数理解析研究所講究録
第 1767 巻 2011 年 103-114

LIFTING OF PAIRS OF ELLIPTIC MODULAR FORMS TO SIEGEL MODULAR FORMS OF HALF-INTEGRAL WEIGHT OF DEGREE TWO

SHUICHI HAYASHIDA

1. INTRODUCTION

The aim of this exposition is to explain our recent work on a certain lifting from pairs of two elliptic modular forms to Siegel modular forms of half-integral weight of degree two [H11a, H11b]. The existence of this lifting has been conjectured by Ibukiyama and the author [HI05] as follows.

Conjecture 1 ([HI05]). *Let k be an integer. Let $f \in S_{2k-2}(SL_2(\mathbb{Z}))$, $g \in S_{2k-4}(SL_2(\mathbb{Z}))$ be elliptic modular forms of weight $2k-2$ and $2k-4$, respectively. We assume that f and g are normalized Hecke eigenforms.*

Then, there exists $F \in S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$ such that F is an eigenform for any Hecke operators, and the L-function of F satisfies the identity

$$L(s,F) \;=\; L(s,f)\,L(s-1,g).$$

Here $S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$ is a generalization of the Kohnen plus space for Siegel modular forms of degree two, and where $L(s,F)$ denotes the L-function of $F \in S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$, and where $L(s,f)$ and $L(s,g)$ denote the usual L-functions of f and g, respectively. We remark that the L-function of modular forms of half-integral weight was first introduced by Shimura [Sh73], and was generalized by Zhuravlev [Zh84] for Siegel modular forms of half-integral weight. The above L-function $L(s,F)$ contains the Euler 2-factor which is introduced in [HI05] for $S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$. Some numerical examples of Euler factors have supported the above conjecture.

The following theorem is the main result of this exposition.

Theorem 2. *Let k be an even integer. Let f and g be as in the above conjecture. Then we obtain a Siegel modular form $\mathcal{F}_{f,g} \in S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$ from the pair of f and g. And if $\mathcal{F}_{f,g}$ does not vanish identically, then $\mathcal{F}_{f,g}$ satisfies the properties in the above conjecture.*

2. Construction of $\mathcal{F}_{f,g}$

From now on we assume that k is an even integer, and we assume that $f \in S_{2k-2}(SL_2(\mathbb{Z}))$ and $g \in S_{2k-4}(SL_2(\mathbb{Z}))$ are normalized Hecke eigenforms, namely, the first Fourier coefficients of f and g are both 1.

The purpose of this section is to explain the construction of the Siegel modular form $\mathcal{F}_{f,g} \in S^+_{k-1/2}(\Gamma_0^{(2)}(4))$ which will satisfy the properties in Theorem 2. This construction was suggested by Prof. T. Ikeda to the author in 2001 at the Hakuba conference.

We denote by $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ the space of Siegel cusp forms of weight k of degree 4, and denote by $S^+_{k-\frac{1}{2}}(\Gamma_0^{(3)}(4))$ the generalized plus space of weight $k-\frac{1}{2}$ of degree 3 (cf. [Ib92]), which is a certain subspace of Siegel modular forms of weight $k-\frac{1}{2}$ of degree three.

Let $I(g) \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be the Duke-Imamoglu-Ikeda lift of g. We consider a Fourier-Jacobi expansion of $I(g)$:

$$I(g)\left(\left(\begin{smallmatrix} \tau_3 & z \\ {}^t z & \omega_1 \end{smallmatrix}\right)\right) = \sum_{n>0} \Psi_n(\tau_3, z)\, e^{2\pi\sqrt{-1}n\omega_1},$$

where $\tau_3 \in \mathfrak{H}_3$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{3,1}(\mathbb{C})$. Here we denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n and by $M_{n,m}(K)$ the matrices of size n by m with entries in a commutative ring K. We remark that the form Ψ_n is a Jacobi form of index n of weight k of degree 3. For the definition of Jacobi forms of higher degree the reader is referred to [Zi89].

By the isomorphism between the generalized plus space and the space of Jacobi forms of index one (cf. [Ib92]), there exists $G \in S^+_{k-\frac{1}{2}}(\Gamma_0^{(3)}(4))$ which corresponds to the Jacobi form Ψ_1 of index 1.

