

# On the Igusa modular form of weight 10

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**Abstract.** This paper is related to the authors' talk at the RIMS conference 2011 on: *Automorphic forms, trace formulas and zeta functions* in Kyoto. The Igusa modular form of weight 10 is the unique Siegel modular form which is a Borcherds and a Saito-Kurokawa lift.

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## 1 Introduction

The Igusa modular form  $\chi_{10}$  appeared first in the famous theorem of Jun-ichi Igusa about the generators of graded algebra of Siegel modular forms of even weight and degree 2 (see [Ig1]). The algebra is equal to

$$(1.1) \quad \mathbb{C}[E_4^2, E_6^2, \chi_{10}, E_{12}^2].$$

We normalized the Siegel type Eisenstein series  $E_k^2$  of weight  $k$  such that the Fourier coefficient related to 0-dim cusp at infinity is one. The Igusa modular form  $\chi_{10}$  is a cusp form of weight 10. Igusa introduced the form in terms of Eisenstein series ([Ig1], page 192).

$$\chi_{10} := -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (E_4^2 E_6^2 - E_{10}^2).$$

It is known that  $\chi_{10}$  is a Saito-Kurokawa lift ([Za]) and a Borcherds lift ([GN1], [GN2]).

The square root of this modular form is related to the denominator formula for a generalized Borcherds-Kac-Moody super algebra (Gritsenko, Nikulin). Moreover it is as a partition function of BPS dyons in the toroidally compactified heterotic string theory. To study a generalized Kac-Moody algebra one has to know the imaginary simple roots and the multiplicities of all positive roots. It is absolutely crucial that the underlying modular form has a degenerate Fourier expansion (Saito-Kurokawa lift) and an infinite product (Borcherds lift). We refer to ([CD], [CV]) for more details. The following theorem states that there are no other Siegel modular forms of degree 2 which are Borcherds and Saito-Kurokawa lifts .

**Theorem** *Let  $F$  be a Siegel modular form of degree 2. If  $F$  is a Borcherds lift and a Saito-Kurokawa lift, then  $F$  is proportional to the Igusa modular form.*

We note that the Borcherds lift is multiplicative and the Saito-Kurokawa lift additive.

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## 2 Siegel modular forms, Witt operator and Taylor expansions

For an introduction to the theory of Siegel modular forms we refer to Klingen's book ([Kl]). Let  $\Gamma_n$  be the Siegel modular group and  $\mathfrak{H}_n$  the upper half space of degree  $n$ :

$$\begin{aligned}\Gamma_n &:= \left\{ \gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t \gamma \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \gamma = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \right\} \\ \mathfrak{H}_n &:= \{ Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^t Z = Z, \mathrm{Im}(Z) > 0 \},\end{aligned}$$

where  $\mathbf{0}_n$  (respectively  $\mathbf{1}_n$ ) is the zero (respectively identity) matrix of degree  $n$ . Then we denote by  $M_k(\Gamma_n)$  the space of Siegel modular forms of weight  $k$  on  $\Gamma_n$  and by  $S_k(\Gamma_n)$  the subspace of cusp forms. In the case  $n = 1$  we usually drop the index and for  $n = 2$  which we are mainly interested in we often write  $(\tau_1, z, \tau_2)$  for a point

$$\begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_2.$$

Next we introduce two useful operators. Let  $F \in M_k(\Gamma_2)$ . Define

$$\begin{aligned}\Phi(F)(\tau) &:= \lim_{y \rightarrow \infty} F(\tau, 0, iy) \quad (\tau \in \mathfrak{H}_1), \\ \mathcal{W}(F)(\tau_1, \tau_2) &:= F(\tau_1, 0, \tau_2) \quad (\tau_1, \tau_2 \in \mathfrak{H}_1).\end{aligned}$$

Then  $\Phi(F) \in M_k(\Gamma)$  and  $\mathcal{W}(F) \in \mathrm{Sym}^2(M_k(\Gamma))$ . The operator  $\Phi$  (respectively  $\mathcal{W}$ ) is called the *Siegel* (respectively *Witt*) operator. Then  $S_k(\Gamma_2) = \{F \in M_k(\Gamma_2) \mid \Phi(F) = 0\}$ .

