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INSTABILITY OF CUSPIDAL EIGENVALUES FOR HYPERBOLIC SURFACES

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ABSTRACT. This is an announcement of the main results in [25] concerning dissolving cusp forms. The detailed proofs will appear there.

1. INTRODUCTION AND BACKGROUND

A hyperbolic surface of finite area can be realized as the quotient $M=\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the hyperbolic upper-half plane

$$\mathbb{H}=\{z=x+iy, y>0\}$$

and Γ is a discrete cofinite subgroup of $\mathrm{SL}_2(\mathbb{R})$. The hyperbolic metric and the Laplace operator are given by

$$\frac{dx^2+dy^2}{y^2}, \quad \Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

Traditionally we write the eigenvalue equation as

$$\Delta u+s(1-s)u=0.$$

We also use the parametrization of the spectrum via $s=1/2+ir$. We are interested in automorphic functions on $\Gamma\backslash\mathbb{H}$, i.e. functions u that satisfy

$$u(\gamma\cdot z)=u(z), \quad \gamma\in\Gamma, \quad z\in\mathbb{H}.$$

Later on other automorphic forms play a crucial role in understanding the instability of the spectrum. We are particularly interested in the Maaß cusp forms, which are functions $u_j(z)$ on $\Gamma\backslash\mathbb{H}$ satisfying the eigenvalue equation with eigenvalue $s_j(1-s_j)$, and $u_j(z)\to 0$, $z\to i\infty$. In terms of notation we will also write $s_j=1/2+ir_j$. The actual eigenvalue is $1/4+r_j^2$ and is $\geq 1/4$, iff $r_j\in\mathbb{R}$. There may exist finitely many other eigenvalues $\lambda_j=1/4+r_j^2$ that are less than $1/4$, called exceptional eigenvalues, with $\lambda_0=0$ always occuring, corresponding to the constant eigenfunction. The eigenvalues which are $\geq 1/4$ are embedded in the continuous spectrum. This consists of the interval $[1/4,\infty)$ with

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multiplicity equal to the number of cusps. The continuous spectrum is provided by Eisenstein series, defined (for the cusp $i\infty$) as

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s, \quad \Re(s) > 1$$

and having eigenvalue $s(1-s)$. The most important properties of this series are (i) they have analytic continuation on $\mathbb{C}$, with the line $\Re(s) = 1/2$ corresponding to the continuous spectrum (ii) they satisfy a functional equation $E(z,s) = \phi(s)E(z,1-s)$, where $\phi(s)$ is the scattering operator and (iii) they have no pole on $\Re(s) = 1/2$. The poles of $E(z,s)$ with $\Re(s) < 1/2$ are called resonances or scattering poles.

2. Weyl's Law and perturbation theory

If the group Γ is cocompact, i.e. $\Gamma\backslash\mathbb{H}$ is compact, i.e. has no cusps, there is no continuous spectrum and we can count the eigenvalues, which are a discrete set accumulating to ∞, by defining the counting function

$$N(T) = \#\{r_j \leq T, \quad r_j \geq 0\}.$$

The result, which holds in much greater generality, is

$$N(T) \sim \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}T^2, \quad T \to \infty.$$

However, for cofinite subgroups, one must take into account the continuous spectrum. For spectral value $1/4 + T^2$ one should consider the winding number of the scattering function given by

$$M(T) = -\frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr.$$

We get the following theorem, due to Selberg.

Theorem 2.1 (Weyl's Law).

$$N(T) + M(T) \sim \frac{\mathrm{Area}\ (\Gamma\backslash\mathbb{H})}{4\pi}T^2, \quad T \to \infty.$$

Theorem 2.1 follows from the Selberg trace formula. The main questions and problems related to Theorem 2.1 are:

(1) Are there infinitely many Maaß cusp forms?
(2) Let the Selberg zeta function be defined for $\Re(s) > 1$ by the infinite product

$$Z(s) = \prod_{\{\gamma_0\}} \prod_{k=0}^{\infty} (1 - N(\gamma_0)^{-(s+k)}).$$

Here γ_0 are the primitive hyperbolic conjugacy classes of Γ and $N(\gamma_0)$ is the norm of of γ_0. This has also analytic continuation on $\mathbb{C}$. Are there infinitely many zeros of the Selberg zeta function on $\Re(s) = 1/2$?