We consider the following expansion of a pullback of G:

$$G\left(\left(\begin{smallmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{smallmatrix}\right)\right) = \sum_{h} \mathcal{F}_{h,g}(\tau_2)\, h(\omega_1),$$

where $\tau_2 \in \mathfrak{H}_2$ and $\omega_1 \in \mathfrak{H}_1$, and where h runs over all elements in a basis which consists of Hecke eigenforms in the Kohnen plus space $S^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$. We see that $\mathcal{F}_{h,g}$ belongs to $S^+_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$ for any h. In particular there exists $\hat{f} \in S^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$ which corresponds to f by the Shimura correspondence. Therefore, we define

$$\mathcal{F}_{f,g} := \mathcal{F}_{\hat{f},g},$$

then the form $\mathcal{F}_{f,g}$ belongs to $S^+_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$.

To complete the proof of Theorem 2 we need to show that the form $\mathcal{F}_{f,g}$ is an eigenform for any Hecke operators under the assumption $\mathcal{F}_{f,g} \not\equiv 0$, and we need to show that the L-function $L(s, \mathcal{F}_{f,g})$ of $\mathcal{F}_{f,g}$ coincides with $L(s-1, g)L(s, f)$. We shall show these facts in the following sections.

3. Fourier-Jacobi expansion of G

Let $G \in S^{+}_{k-\frac{1}{2}}(\Gamma_0^{(3)}(4))$ be the form defined in the previous section. We consider a Fourier-Jacobi expansion of G:

$$G(\left(\begin{smallmatrix} \tau_2 & z \\ {}^t z & \omega_1 \end{smallmatrix}\right)) = \sum_{m>0} \phi_m(\tau_2, z)\, e^{2\pi\sqrt{-1}m\omega_1},$$

where $\tau_2 \in \mathfrak{H}_2$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{2,1}(\mathbb{C})$.

The purpose of this section is to give a certain relation among ϕ_m, which plays an important role in the proof of Theorem 2. We remark that the form ϕ_m is a Jacobi form of index m of weight $k-\frac{1}{2}$. Here, Jacobi forms of index m of weight $k-\frac{1}{2}$ of degree 2 are holomorphic functions on $\mathfrak{H}_2 \times M_{2,1}(\mathbb{C})$ which satisfy a certain transformation formula (cf. [H11a].) We denote by $J^{(2)}_{k-\frac{1}{2},m}$ the space of Jacobi forms of index m of weight $k-\frac{1}{2}$ of degree 2.

For any prime p we define two maps $V^{(2)}_{1,p}$ and $V^{(2)}_{2,p}$ which are maps from $J^{(2)*}_{k-\frac{1}{2},m}$ to $J^{(2)}_{k-\frac{1}{2},mp^2}$ (the reader is referred to [H11a] for the precise definition of $V^{(2)}_{i,p}$), where $J^{(2)*}_{k-\frac{1}{2},m}$ is a certain subspace of $J^{(2)}_{k-\frac{1}{2},m}$. These $V^{(2)}_{1,p}$ and $V^{(2)}_{2,p}$ are certain generalizations of the V_l-operator in [EZ85, p.43]. We also define the map $U_l : J^{(2)}_{k-\frac{1}{2},m} \to J^{(2)}_{k-\frac{1}{2},ml^2}$ defined by $(\phi|U_l)(\tau_2, z) := \phi(\tau_2, lz)$ for any $\phi \in J^{(2)}_{k-\frac{1}{2},m}$.

Now we obtain the following relations.

Theorem 3. *Let ϕ_m be as above. For any prime p we have*

$$\phi_m|V^{(2)}_{1,p} = p\, b(p)\, \phi_m|U_p + \phi_{mp^2} + \left(\frac{-m}{p}\right) p^{k-2}\, \phi_m|U_p + p^{2k-3}\, \phi_{\frac{m}{p^2}}|U_{p^2}$$

and

$$\begin{aligned}\phi_m|V^{(2)}_{2,p} &= b(p)\left(\phi_{mp^2} + \left(\frac{-m}{p}\right) p^{k-2}\, \phi_m|U_p + p^{2k-3}\, \phi_{\frac{m}{p^2}}|U_{p^2}\right) \\ &\quad + (p^{2k-4} - p^{2k-6})\, \phi_m|U_p,\end{aligned}$$

where $b(p)$ is the p-th Fourier coefficient of g.

We remark that the above identities can be translated to relations among Fourier coefficients of G. The outline of the proof of Theorem 3 will be given in Section 5. We also remark that the above identities can be regarded as certain generalizations of the Maass relation for Siegel modular forms of half-integral weight of degree three.

