D(2)$, and this second result make a bridge between the standard functions and the co-standard functions, and we have done.

Actually, we find the case $D(6) \cup D(4)$ is enough since the other cases can be expressed by the formers up to sign. Thus we need only $(k+1)(l+2)$ standard functions.

The precise coefficients referred in Theorem 6.3 are discussed below.

6.2. **The non-standard parts.** Let us define *the double binomial coefficient* $\left\{ \begin{matrix} d; f \\ d-s \end{matrix} \right\}$ with $d \leq f$ and $0 \leq s \leq d$ by $\left\{ \begin{matrix} d; f \\ d-s \end{matrix} \right\} := \binom{d}{d-s}\binom{f}{d-s}$. Let $[z]_{\{d\}}$ be the Pochhammer symbol defined by $[z]_{\{d\}} := \prod_{l=1}^{d}(z+l-1)$. For our purpose, it is convenient to introduce the double Pochhammer symbol: $[[z]]_{\{d_1;d_2\}} := [z]_{\{d_1\}}[z]_{\{d_2\}}$. When $d=0$, we set $[z]_{\{0\}} = 1$.

**Notation 6.4.** (1) *(the upper case) Assume that $0 \leq i \leq l+1$. Then we define*

$$\gamma(i) = \begin{cases} c(i) & \text{if } wt(M_{(a)}) \in D(6) \cup D(4), \\ 0 & \text{if } wt(M_{(a)} \in D(3) \cup D(5). \end{cases}$$

(2) *(the lower case) Assume that $k+l+1 \geq i \geq l+1 = \mu_2 - \mu_3$. Then we define*

$$\gamma(i) = \begin{cases} 0, & \text{if } wt(M_{(b)}) \in D(6) \cup D(4), \\ c(i), & \text{if } wt(M_{(b)}) \in D(3) \cup D(5). \end{cases}$$

**Theorem 6.5.** *Let $S_{i,a}$, $S_{i,b}$ be standard functions and let $w \in W(\mu)$ be the weight of $\tau_\mu$.*

(1) *(The upper non-central cases) For $d \leq f \leq m(w)$, we have*

$$c(M_{(a)}[-d] \boxtimes \widehat{M_{(a)}}[-f]; a_r) = c(M_{(a)}[-f] \boxtimes \widehat{M_{(a)}}[-d]; a_r)$$
$$= (-1)^f \sum_{s=0}^{d} (-1)^s \left\{ \begin{matrix} d; f \\ d-s \end{matrix} \right\} \frac{[[\gamma(i)+a+d-s+1]]_{\{s;f-d+s\}}}{[[c(i)+2(a+d-s+1)]]_{\{s;f-d+s\}}} \frac{1}{\binom{c(i)+2a+2d-2s}{d-s}} S_{i,d-s}(r).$$

(2) *(The lower non-central cases)*

$$c(M_{(b)}[-d] \boxtimes \widehat{M_{(b)}}[-f]; a_r) = c(M_{(b)}[-f] \boxtimes \widehat{M_{(b)}}[-d]; a_r)$$
$$= (-1)^f \sum_{s=0}^{d} (-1)^s \left\{ \begin{matrix} d; f \\ d-s \end{matrix} \right\} \frac{[[\gamma(i)+b+d-s+1]]_{\{s;f-d+s\}}}{[[c(i)+2(b+d-s+1)]]_{\{s;f-d+s\}}} \frac{1}{\binom{c(i)+2b+2d-2s}{d-s}} S_{i,b+d-s}(r).$$

(3) *(The upper central cases)*

$$c(M_{[a]}[-d] \boxtimes \widehat{M_{[a]}}[-f]; a_r) = c(M_{[a]}[-f] \boxtimes \widehat{M_{[a]}}[-d]; a_r)$$
$$= (-1)^{a+f} \sum_{s=0}^{d} (-1)^s \begin{Bmatrix} d; f \\ d-s \end{Bmatrix} \frac{[[c(i) - a + d - s + 1]]_{\{s; f-d+s\}}}{[[c(i) + 2(d - s + 1))]]_{\{s; f-d+s\}}} \frac{1}{\binom{c(i)+2(d-s)}{a+d-s}} \mathcal{S}_{i,d-s}(r).$$