(3) A function is called cuspidal if $\int_\gamma f = 0$ for all horocycles γ. Is the space of continuous cuspidal functions infinite dimensional?

(4) Which of the two terms $N(T)$ or $M(T)$ dominate the asymptotics in Theorem 2.1?

To be able to separate $N(T)$ from $M(T)$ in Weyl's law, one needs to know estimates on the scattering function (determinant). For $\mathrm{SL}_2(\mathbb{Z})$ a classical calculation [5] gives

$$\phi(s) = \frac{\pi^{2s-1}\Gamma(1-s)\zeta(2-2s)}{\Gamma(s)\zeta(2s)},$$

where $\zeta(s)$ is the Riemann zeta function and $\Gamma(s)$ is the Gamma function. Using classical estimates of $\zeta(s)$ on the line of absolute convergence $\Re(s) = 1$, as is required here, we can estimate the continuous spectrum contribution by $M(T) = O(T)$ and this implies

$$N(T) \sim \frac{\text{Area }(\Gamma\backslash\mathbb{H})}{4\pi}T^2. \tag{2.1}$$

For other congruence subgroups, like $\Gamma_0(N)$, Dirichlet L-series show up in det $\phi_{ij}(s)$ but the result is the same. The question whether Weyl's law in the form of (2.1) holds for all cofinite subgroups was left open for a long time. Groups where infinitely many cusp forms exist include e.g. cycloidal groups (Venkov). In the beginning eighties researchers lost faith in (2.1). In fact, Phillips and Sarnak initiated a project to disprove Weyl's law for cuspidal eigenvalues by using perturbation theory. The general philosophy is that for a cofinite subgroup Maaß cuspidal eigenvalues, i.e. the embedded eigenvalues of the Laplace operator Δ in the continuous spectrum are unstable. Unless symmetries force them to remain cusp forms, they tend to become resonances.

This is a good point to introduce other types of automorphic forms, L-functions and Fourier expansions. We assume that the cusp at infinity has width 1. A Maaß cusp form u_j has a Fourier expansion

$$u_j(z) = \sum_{n\neq 0} b_n\sqrt{y}K_{s_j-1/2}(2\pi|n|y)e^{2\pi inx},$$

where $K_s(y)$ is the McDonald-Bessel function. We normalize u_j to have L^2 norm equal to 1. We also need holomorphic cusp forms of weight k. These are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = (cz+d)^k f(z), \quad \gamma \in \Gamma,$$

with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

In the framework of perturbation theory we only need $k = 2$ and $k = 4$. For $k = 2$, which is the simplest case, we introduce the antiderivative

$$F(z) = \int_{i\infty}^{z} f(w)\, dw = \sum_{n=1}^{\infty} \frac{a_n}{2\pi i n} e^{2\pi i n z}.$$

If f and g are holomorphic cusp forms of weight k, then the Rankin–Selberg convolution of f and $g \in S_k(\Gamma)$ is given by

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \frac{a_n c_n}{n^s}, \quad \text{if} \quad g(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}.$$

Similarly, we define

$$L(f \otimes u_j, s) = \sum_{n=1}^{\infty} \frac{a_n b_{-n}}{n^s}.$$

More recently, Deitmar and Diamantis [7], Diamantis, Knopp, Mason, and O'Sullivan [8] have defined L-functions for the product of the antiderivative $F(z)$ and $g(z)$, which has the Fourier expansion

$$F(z) \cdot g(z) = \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \frac{a_j c_{n-j}}{j} e^{2\pi i n z}.$$

The new L-function is

$$L(F \cdot g, s) = \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \frac{a_j c_{n-j}}{j} \frac{1}{n^s}.$$

In the Phillips-Sarnak framework two kinds of perturbation are considered:

(1) Character varieties generated by $\alpha \in H^1(M, \mathbb{R})$. Let $\mathfrak{F}$ be a fundamental domain of Γ. Instead of $L^2(\Gamma\backslash\mathbb{H})$, we consider the following space

$$L^2(\Gamma\backslash\mathbb{H}, \chi(\epsilon)) = \{h \in L^2(\mathfrak{F}), \quad h(\gamma z) = \chi(\epsilon, \gamma) h(z),\ \gamma \in \Gamma\},$$

where

$$\chi(\epsilon, \gamma) = \exp\left(2\pi i \epsilon \int_{z_0}^{\gamma z_0} \alpha\right).$$