4. Proof of Theorem 2

In this section we assume Theorem 3 and shall prove Theorem 2. We use the same symbols in Section 2, 3.

For a prime p, let $X_1(p)$ and $X_2(p)$ be Hecke operators which generate the local Hecke ring acting on $S^+_{k-\frac{1}{2}}(\Gamma_0^{(2)}(4))$. For the precise definition of these Hecke operators the reader is referred to [HI05, p.513]. By the virtue of the definitions of $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$, we can obtain

$$\left(\phi|V_{1,p}^{(2)}\right)(\tau_2,0) = p^{-k+\frac{7}{2}}\phi(\tau_2,0)|X_1(p)$$

and

$$\left(\phi|V_{2,p}^{(2)}\right)(\tau_2,0) = \phi(\tau_2,0)|X_2(p)$$

for any $\phi \in J^{(2)*}_{k-\frac{1}{2},m}$, and where we regard $\phi(\tau_2,0)$ as a function of $\tau_2 \in \mathfrak{H}_2$.

Therefore, due to Theorem 3, we obtain

$$\begin{aligned}
& p^{-k+\frac{7}{2}}G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right)|X_1(p)\\
&= \sum_m p^{-k+\frac{7}{2}}((\phi_m(\tau_2,0))|X_1(p))\, e^{2\pi\sqrt{-1}m\omega_1}\\
&= \sum_m \left((\phi_m|V_{1,p}^{(2)})(\tau_2,0)\right) e^{2\pi\sqrt{-1}m\omega_2}\\
&= \sum_m \{p\, b(p)\, \phi_m(\tau_2,0) + \phi_{mp^2}(\tau_2,0)\\
&\qquad + \left(\frac{-m}{p}\right)p^{k-2}\phi_m(\tau_2,0) + p^{2k-3}\phi_{\frac{m}{p^2}}(\tau_2,0)\Big\}\, e^{2\pi\sqrt{-1}m\omega_1}\\
&= p\, b(p)\, G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right) + G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right)|T_1(p^2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right)|X_2(p)\\
&= b(p)\, G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right)|T_1(p^2) + (p^{2k-4} - p^{2k-6})\, G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right).
\end{aligned}$$

Because $G\left(\left(\begin{smallmatrix}\tau_2 & 0\\ 0 & \omega_1\end{smallmatrix}\right)\right) = \sum_h \mathcal{F}_{h,g}(\tau_2)\, h(\omega_1)$ and because of the above identities, we obtain

$$p^{-k+\frac{7}{2}}\,\mathcal{F}_{f,g}|X_1(p) = (p\,b(p)+a(p))\,\mathcal{F}_{f,g}$$

and

$$\mathcal{F}_{f,g}|X_2(p) = \left(a(p)b(p)+p^{2k-4}-p^{2k-6}\right)\mathcal{F}_{f,g},$$

where $a(p)$ is the p-th Fourier coefficient of f. Hence $\mathcal{F}_{f,g}$ is an eigenform for any Hecke operators. Moreover the eigenvalues of $\mathcal{F}_{f,g}$ follow from the above identities. Let $\lambda(p) = p^{k-\frac{7}{2}}\,(p\,b(p)+a(p))$ and $\omega(p) = a(p)b(p)+p^{2k-4}-p^{2k-6}$ be the eigenvalues of $\mathcal{F}_{f,g}$ with respect to $X_1(p)$ and $X_2(p)$, respectively. Then we obtain

$$\begin{aligned} L(s,\mathcal{F}_{f,g}) &= \prod_p \Big(1-\lambda(p)p^{-k+\frac{7}{2}-s}+(p\,\omega(p)+p^{2k-5}(1+p^2))p^{-2s} \\ &\qquad -\lambda(p)p^{k+\frac{1}{2}-3s}+p^{4k-6-4s}\Big)^{-1} \\ &= L(s,f)\,L(s-1,g). \end{aligned}$$

□

5. Proof of Theorem 3

In this section we shall give the outline of the proof of Theorem 3. The steps for the proof of Theorem 3 are as follows.

(1) By the virtue of the Duke-Imamoglu-Ikeda lift, it is enough to show Theorem 3 for the case of generalized Cohen-Eisenstein series. Here, the generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, which are not cusp forms (cf. [Co75] for degree one, and [Ar98] for general degree).

(2) Show certain linear isomorphisms between the space of certain Jacobi forms of half-integral weight of integer index and the space of Jacobi forms of integral weight of matrix index.