(4) *(The lower central cases)*

$$c(M^{[b]}[-d] \boxtimes \widehat{M^{[b]}}[-f]; a_r) = c(M^{[b]}[-f] \boxtimes \widehat{M^{[b]}}[-d]; a_r)$$
$$= (-1)^{b+f} \sum_{s=0}^{d} (-1)^s \begin{Bmatrix} d; f \\ d-s \end{Bmatrix} \frac{[[b + d - s + 1]]_{\{s; f-d+s\}}}{[[c(i) + 2d - 2s + 2]]_{\{s; f-d+s\}}} \frac{1}{\binom{c(i)+2(d-s)}{b+d-s}} \mathcal{S}_{i,d-s}(r).$$

6.3. **The Cartan decomposition.** By the previous section, it is enough to detect the standard functions. As mentioned before, the matrix coefficients are defined by its radial components. We specify the coordinate expression of $A$ as follows now.

**Notation 6.6.** *Let $t = \log r$ with $r > 0$, and let $\mathrm{sh}(t)$, $\mathrm{ch}(t)$ are hyperbolic (co)sine functions. We define $\alpha(t) = \dfrac{1}{\mathrm{sh}(t)}$, $\beta(t) = \dfrac{\mathrm{ch}(t)}{\mathrm{sh}(t)}$ and put $a_r = a(t) = \begin{pmatrix} \mathrm{ch}(t) & 0 & 0 & \mathrm{sh}(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{sh}(t) & 0 & 0 & \mathrm{ch}(t) \end{pmatrix}$.*

**Proposition 6.7.** *Let $H_\mathfrak{a} = E_{14} + E_{41}$. Then we have*

$$E_{14} = \frac{1}{2} H_\mathfrak{a} + \frac{1}{2}\alpha(2t)\,\mathrm{Ad}(a_r^{-1}) H_{14} - \frac{1}{2}\beta(2t) H_{14},$$
$$E_{41} = \frac{1}{2} H_\mathfrak{a} - \frac{1}{2}\alpha(2t)\,\mathrm{Ad}(a_r^{-1}) H_{14} + \frac{1}{2}\beta(2t) H_{14}.$$

*Moreover for $i = 2$ or $i = 3$,*

$$E_{i4} = \alpha(t)\,\mathrm{Ad}(a_r^{-1}) E_{i1} - \beta(t) E_{i1}, \qquad E_{4i} = -\alpha(t)\,\mathrm{Ad}(a_r^{-1}) E_{1i} + \beta(t) E_{1i}.$$

We have the obvious realization: $H_\mathfrak{a} \mapsto \varepsilon_r := r\frac{\partial}{\partial r}$.

6.4. **Solutions of the Dirac-Schmid equations for the standard functions.** In this section we find two results.
(1) any standard function is a isobaric $\mathbb{Q}[\frac{\beta(t)}{\alpha(t)}]$-linear combination of certain $(k + l + 2)$ 'backbone' functions $F_i(r)$ $(0 \le i \le k + l + 1)$, which are Gaussian hypergeometric functions with adequate parameters modulo some simple multipliers. Here we note only that

$$F_i(r) := (l + 2 - i)\mathcal{S}_{i,0}(t)$$

if $i \le l + 1$ (the upper case). For $i \ge l + 2$, we give the definition later; (2) for each $i = 0, \cdots, k + l$, the vector of pair $F_i$, $F_{i+1}$ satisfies a differential equation

$$r\frac{d}{dr}\begin{pmatrix} F_i \\ F_{i+1} \end{pmatrix} = A(r) \begin{pmatrix} F_i \\ F_{i+1} \end{pmatrix}.$$

which is equivalent to a hypergeometric equation. Thus any standard function $\mathcal{S}_{j,a}(r)$ of level $j$ is a $\mathbb{Q}$-linear combination of the isobaric functions

$$F_j(r),\ \left(\frac{\beta}{\alpha}\right)F_{j-1}(r),\cdots,\left(\frac{\beta}{\alpha}\right)^s F_{j-s}(r),$$

of adequate length $s$.