For each $\chi(\epsilon)$ we consider the associated Laplace operator $\Delta(\epsilon)$, acting on $L^2(\Gamma\backslash\mathbb{H}, \chi(\epsilon))$. This corresponds to the Laplacian acting on sections of a flat line bundle over $\Gamma\backslash\mathbb{H}$. For $\epsilon = 0$

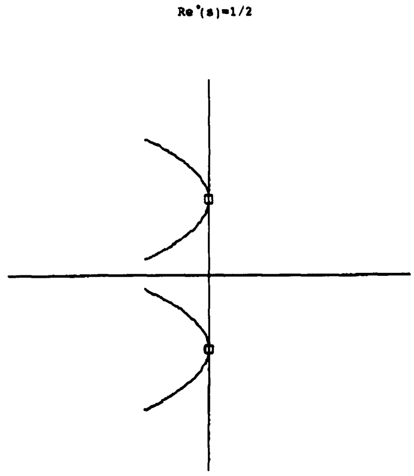

FIGURE 1. Fermi's Golden Rule

we get the standard Laplacian. The connection to automorphic forms lies on the fact that α can be take to be $\Re(f(z)dz)$, where $f(z)$ is a cusp form of weight 2.

(2) Teichmüller space of Γ, generated by $f \in S_4(\Gamma)$. Here we are working with the moduli of Riemann surfaces of the same signature as $\Gamma\backslash\mathbb{H}$. Roughly speaking we perturb the fundamental domain in such a way that it remains fundamental domain of a Fuchsian group. Alternatively we are considering a family of inequivalent embeddings ϕ_ϵ of an abstract group Γ into $\mathrm{SL}_2(\mathbb{R})$. Let $\Gamma_\epsilon = \phi_\epsilon(\Gamma)$. We consider $\Delta(\epsilon)$ on $L^2(\Gamma_\epsilon \setminus \mathbb{H})$. The global theory of Teichmüller space is not important here. It is the local theory that we need.

There are formulas for the lower perturbation terms of the perturbation series

$$\Delta(\epsilon) = \Delta + \epsilon\Delta^{(1)} + \epsilon^2\Delta^{(2)}/2 + \cdots$$

For case (1) $\Delta(\epsilon)h = \Delta h + 4\pi i\epsilon\langle dh, \alpha\rangle - 4\pi^2\epsilon^2\langle\alpha,\alpha\rangle h$, where $\langle\cdot,\cdot\rangle$ is the inner product of 1-forms. This is not completely accurate, as perturbation theory requires that we work on a fixed Hilbert space, like $L^2(\Gamma\backslash\mathbb{H})$, and then the series above correspond to the conjugated Laplacian $U(\epsilon)^{-1}\Delta U(\epsilon)$ by appropriate unitary operators $U(\epsilon)$. The exact formulas can be found in [26, 28, 29].

We now describe the result in [26, 29] Let $\lambda_j = 1/4 + r_j^2$ be an embedded eigenvalue with $\lambda_j > 1/4$, so that $E(z, 1/2+ir_j)$ corresponds in the same eigenvalue. In [26] Phillips and Sarnak identified a condition that turns λ_j into a resonance in Teichmüller space, i.e. dissolving λ_j

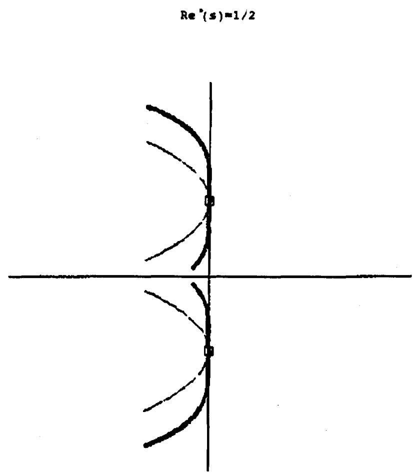

FIGURE 2. Second order contact (thin line) vs. fourth order contact (thick line)

into a resonance. In [29] Sarnak identified a similar condition for character varieties. Let $\Delta^{(1)}$ denote the infinitesimal variation of the family of Laplacians in either perturbation. Then the dissolving condition – usually called the Phillips–Sarnak condition – is