(3) Show a compatibility between the linear isomorphisms in (2) and certain operators which shift the indices of Jacobi forms.

(4) We need a certain relation between Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series of matrix index shown by S.Boecherer [Bo83].

(5) Calculate the action of shift operators on Jacobi-Eisenstein series of matrix index explicitly.

(6) Show Theorem 3 for the case of generalized Cohen-Eisenstein series by using (2), (3), (4) and (5).

In this section we will explain the above steps more precisely.

5.1. **Generalized Cohen-Eisenstein series.** Let k' be an even integer. We denote by $\mathcal{H}^{(3)}_{k'-\frac{1}{2}}$ the generalized Cohen-Eisenstein series of weight $k'-\frac{1}{2}$ of degree 3, which is a certain Siegel modular form of weight $k'-\frac{1}{2}$ of degree 3 (cf. [Co75, Ar98]). More precisely, the form $\mathcal{H}^{(3)}_{k'-\frac{1}{2}}$ corresponds to the Jacobi-Eisenstein series $E^{(3)}_{k',1}$ of index 1 of weight k' of degree 3 by the isomorphism between Siegel modular forms of half-integral weight and Jacobi forms of index one (cf. [EZ85, Ib92].)

By the virtue of the Duke-Imamoglu-Ikeda lift, the Fourier coefficients of G satisfy the similar relation which the Fourier coefficients of $\mathcal{H}^{(3)}_{k'-\frac{1}{2}}$ satisfy. Hence, it is enough to show Theorem 3 for the case of $\mathcal{H}^{(3)}_{k'-\frac{1}{2}}$ for sufficiently many k'.

We take a Fourier-Jacobi expansion of $\mathcal{H}^{(3)}_{k-\frac{1}{2}}$:

$$\mathcal{H}^{(3)}_{k-\frac{1}{2}}\left(\begin{pmatrix} \tau_2 & z \\ {}^t z & \omega_1 \end{pmatrix}\right) = \sum_{m\geq 0} e^{(2)}_{k-\frac{1}{2},m}(\tau_2, z)\, e^{2\pi\sqrt{-1}m\omega_1},$$

where $\tau_2 \in \mathfrak{H}_2$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{2,1}(\mathbb{C})$. The form $e^{(2)}_{k-\frac{1}{2},m}$ is a Jacobi form of index m of weight $k-\frac{1}{2}$ of degree 2, which belongs to $J^{(2)*}_{k-\frac{1}{2},m}$.

We need to show the following theorem.

Theorem 4. *Let $k > 5$ be an even integer and let $e^{(2)}_{k-\frac{1}{2},m}$ be as above. For any prime p we have*

$$\begin{aligned} e^{(2)}_{k-\frac{1}{2},m}|V^{(2)}_{1,p} &= p\,(1+p^{2k-5})\,e^{(2)}_{k-\frac{1}{2},m}|U_p + e^{(2)}_{k-\frac{1}{2},mp^2} \\ &\quad + \left(\frac{-m}{p}\right) p^{k-2}\, e^{(2)}_{k-\frac{1}{2},m}|U_p + p^{2k-3}\, e^{(2)}_{k-\frac{1}{2},\frac{m}{p^2}}|U_{p^2} \end{aligned}$$

and

$$\begin{aligned} e^{(2)}_{k-\frac{1}{2},m}|V^{(2)}_{2,p} &= (1+p^{2k-5})\left\{ e^{(2)}_{k-\frac{1}{2},mp^2} + \left(\frac{-m}{p}\right) p^{k-2}\, e^{(2)}_{k-\frac{1}{2},m}|U_p \right. \\ &\quad \left. + p^{2k-3}\, e^{(2)}_{k-\frac{1}{2},\frac{m}{p^2}}|U_{p^2} \right\} + (p^{2k-4}-p^{2k-6})\, e^{(2)}_{k-\frac{1}{2},m}|U_p. \end{aligned}$$

Theorem 3 follows from this theorem.

5.2. **An isomorphism between two spaces of Jacobi forms.** Let $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ be the space of Siegel modular forms of weight k of even degree $n+2$, and let $M^*_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ be the subspace of $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ which consists of all Duke-Imamoglu-Ikeda lifts in $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$. We remark that the subspace $M^*_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ contains the Siegel-Eisenstein series.