Let  $f_1, f_2, \dots, f_d$  be a basis of newforms of  $S_k$  and  $f_0 = e_k$ . Here  $e_k$  denotes the elliptic Eisenstein series with constant term  $a(0) = 1$ .

Then we define

$$(2.1) \quad \mathrm{Sym}^2(M_k(\Gamma))^D := \left\{ \sum_{i=0}^d \alpha_i f_i \otimes f_i \mid \alpha_i \in \mathbb{C} \right\}.$$

By  $\mathrm{Sym}^2(S_k(\Gamma))^D$  we denote the cuspidal part.

A Siegel modular form  $F \in M_k(\Gamma_2)$  admits the Fourier expansion

$$F(\tau_1, z, \tau_2) = \sum_{n,r,m \in \mathbb{Z}} A_F(n, r, m) \mathbf{e}(n\tau_1 + rz + m\tau_2),$$

where we put  $\mathbf{e}(z) = \exp(2\pi iz)$  for  $z \in \mathbb{C}$ . Note that  $A_F(n, r, m) = 0$  unless  $n, m, 4nm - r^2 \geq 0$ . We also use the following shortcuts:  $q := \mathbf{e}(\tau)$ ,  $q_1 := \mathbf{e}(\tau_1)$ ,  $\zeta := \mathbf{e}(z)$ ,  $q_2 := \mathbf{e}(\tau_2)$ . It is easy to see that:

$$(2.2) \quad \Phi(F)(\tau) = \sum_{n=0}^{\infty} A_F(n, 0, 0) q^n$$

$$(2.3) \quad \mathcal{W}(F)(\tau_1, \tau_2) = \sum_{n,m=0}^{\infty} \left( \sum_r A_F(n, r, m) \right) q_1^n q_2^m.$$

We define the order of the  $q$ -expansion of a modular form  $F \in M_k(\Gamma_2)$  by

$$\text{ord}(F) := \min \{n \in \mathbb{N}_0 \mid A_F(n, r, m) \neq 0\}.$$

**Remark 2.1.** If  $\text{ord}(F) \geq 2$ , then  $F \notin \text{Sym}^2(M_k(\Gamma))^D$ .

Let  $k$  be even. Then  $F \in M_k(\Gamma)$  has the Taylor expansion

$$(2.4) \quad F(\tau_1, z, \tau_2) = \sum_{l=0}^{\infty} \Psi_{2l}(\tau_1, \tau_2) z^{2l}.$$

It is clear that  $\Psi_0$  is the image of the Witt operator and an element of  $\text{Sym}^2(M_k(\Gamma))$ . Moreover if  $\Psi_0$  is identically zero then  $\psi_2 \in \text{Sym}^2(S_{k+2}(\Gamma))$ .

Finally let  $E_k^n$  denote the Siegel-type Eisenstein series on  $\Gamma_n$ , normalized by  $\Phi^n(E_k^n) = 1$ . Here  $\Phi^n$  denotes the  $n$ -th iteration of the  $\Phi$  operator. Let  $E_k^n(f)$  denote the Klingen Eisenstein series attached to  $f \in S_k(\Gamma), f \neq 0$ . Note that  $\Phi^{n-1}(E_k^n(f)) = f$ . Let further  $M_k^{2,0}$  be the 1-dim space generated by Siegel Eisenstein series of weight  $k$  and degree 2, let  $M_k^{2,1}$  be the space generated by all Klingen type Eisenstein series of weight  $k$  and degree 2 and let  $M_k^{2,2} = S_k(\Gamma_2)$ . Then

$$(2.5) \quad M_k(\Gamma_2) = M_k^{2,0} \oplus M_k^{2,1} \oplus M_k^{2,2}.$$

The direct sum is related to the Petersson scalar product. Moreover this decomposition is respected by the Siegel  $\Phi$  operator. Let  $F \in M_k(\Gamma_2)$  with decomposition  $F_0 + F_1 + F_2$ . Then

$$(2.6) \quad \Phi(F) = \Phi(F_0) + \Phi(F_1) + \Phi(F_2)$$

$$(2.7) \quad = c_1 E_k + c_2 f \quad (c_1, c_2 \in \mathbb{C}, f \in S_k(\Gamma)).$$