$$\langle \Delta^{(1)} u_j, E(z, 1/2 + ir_j)\rangle \neq 0. \tag{2.2}$$

In [27] Phillips and Sarnak identified the dissolving condition in terms of the speed that the cuspidal eigenvalue leaves the line $\Re(s) = 1/2$ to become a resonance to the left half-plane. This is Fermi's Golden Rule. If $s_j(\epsilon)$ denotes the position of the resonance or embedded cusp form, with perturbation series

$$s_j(\epsilon) = s_j + s_j^{(1)}(0)\epsilon + \frac{s_j^{(2)}(0)}{2!}\epsilon^2 + \cdots , \tag{2.3}$$

then

$$\Re s_j^{(2)}(0) = -\frac{1}{4r_j^2}\left|\langle \Delta^{(1)} u_j, E(z, 1/2 + ir_j)\rangle\right|^2 . \tag{2.4}$$

In this paper we annouce the results of our investigation of the following issue: what happens when the expression (2.4) vanishes, or equivalently: what happens if the Phillips-Sarnak condition is *not* satisfied.

The proof of (2.4) in [27] uses the Lax-Phillips scattering theory as developed for automorphic functions, see [18]. The crucial ingredient is provided by the cut-off wave operator B. Its spectrum (on appropriate spaces) coincides with the singular set (counting multiplicities). It includes the embedded eigenvalues and the resonances. The motion of an embedded eigenvalue depending on the perturbation parameter ϵ on the complex place $\mathbb{C}$ can be identified as the motion of an eigenvalue of B. Given that Phillips and Sarnak proved that regular perturbation theory applies to this setting, it follows that an embedded eigenvalue moves (with at most algebraic singularities) as function of ϵ, see Theorem 4.1 below, either remaining a cuspidal eigenvalue or becoming a resonance. Eq. (2.4) follows using standard perturbation theory techniques. Balslev provided a different proof of Eq. (2.4) in [2] by introducing the technique of analytic dilations and imitating the setting of Fermi's Golden Rule for the helium atom, see [30]. A slightly modified version of the application of perturbation theory is provided in [22], using the formulas in [17, p. 79].

Once the dissolving condition had been identified, Phillips and Sarnak [26] showed that it is proportional to the special value of a Rankin–Selberg convolution of u_j with the holomorphic cusp form f generating the deformation $L(u_j \otimes f, s_j + 1/2)$. (The normaization of the Rankin-Selberg convolution is such that $\Re(s) = 1$ is the critical line.) These special values have been subsequently studied [9, 10, 19] with the aim of showing that a generic surface with cusps has 'few' embedded eigenvalues in the sense of Weyl's law. The best result here is due to Luo [19], who proved that a positive proportion of the Rankin-Selberg values is nonzero. This implies that, under the hypothesis that the multiplicities of the eigenvalues of $\Gamma_0(p)$ are bounded, the generic Γ_ϵ in the Teichmüller space of $\Gamma_0(p)$ fails Weyl's law in the form (2.1).

A different line of approach has been to develop alternate perturbation settings, where the condition to check is easier to understand. Wolpert, Phillips and Sarnak, and Balslev and Venkov succeded in investigating Weyl's law this way. [31, 28, 3, 4].

3. Recent developments and results

A more recent development came through the numerical investigation of the poles of Eisenstein series by Avelin [1]. Working with the Teichmüller space of $\Gamma_0(5)$, she found a fourth order contact of $s_j(\epsilon)$ with the unitary axis $\Re(s) = 1/2$. This is illustrated in Fig. 2.3, taken from Avelin's thesis. It is easy to explain why certain directions in the moduli space will not satisfy the Phillips-Sarnak condition (2.2): If the dimension of the moduli space is at least 2, then the map $f \to \langle \Delta^{(1)} u_j, E(z, 1/2 + ir_j) \rangle$ is linear, therefore, is has nontrivial kernel. Avelin also identified numerically the most suitable curve that the singular point follows in the left half-plane. Notice that the comparison of the two graphs in Fig. 2.3 shows that the fourth order contact is