Let $J_{k,1}^{(n+1)*}$ be the subspace of $J_{k,1}^{(n+1)}$ which consists of all Jacobi forms of index 1 obtained by the Fourier-Jacobi expansion of the Siegel modular forms in $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$. We denote by $M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$ the subspace of the generalized plus space $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ of degree $n+1$ which corresponds to $J_{k,1}^{(n+1)*}$ by the isomorphism between two spaces $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ and $J_{k,1}^{(n+1)}$. Now, we have the following diagram.

$$\begin{array}{ccccc} M_k(\mathrm{Sp}_{n+2}(\mathbb{Z})) & \supset & M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z})) & & \\ \downarrow & & \downarrow & & \\ J_{k,1}^{(n+1)} & \supset & J_{k,1}^{(n+1)*} & \cong & M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \end{array}$$

Let $\phi_1 \in J_{k,1}^{(n+1)}$. We regard the function $\phi_1(\tau_{n+1}, z)\, e^{2\pi\sqrt{-1}\omega_1}$ as a function on $\begin{pmatrix} \tau_{n+1} & z \\ {}^t z & \omega_1 \end{pmatrix} \in \mathfrak{H}_{n+2}$, where $\tau_{n+1} \in \mathfrak{H}_{n+1}$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{n+1,1}(\mathbb{C})$. We consider the expansion:

$$\phi_1(\tau_{n+1}, z)\, e^{2\pi\sqrt{-1}\omega_1} = \sum_{\substack{\mathcal{S}\in Sym_2^* \\ \mathcal{S}=\begin{pmatrix} * & * \\ * & 1 \end{pmatrix}}} \phi_{\mathcal{S}}(\tau_n, z')\, e^{2\pi\sqrt{-1}\,tr(\mathcal{S}\omega_2)}$$

where Sym_2^* denotes the set of all half-integral symmetric matrices of size 2, and where $\begin{pmatrix} \tau_{n+1} & z \\ {}^t z & \omega_1 \end{pmatrix} = \begin{pmatrix} \tau_n & z' \\ {}^t z' & \omega_2 \end{pmatrix} \in \mathfrak{H}_{n+2}$, $\tau_n \in \mathfrak{H}_n$, $\omega_2 \in \mathfrak{H}_2$ and $z' \in M_{n,2}(\mathbb{C})$. We remark that $\phi_{\mathcal{S}} \in J_{k,\mathcal{S}}^{(n)}$, where $J_{k,\mathcal{S}}^{(n)}$ is the space of Jacobi forms of weight k of index $\mathcal{S}$ of degree n (cf. [Zi89], [H11a]). We consider the map

$$\mathrm{FJ}_{1,\mathcal{S}} : J_{k,1}^{(n+1)} \to J_{k,\mathcal{S}}^{(n)} \quad \text{via} \quad \phi_1 \mapsto \phi_{\mathcal{S}}.$$

We denote by $J_{k,\mathcal{S}}^{(n)*} \subset J_{k,\mathcal{S}}^{(n)}$ the image of $J_{k,1}^{(n+1)*}$ by the map $\mathrm{FJ}_{1,\mathcal{S}}$. Namely, the Jacobi forms in $J_{k,\mathcal{S}}^{(n)*}$ are obtained by the Duke-Imamoglu-Ikeda lifting and by the Fourier-Jacobi expansion.

Let m be an integer such that $-m \equiv 0, 1 \mod 4$. We take a matrix $\mathcal{M} \in \mathrm{Sym}_2^*$, such that $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$ and $\det(2\mathcal{M}) = m$.

Lemma 5. *The space $J_{k,\mathcal{M}}^{(n)*}$ is linearly isomorphic to the space $J_{k-\frac{1}{2},m}^{(n)*}$ and this isomorphism is given by a correspondence between Fourier coefficients. We denote by $\iota_{\mathcal{M}}$ this map. Moreover, the following diagram* *is commutative.*

$$\begin{array}{ccc} J_{k,1}^{(n+1)*} & \longrightarrow & M^*_{k-\frac{1}{2}}(\Gamma_0^{(n+1)}(4)) \\ {\scriptstyle FJ_{1,\mathcal{M}}}\downarrow & & \downarrow \\ J_{k,\mathcal{M}}^{(n)*} & \xrightarrow{\iota_{\mathcal{M}}} & J_{k-\frac{1}{2},m}^{(n)*}, \end{array}$$

where the map of the down arrow in the right hand side is given by the Fourier-Jacobi expansion.