We have a definite result when we restrict ourselves to treat the case $i \leq \inf\{k, l+1\}$. For other cases, see [HKO3]. As for the result (1): we have the following:

**Proposition 6.8.** *Assume that* $0 \leq i \leq \inf\{k, l+1\}$, $j+d \leq l+1$ *and* $d \leq k$. *Then we have*

$$(-1)^d\binom{k}{d}\mathcal{S}_{j+d,d}(r) = \sum_{p=0}^{d}\binom{d}{p}(l+2-j-d+p)\frac{\prod_{s=0}^{d-p-1}(k+l+2-j-s)}{\prod_{s=0}^{d-p}(l+2+d-j-s)}\left(\frac{\beta(t)}{\alpha(t)}\right)^p\mathcal{S}_{j+d-p,0}(r).$$

So we only consider the standard functions of the form $\mathcal{S}_{i,0}(r)$. The results corresponding to (2) is given below. We show the standard functions are actually the hypergeometric functions.

6.5. **Constrution of hypergeometric pairs.** We deduce pairs of differential relations between two matrix coefficients associated with the $G$-patterns $\mathbf{m}_0\binom{0,-i}{-i}$ and $\mathbf{m}_0\binom{0,-i-1}{-i-1}$ $(0 \leq i \leq l)$ with weights $(k+l+1-i, l+1, i)$ and $(k+l-i, l+1, i+1)$ respectively. Among others when $i=0$, this gives a differential equation of rank 2 for $c(\mathbf{m}_0 \boxtimes \widehat{\mathbf{m}_0}; a_r)$.

**Corollary 6.9.** *Let* $0 \leq i \leq l$. *Then we have a pair of the forward relation:*

$$(+D):\quad \{\rho_A(\mathcal{R}_{E_{41}}) + (i+1)\beta(t)\}\,\mathcal{S}_{i,0}(r) = \frac{l+1-i}{l+2-i}\cdot(k+l+2-i)\mathcal{S}_{i+1,0}(r)$$

*and the backward equation:*

$$(-D):\quad \{\rho_A(\mathcal{R}_{E_{14}}) + (k+l+2-i)\beta(t)\}\,\mathcal{S}_{i+1,0}(r) = \frac{l+2-i}{l+1-i}\cdot(i+1)\alpha(t)\mathcal{S}_{i,0}(r).$$

*Proof.*) These are immediate consequences of Proposition 5.3. □

Put $m = k+l+2$, and let $\varepsilon_r$ be the Euler operator $r\frac{d}{dr}$. Then formulas in the previous Corollary is rewritten in the following form.

**Notation 6.10.** *We introduce the atomic functions* $F_i(r)$ *by*

$$F_i(r) = (l+2-i)\mathcal{S}_{i,0}(r)$$

*for* $0 \leq i \leq l+1$.

**Lemma 6.11.** *Recall* $r = \exp(t)$, *and let* $m = k+l+2$. *For* $i$ *satisfying* $0 \leq i \leq l+1$, *we have a pair of equations*

$$\frac{1}{2}\{\varepsilon_r - (m-i-1)\alpha(2t) + (m-i-1)\beta(2t) + 2(i+1)\beta(t)\}F_i(r) = (m-i)\alpha(t)F_{i+1}(r),$$

$$\frac{1}{2}\{\varepsilon_r + (m-i-2)\alpha(2t) - (m-i-2)\beta(2t) + 2(m-i)\beta(t)\}F_{i+1}(r) = (i+1)\alpha(t)F_i(r).$$

Now we introduce new variables $p$ and $z$ by

$$p := \mathrm{ch}^2(t) = 1 - z.$$

This system has three regular singularities at $p = 0, \; = 1, \; = \infty$. We determine the exponents of the characteristic equations at these points.

**Lemma 6.12.** *The function $F_i(p)$ belongs to the Riemann's $P$-scheme :*

$$\mathfrak{P}\left\{\begin{matrix} 0 & 1 & \infty \\ -\frac{1}{2}(m-i-1) & 0 & \frac{1}{2}(m+i+1) \\ +\frac{1}{2}(m-i-1) & -(m+1) & \frac{1}{2}(m-i+3) \end{matrix}\right\}.$$

*Among others the unique solution regular at $r = 1$ for $F_i$ is of the form*

$$F_i(r) = const. \cdot \mathrm{ch}(t)^{(m-i-1)} {}_2F_1(m-i+1, m, m+2; 1-p)$$

*with the Gaussian hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ with parameters $\alpha = m - i + 1$, $\beta = m$ and $\gamma = m + 2$.*

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