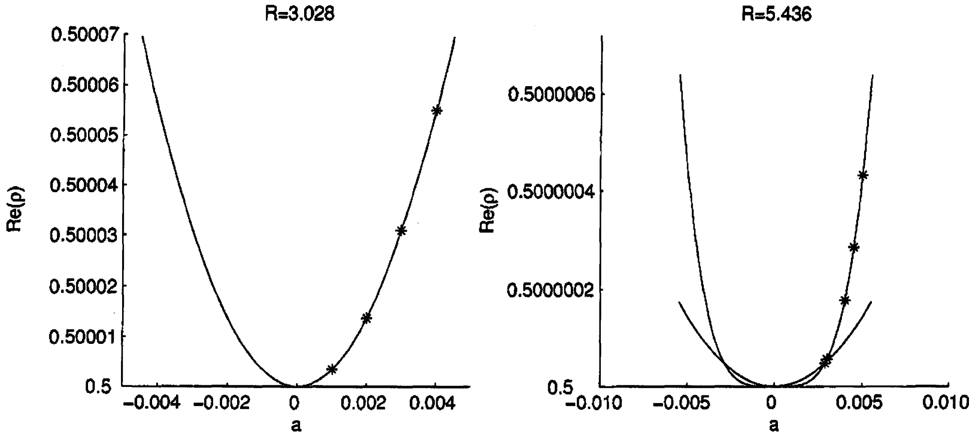

Figure 2.3: The stars correspond to poles of $\varphi(s)$ found near the cusp forms having $R_0 = 3.028$ and $R_0 = 5.436$. $\mathrm{Re}(\rho) = 1/2 + \eta$ is plotted as a function of the parameter a. The curve in the first plot is $1/2 + 3.4389a^2$. The curves in the second plot are $1/2 + 0.0144a^2$ and $1/2 + 1744.6a^4$. We see that the stars appear to lie on the fourth order curve.

not due to parametrization: one can substitute ϵ^2 for ϵ. The graph on the right shows that the behaviour is consistent with the parameter being the square of the graph on the left. Farmer and Lemurell [12] investigated a related issue: the investigated whether a given cuspidal eigenvalue for a congruence groups survives in some subvariety of Teichmüller space. These developments motivated us to investigate whether one can identify higher order Fermi-type conditions that will explain what happens when the Phillips-Sarnak condition fails. We find conditions that guarantee that an embedded eigenvalue (cuspidal eigenvalue) becomes a resonance (scattering pole).

For this purpose we introduce the perturbation series of the generalized eigenfunctions $D(z,s,\epsilon)$, with $D(z,s,0) = E(z,s)$:

$$D(z,s,\epsilon) = D(z,s,0) + D^{(1)}(z,s)\epsilon + \frac{D^{(2)}(z,s)}{2!}\epsilon^2 + \cdots . \tag{3.1}$$

Theorem 3.1. *Assume that for $k = 0,1,\dots,n-1$ the functions $D^{(k)}(z,s)$ are regular at a simple cuspidal eigenvalue $s_j = 1/2 + ir_j$. Then $D^{(n)}(z,s)$ has at most a first order pole at s_j.*

1. *If $D^{(n)}(z,s)$ has a pole at s_j, then the embedded eigenvalue becomes a resonance.*
2. *Moreover, with $\|\cdot\|$ the standard L^2-norm,*

$$\Re s_j^{(2n)}(0) = -\frac{1}{2}\binom{2n}{n}\left\|\operatorname*{res}_{s=s_j} D^{(n)}(z,s)\right\|^2,$$

and this is the leading term in the expansion of $\Re s_j(\epsilon)$, *i.e.* $\Re s_j^{(j)}(0) = 0$ *for* $j < 2n$.

Corollary 3.2. *An embedded simple eigenvalue* s_j *becomes a resonance if and only if for some* $m \in \mathbb{N}$ *the function* $D^{(m)}(z,s)$ *has a pole at* s_j.

Remark 3.3. For $n = 1$ the condition in the theorem is the classical Fermi's Golden Rule. Our method provides a new proof of this well-known result without using energy inner products, see [27] but assuming Theorem 4.1.

Remark 3.4. The assumptions of the theorem may equivalently be stated as $\Re(s^{(j)}(0)) = 0$ for $j = 1, \dots, 2n-1$. So in the theorem we are really assuming that the embedded eigenvalue does not become a resonance to order *less* than $2n$.