Due to the above lemma the m-th Fourier-Jacobi coefficient $e_{k-\frac{1}{2},m}^{(n)}$ of $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ corresponds to a certain Jacobi form $e_{k,\mathcal{M}}^{(n)}$ of index $\mathcal{M}$. Here $e_{k-\frac{1}{2},m}^{(n)} \in J_{k-\frac{1}{2},m}^{(n)*}$, $e_{k,\mathcal{M}}^{(n)} \in J_{k,\mathcal{M}}^{(n)*}$ and $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)} \in M^*_{k-\frac{1}{2}}(\Gamma_0^{(n+1)}(4))$. In particular, we can say that the form $e_{k,\mathcal{M}}^{(n)}$ is the $\mathcal{M}$-th Foureir-Jacobi coefficient of the Siegel-Eisenstein series of weight k of degree $n+2$ (see §5.4 below).

Now we want to translate the relation among $e_{k-\frac{1}{2},m}^{(2)}$ in Theorem 4 to the relation among $e_{k,\mathcal{M}}^{(2)}$. We need to show a compatibility between the above isomorphism $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)*} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$ and certain operators acting on each space. In the next subsection we will explain this compatibility.

5.3. **Compatibility between isomorphisms of Jacobi forms and certain operators.** We can define certain operators

$$V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)} : J_{k,\mathcal{M}}^{(n)} \to J_{k,\mathcal{M}[\begin{pmatrix} p & \\ & 1\end{pmatrix}]}^{(n)} \qquad (i = 1, ..., n).$$

Namely, the operator $V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)}$ changes the matrix of the index of Jacobi forms. The operator $V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)}$ is a generalization of V_l-operator in [EZ85, p.43]. For the precise definition of $V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)}$ the reader is referred to [H11a].

Lemma 6. *Let $\mathcal{M}$ be a matrix in Lemma 5. Then, the form*

$$\iota_{\mathcal{M}[\begin{pmatrix} p & \\ & 1\end{pmatrix}]}(e_{k,\mathcal{M}}^{(n)} | V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)})$$

is the same to $e_{k-\frac{1}{2},m}^{(n)} | V_{i,p}^{(n)}$ up to constant. Namely, the isomorphism $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)*} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$ and the operators $V_{i,p}^{(n)}$, $V_{i,\begin{pmatrix} p & \\ & 1\end{pmatrix}}^{(n)}$ are compatible.

We remark that the form $e^{(n)}_{k,\mathcal{M}}|V^{(n)}_{i,\binom{p}{\ 1}} \in J^{(n)}_{k,\mathcal{M}[\binom{p}{\ 1}]}$ *may not be in* $J^{(n)*}_{k,\mathcal{M}[\binom{p}{\ 1}]}$*, but we can extend the map* $\iota_{\mathcal{M}[\binom{p}{\ 1}]}$ *to* $J^{(n)}_{k,\mathcal{M}[\binom{p}{\ 1}]}$.

Due to Lemma 5 and 6, we can translate the relation among $e^{(2)}_{k-\frac{1}{2},m}$ ($m \in \mathbb{Z}$) in Theorem 4 to the relation among $e^{(2)}_{k,\mathcal{M}}$ ($\mathcal{M} \in \mathrm{Sym}_2^*$). Hence, it is enough to show the same relation in Theorem 4 for $e^{(2)}_{k,\mathcal{M}}$. For the calculation of $e^{(2)}_{k,\mathcal{M}}|V^{(2)}_{i,\binom{p}{\ 1}}$, we need relations between $e^{(2)}_{k,\mathcal{M}}$ and Jacobi-Eisenstein series. We will explain this relation in the next subsection.

5.4. **Fourier-Jacobi coefficients of Siegel-Eisenstein series.** We denote by $E^{(n+2)}_k$ the Siegel-Eisenstein series of weight k of degree $n+2$. We take the Fourier-Jacobi expansion:

$$E^{(n+2)}_k\left(\begin{pmatrix} \tau_n & z \\ {}^tz & \omega_2 \end{pmatrix}\right) = \sum_{\mathcal{M}\in Sym_2^*} e^{(n)}_{k,\mathcal{M}}(\tau_n, z)\, e^{2\pi\sqrt{-1}\,tr(\mathcal{M}\omega_2)},$$

where $\tau_n \in \mathfrak{H}_n$, $\omega_2 \in \mathfrak{H}_2$ and $z \in M_{n,2}(\mathbb{C})$.