Remark 3.5. At first glance it may seem that the condition identifies one perturbation object with another, equally unknown. However, the condition can surprisingly also be expressed as the nonvanishing at a special point of a Dirichlet series. The relevant series is more complicated than the standard Rankin–Selberg convolution. In the case of character varieties and $n = 2$ this Dirichlet series is

$$L(u_j \otimes F^2, s) = \sum_{n=1}^{\infty} \left(\sum_{k_1 + k_2 = n} \frac{a_{k_1}}{k_1} \frac{a_{k_2}}{k_2} b_{-n} \right) \frac{1}{n^{s-1/2}}, \tag{3.2}$$

where a_n are the Fourier coefficients of f, and b_n are the coefficients of u_j.

Even more, $D^{(n)}(z,s)$ has been the object of intense investigation by Goldfeld, O'Sullivan, Chinta, Diamantis, the authors, Jorgenson et. al. [13, 14, 6, 11, 20, 23, 24, 16]. It can be defined for $\Re(s) > 1$ as

$$D^{(n)}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(2\pi i \int_{i\infty}^{\gamma z} \Re f(w)\, dw \right)^n \Im(\gamma z)^s. \tag{3.3}$$

In fact, in [21, 23] it was proved that

$$\mathrm{Res}_{s=s_j} D^{(1)}(z,s) = \langle \Delta^{(1)} u_j, E(z, 1/2 + ir_j) \rangle u_j(z),$$

which gives the Phillips–Sarnak condition when one takes the L^2-norm. This motivated us to investigate the residues of $D^{(n)}(z,s)$ and derive Theorem 3.1. The functions $D^{(n)}(z,s)$ have been used to prove the Gaussian distribution of periods of weight 2-cusp forms [24].

Remark 3.6. Formulas for higher order approximation in perturbation theory can be found in [17]. Fermi's Golden Rule is tied to the formula

$$\lambda^{(2)} = \mathrm{tr}(T^{(2)}P + T^{(1)} R_0 T^{(1)} P),$$

where $T^{(1)}, T^{(2)}$ is the first and second variation of the family of operators $T(\epsilon)$, P is the spectral projection at the given eigenvalue λ_j and R_0 is the reduced resolvent at λ_j. Implicitly all proofs of (2.4) use a variation of this formula. Formulas for higher approximation are much more complicated, here is the fourth order one:

$$\begin{aligned}
\lambda^{(4)} &= \operatorname{tr}[T^{(4)}P - T^{(1)}ST^{(3)}P - T^{(2)}ST^{(2)}P - T^{(3)}ST^{(1)}P + \\
&+ T^{(1)}ST^{(1)}ST^{(2)}P + T^{(1)}ST^{(2)}ST^{(1)}P + T^{(2)}ST^{(1)}ST^{(1)}P - \\
&- T^{(1)}S^2T^{(1)}PT^{(2)}P - T^{(1)}S^2T^{(2)}PT^{(1)}P - T^{(2)}S^2T^{(1)}PT^{(1)}P \\
&- T^{(1)}ST^{(1)}ST^{(1)}ST^{(1)}P + T^{(1)}S^2T^{(1)}ST^{(1)}PT^{(1)}P + \\
&+ T^{(1)}ST^{(1)}S^2T^{(1)}PT^{(1)}P + T^{(1)}S^2T^{(1)}PT^{(1)}ST^{(1)}P - \\
&- T^{(1)}S^3T^{(1)}PT^{(1)}PT^{(1)}P].
\end{aligned}$$

This kind of formula is rather unmanageable. In fact, Simon [30] remarks that: One would like to say that the higher order terms in our series agree with the 'usual' terms in the physicists time-dependent perturbation series. There is unfortunately great confusion in the physics literature concerning the 'correct' higher order terms.

Remark 3.7. This theorem gives an algorithmic method of checking whether in a particular direction of moduli space an embedded eigenvalue becomes a resonance. If $D^{(1)}(z, s)$ is regular at s_j, which is equivalent to the vanishing of the Phillips–Sarnak condition, then the embedded eigenvalue stays an eigenvalue to second order and we need to check the higher order condition $D^{(2)}(z, s)$. If this is regular one looks at the next term in the perturbation series of $D(z, s, \epsilon)$ etc.

Remark 3.8. The simplicity of s_j is not important. One can deal with higher order multiplicities.

3.1. **Generic dissolving.** Recall

$$L(u_j \otimes F^2, s) = \sum_{n=1}^{\infty}\left(\sum_{k_1+k_2=n}\frac{a_{k_1}}{k_1}\frac{a_{k_2}}{k_2}b_{-n}\right)\frac{1}{n^{s-1/2}}.$$

The following theorem gives the analytic properties of this Dirichlet series.

Theorem 3.9. *$L(u_j \otimes F^2, s)$ admits meromorphic continuation with possible poles at s_k, s_k cuspidal eigenvalue or pole of $\phi(s)$. The poles are of order ≤ 1. The completed L-function*

$$\Lambda(u_j \otimes F^2, s) = \frac{\Gamma(s+s_j-1)\Gamma(s-s_j)}{(4\pi)^s\Gamma(s)}L(u_j \otimes F^2, s)$$

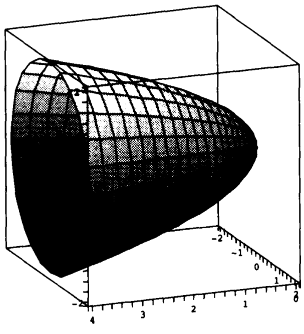

FIGURE 3. Cuspidal eigenvalue dissolves into a resonance in a two dimentional variety.

satisfies

$$\Lambda(u_j \otimes F^2, s) = \phi(s)\Lambda(u_j \otimes F^2, 1-s).$$

Phillips and Sarnak [26] proved that, if $f \in S_4(\Gamma)$ generate a one (complex) dim subspace $V \subset T(\Gamma)$ in Teichmüller space, then a sufficient condition that u_j is dissolved in *almost* all directions in V is that

$$L(u_j \otimes f, s_j + 1/2) \neq 0.$$

We can be more precise, at least in the character variety case.

Theorem 3.10. *Let $f \in S_2(\Gamma)$. Assume in both directions $\omega_1 = \Re(f(z)\,dz)$ and $\omega_2 = \Im(f(z)\,dz)$ the Phillips-Sarnak condition fails. Then*

(a) $L(u_j \otimes F^2, s)$ is holomorphic at s_j.

(b) If $L(u_j \otimes F^2, s_j) \neq 0$, then for all directions ω in the (real) span of ω_1, ω_2 with at most two exceptions we have

$$\Re s_j^{(4)}(0, \omega) \neq 0,$$

i.e. the cusp form dissolved in the direction ω.

Remark 3.11. There are appropriate generalizations to all orders.

3.2. **Are Maaß Cusp Forms Isolated?** Clearly $\Re s(\epsilon)$ is real analytic with maximum at $\epsilon = 0$. Its Hessian can be computed. Because of symmetries it is often degenerate. For example for $\Gamma_0^*(37)$, an extension of $\Gamma_0(37)$ with genus $g = 1$, and u_j even, we can even take $f(z)$ to be a theta series. Let $\alpha_1 = \Re(f(z)dz)$, $\alpha_2 = \Im(f(z)dz)$. Then the Hessian of $\Re s(\epsilon)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & -\left\|\mathrm{Res}_{s=s_j} D^{(1)}(z, s, \alpha_2)\right\|^2 \end{pmatrix}$$

We expect the residue to be nonzero for many s_j's, according to [19]. We get the following theorem.

Theorem 3.12. *Assume that*

$$\frac{d^2}{d\epsilon_2^2}\Re(s_j(0,0)) \neq 0$$

and that

$$\exists m, \quad \frac{d^m}{d\epsilon_1^m}\Re(s_j(0,0)) \neq 0.$$

Then

$$\Re(s_j(\epsilon_1, \epsilon_2)) < 1/2$$

in a punctured neighborhood of $(0,0)$, *i.e. the cuspidal eigenvalue becomes a resonance in this punctured neighborhood.*

Remark 3.13. If a cusp form remains on a real analytic subvariety of the deformation space, then this subvariety is given by $\epsilon_1 = 0$ for all even cusp forms!

4. Methods of proof

The important theorem about the singular set needed is the following:

Theorem 4.1. [27, Corollary 5.2] *If* $s_j(0)$ *is in the singular set* $\sigma(0)$ *for* $\epsilon = 0$ *and has multiplicity* 1, *then it moves real analytically in* ϵ *for* $|\epsilon|$ *sufficiently small. If the multiplicity is greater than one, then the singular points decompose into a finite system of real analytic functions having at most algebraic singularities.*