We denote by Sym_2^+ the set of all positive-definite half-integral symmetric matrices of size 2. We let $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \mathrm{Sym}_2^+$. We put $m = \det(2\mathcal{M})$, and let D_0 be the discriminant of $\mathbb{Q}(\sqrt{-m})$, and put $f = \sqrt{-\dfrac{m}{D_0}}$. We remark that f is a natural number.

We set $g_k(m) := \sum_{d|f} \mu(d)\, h_{k-\frac{1}{2}}\left(\dfrac{m}{d^2}\right)$, where $\mu(d)$ is the Möbius function and $h_{k-\frac{1}{2}}(d)$ denotes the d-th Fourier coefficient of the Cohen-Eisenstein series of weight $k-\frac{1}{2}$ (cf. [Co75].)

We denote by $E^{(n)}_{k,\mathcal{S}}$ the Jacobi-Eisenstein series of index $\mathcal{S} \in \mathrm{Sym}_2^+$ of weight k of degree n (cf. [Zi89].) The following proposition follows from [Bo83, Satz 7].

Proposition 7. *For* $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in Sym_2^+$ *we put* $m = \det(2\mathcal{M})$. *Let* D_0*,* f *be as above. If* $k > n+3$*, then*

$$e^{(n)}_{k,\mathcal{M}}(\tau, z) = \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E^{(n)}_{k,\mathcal{M}[{}^tW_d{}^{-1}]}(\tau, zW_d),$$

where we chose a matrix $W_d \in GL_2(\mathbb{Q}) \cap M_{2,2}(\mathbb{Z})$ *for each* d *which satisfies the conditions* $\det(W_d) = d$ *and* $W_d^{-1}\mathcal{M}\,{}^tW_d{}^{-1} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in Sym_2^+$. *The above summation is well-defined, namely it does not dependent on the choice of matrix* W_d.

Hence, relations among $e^{(2)}_{k,\mathcal{M}}$ is translated to the relations among $E^{(2)}_{k,\mathcal{M}}$. Namely, for the calculation of $e^{(2)}_{k,\mathcal{M}}|V^{(2)}_{i,\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)}$ it is enough to calculate $E^{(2)}_{k,\mathcal{M}[{}^tW_d{}^{-1}]}|V^{(2)}_{i,\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)}$, because $V^{(2)}_{i,\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)}$ is a linear map.

5.5. **Jacobi-Eisenstein series $E^{(2)}_{k,\mathcal{M}}$ and operators $V^{(2)}$.** Let $E^{(2)}_{k,\mathcal{M}}$ be the Jacobi-Eisenstein series of weight k of index $\mathcal{M} \in \mathrm{Sym}_2^+$ of degree 2, which is a holomorphic function on $\mathfrak{H}_2 \times \mathbb{C}^2$.

Proposition 8. *We assume $\mathcal{M} = \left(\begin{smallmatrix} * & * \\ * & 1\end{smallmatrix}\right)$. Then, the form $E^{(2)}_{k,\mathcal{M}}|V^{(2)}_{i,\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)}$ ($i = 1, 2$) is written as a linear combination of three forms*

$$E^{(2)}_{k,\mathcal{M}}(\tau, z\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)),\ E^{(2)}_{k,\mathcal{M}[\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)]}(\tau, z),$$

and

$$E^{(2)}_{k,\mathcal{M}[X^{-1}\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1}]}(\tau, z\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right){}^tX\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)),$$

explicitly. Here, if $p|f$, then $X = \left(\begin{smallmatrix} 1 & 0 \\ x & 1\end{smallmatrix}\right)$ is a 2×2 matrix with an integer x, such that $\mathcal{M}[X^{-1}\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1}] \in Sym_2^+$, and if $p\nmid f$, then the third form of the above does not appear.

By Lemma 5, 6 and Proposition 7, 8, we obtain Theorem 4. Thus, we can conclude Theorem 3.

6. Examples of non-vanishing of $\mathcal{F}_{f,g}$

In Theorem 2 we need the assumption that $\mathcal{F}_{f,g}$ does not identically vanish. In this final section we shall give some examples of non-vanishing of $\mathcal{F}_{f,g}$.

For even integer $k \le 24$, the dimensions of the spaces $S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$, $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $S^+_{k-1/2}(\Gamma_0^{(2)}(4))$ are given as follows.

k	2, 4, 6	8	10	12	14	16	18	20	22	24
$\dim S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$	0	1	1	1	2	2	2	3	3	3
$\dim S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$	0	0	1	1	1	2	2	2	3	3
$\dim S^+_{k-1/2}(\Gamma_0^{(2)}(4))$	0	0	1	1	2	4	4	6	9	10

We remark that the dimension formula for $S^+_{k-1/2}(\Gamma_0^{(2)}(4))$ is given in [HI05].

Lemma 9. *Let $k = 10$, 12 or 14. Then, the form $\mathcal{F}_{f,g} \in S^+_{k-1/2}(\Gamma_0^{(2)}(4))$ does not vanish identically for any normalized Hecke eigenforms $f \in S_{2k-2}(SL_2(\mathbb{Z}))$ and $g \in S_{2k-4}(SL_2(\mathbb{Z}))$.*

Proof. We assume $k = 10$, 12 or 14. Then, $\dim S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) = 1$.

Let $g \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and let $G \in S^+_{k-1/2}(\Gamma_0^{(3)}(4))$ be the Siegel modular form of weight $k-1/2$ of degree 3 which is constructed from g in §2. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and let $h \in S^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$ be the modular form of weight $k - \frac{1}{2}$ which corresponds to f by the Shimura correspondence. Here $S^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$ denotes the Kohnen plus space.

Because $\dim S^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4)) = 1$, we have

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \mathcal{F}_{f,g}(\tau_2)h(\omega_1),$$

where $\tau_2 \in \mathfrak{H}_2$ and $\omega_1 \in \mathfrak{H}_1$. Hence, it is enough to show that $G\left(\left(\begin{smallmatrix} \tau_2 & 0 \\ 0 & \omega_2 \end{smallmatrix}\right)\right)$ does not vanish identically. We take the expansion

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \sum_{N \in Sym_2^+,\, m \in \mathbb{Z}} K(N,m)\, e^{2\pi\sqrt{-1}\, tr(N\tau_2)} e^{2\pi\sqrt{-1} m\omega_2},$$

and we take the Fourier expansion of G:

$$G(Z) = \sum_{M \in Sym_3^+} C(M)\, e^{2\pi\sqrt{-1}\, tr(MZ)}.$$

Then

$$K(N,m) = \sum_{\substack{l \in M_{2,1}(\mathbb{Z}) \\ 4Nm - l^t l > 0}} C\left(\begin{pmatrix} N & \frac{1}{2}l \\ \frac{1}{2}{}^t l & m \end{pmatrix}\right).$$

The Fourier coefficients of G can be calculated by the formula of the Fourier coefficients of the Duke-Imamoglu-Ikeda lift. By numerical calculations we obtain $K\left(\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right), 3\right) \neq 0$. Therefore, we conclude $\mathcal{F}_{f,g} \not\equiv 0$. $\square$

References

[Ar98] T. Arakawa: Köcher-Maass Dirichlet Series Corresponding to Jacobi forms and Cohen Eisenstein Series, *Comment. Math. Univ. St. Paul.* **47** No.1 (1998), 93–122.

[Bo83] S. Boecherer: Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math. Z.* **183** (1983), 21–46

[Co75] H. Cohen: Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217** (1975), 171–185.

[EZ85] M. Eichler and D. Zagier: Theory of Jacobi Forms, Progress in Math. **55**, Birkhäuser, Boston-Basel-Stuttgart, (1985).

[HI05] S. Hayashida and T. Ibukiyama: Siegel modular forms of half integral weights and a lifting conjecture, *Journal of Kyoto Univ,* **45** No.3 (2005), 489-530.

[H11a] S. Hayashida: A Maass relation for generalized Cohen-Eisenstein series of degree two and of degree three, *in preparation.*

[H11b] S. Hayashida: On the lifting of pairs of elliptic modular forms to Siegel modular forms of half-integral weight of degree two, *in preparation.*

[Ib92] T. Ibukiyama: On Jacobi forms and Siegel modular forms of half integral weights, *Comment. Math. Univ. St. Paul.* **41** No.2 (1992), 109–124.

[Sh73] G. Shimura : On modular forms of half integral weight, *Ann. of Math* (2) **97** (1973), 440–481.

[Zh84] V. G. Zhuravlev: Euler expansions of theta transforms of Siegel modular forms of half-integral weight and their analytic properties, *Math. sbornik.* **123 (165)** (1984), 174–194.

[Zi89] C. Ziegler : Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg.* **59** (1989), 191–224.

Shuichi Hayashida
International College/ Department of Mathematics
Osaka University
Machikaneyama 1-30, Toyonaka, Osaka 560-0043, Japan
e-mail hayashida@math.sci.osaka-u.ac.jp