Contour integration gives

$$s(\epsilon) - s_j = -\frac{1}{2\pi i}\int_\gamma (s - s_j)\frac{\frac{d}{ds}\phi(s,\epsilon)}{\phi(s,\epsilon)}ds + \sum_{s_j(\epsilon)\,\text{cuspidal}} (s_j(\epsilon) - s_j),$$

where $\phi(s,\epsilon)$ is the scattering determinant and γ is the boundary of the left-half disc centered at $s_j(0)$. This implies that

$$\Re(s(\epsilon) - s_j) = -\frac{1}{2\pi i}\int_O (s - s_j)\frac{\frac{d}{ds}\phi(s,\epsilon)}{\phi(s,\epsilon)}ds,$$

where O is a circle enclosing only $s_j(0)$ from the singular set. By differentiation we get

$$\frac{d^n}{d\epsilon^n}\Re(s(\epsilon) - s_j) = -\frac{1}{2\pi i}\int_O (s - s_j)\frac{d^n}{d\epsilon^n}\frac{\frac{d}{ds}\phi(s,\epsilon)}{\phi(s,\epsilon)}ds. \tag{4.1}$$

Therefore, we need to understand

$$\frac{d^k}{d\epsilon^k}\phi(s,\epsilon)\bigg|_{\epsilon=0}$$

for $k = 1, \ldots, 2n$.

A major player in the investigation is the resolvent operator and its analytic continuation $R(s) = (\Delta + s(1-s))^{-1}$. Use perturbation theory to get

$$D^{(1)}(z,s) = -R(s)\Delta^{(1)}E(z,s)$$

and more generally

$$D^{(n)}(z,s) = -R(s)\sum_{i=1}^{n}\binom{n}{i}\Delta^{(i)}D^{(n-i)}(z,s). \tag{4.2}$$

The functional equation $E(z,s,\epsilon) = \phi(s,\epsilon)E(z,1-s,\epsilon)$ gives

$$D^{(n)}(z,s) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{d\epsilon^k}\phi(s,\epsilon)\bigg|_{\epsilon=0} D^{(n-k)}(z,1-s).$$

Theorem 4.2. *Assume that for all $k = 1, \ldots, n-1$, $D^{(k)}(z,s)$ is has a removable singularity at s_j. Then*

- $\frac{d^j}{d\epsilon^k}\phi(s,\epsilon)\Big|_{\epsilon=0}$ *has a removable singularity at s_j for $k = 1, \ldots, 2n-1$.*
- $\frac{d^{2n}}{d\epsilon^{2n}}\phi(s,\epsilon)\Big|_{\epsilon=0}$ *has at most a simple pole at s_j with residue*

$$\operatorname{res}_{s=s_j}\frac{d^{2n}}{d\epsilon^{2n}}\phi(s,\epsilon)\bigg|_{\epsilon=0} = -\phi(s_j)\binom{2n}{n}\left\|\operatorname{res}_{s=s_j}D^{(n)}(z,s)\right\|^2.$$

Using this theorem, one can study the singularities and residue of the integrand in (4.1).

Here is an argument that explains why a singularity of $D^{(n)}(z,s)$ at s_j is connected to dissolving cusp forms. The function $D(z,s,\epsilon)$ is an Eisenstein series. Let us assume that $D^{(k)}(z,s)$ is regular at s_j for $k = 1, \ldots, n-1$. Assume u_j is a simple cusp form and that $u_j(\epsilon)$ remains a cusp form with $u_j(0) = u_j$. It is known that cusp forms are perpendicular to the Eisenstein series $D(z,s,\epsilon)$ for all s. This gives

$$\langle u_j(\epsilon), D(z,s,\epsilon)\rangle = 0. \tag{4.3}$$

Phillips and Sarnak [27] proved the real analyticity of $u_j(\epsilon)$. We differentiate (4.3) to get

$$\sum_{k=0}^{n}\binom{n}{k}\left\langle u_j^{(n-k)}, D^{(k)}(z,s)\right\rangle = 0$$

for s close to s_j. By the assumptions the term with $k = n$ should be a regular function at s_j. Under the same assumptions, using (4.2), we see that $D^{(n)}(z,s)$ has at most a first order pole at s_j with residue a multiple of $u_j(0)$. By regularity of $\langle u_j, D^{(n)}(z,s)\rangle$ this residue has to vanish. This approach does not prove Corollary 3.2 but shows the sufficiency of the condition that some $D^{(n)}(z,s)$ has a pole to conclude that s_j becomes a resonance. Corollary 3.2 shows that this is also necessary.

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