### 3 Saito-Kurokawa lifts

One can find an overview in Zagier's Bourbaki article [Za]. Let  $k$  be an even integer. Then there exists an injective linear map

$$(3.1) \quad SKL : M_{2k-2}(\Gamma) \longrightarrow M_k(\Gamma_2),$$

where Hecke eigenforms  $f$  map to Hecke eigenforms  $F = SKL(f)$ . For a Hecke eigenform  $f$ , the spinor L-function  $Z(SKL(f), s)$  is given by

$$Z(SKL(f), s) = \zeta(s - k + 1) \zeta(s - k + 2) L(f, s),$$

where  $L(f, s)$  is the Hecke L-function of  $f$  and  $\zeta(s)$  denotes the Riemann zeta function. We are interested in the image of the lifting, which is given by the so-called Maass Spezialschar:

$$(3.2) \quad M_k^{Spez} := \left\{ F \in M_k(\Gamma_2) \mid A_F(n, r, m) = \sum_{d \in \mathbb{N}, d \mid (n, r, m)} d^{k-1} A_F\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right) \right\}.$$

Here  $(n, r, m)$  denotes the greatest common divisor of  $n, r, m$  (We put  $1 := (0, 0, 0)$ ). To prove our main result we use the following properties of the Maass Spezialschar. If  $F \in M_k^{\text{Spez}}$ , then  $F$  is non-trivial iff  $\Psi_0^F$  or  $\Psi_2^F$  is not identically zero. Moreover

$$(3.3) \quad \Psi_0^F \in \text{Sym}^2(M_k(\Gamma))^D.$$

If  $\Psi_0^F$  is identically zero then

$$(3.4) \quad \Psi_2^F \in \text{Sym}^2(S_{k+2}(\Gamma))^D.$$

**Remark 3.1.** Let  $F \in M_k(\Gamma_2)$  has the decomposition  $F_0 + F_1 + F_2$  as described before. If  $F_1$  is non-trivial, then  $F$  is not in the Spezialschar.

## 4 Borchers lifts

Roughly speaking a Borchers lift BL is a correspondence between modular forms of weight  $1 - \frac{m}{2}$  on  $\mathfrak{H}$  with possible singularities at the cusps and certain meromorphic automorphic forms with possible character on symmetric domains of type IV related to orthogonal groups  $O(2, m)$  ( $m \in \mathbb{N}$ ) ([Bo1],[Bo2], [Bo3]). We note that

$$BL(f + g) = BL(f) \cdot BL(g).$$

Lifts to Siegel modular forms of degree 2 are related to the case  $m = 3$ , where the image is uniquely (up to a scalar) determined by the divisor

$$(4.1) \quad \text{div}(BL(f)) = \sum_{d \in \mathcal{D}} n_d H_d.$$

Here  $\mathcal{D}$  is the set of all positive integers congruent to 0 or 1. The sum is finite and  $n_d \in \mathbb{Z}$ . The  $H_d$  are the Humbert surfaces (see also the following subsection), for general  $m$  they are called Heegner divisor. The image could be an element of  $M_k(\Gamma_2, \nu)$ , a Siegel modular form with the unique non-trivial character  $\nu$  on  $\Gamma_2$ .

**Remark 4.1.** The coefficients of the principal part of the input function are related to the  $n_d$ . A priori it is not clear when the nontrivial character in the image occurs. Moreover even when not all coefficients in the principal part are non-negative, the image could be holomorphic.

### 4.1 Humbert surfaces

Let

$$Q := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -2 & & \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

Put  $Q(X, Y) := {}^t X Q Y$  and  $Q[X] := Q(X, X)$  for  $X, Y \in \mathbb{C}^5$ . For  $Z = (\tau_1, z, \tau_2) \in \mathfrak{H}_2$  put  $\tilde{Z} := {}^t(-\tau_1 \tau_2 + z^2, \tau_1, z, \tau_2, 1) \in \mathbb{C}^5$ . Note that  $Q[\tilde{Z}] = 0$  and  $Q(\tilde{Z}, \tilde{Z}) = 4 \det(\text{Im}(Z)) > 0$ . There exists a homomorphism  $\iota: \text{Sp}_2(\mathbb{R}) \rightarrow O(Q)_{\mathbb{R}}$  such that  $\widetilde{g\langle Z \rangle} = j(g, Z)^{-1} \iota(g) \tilde{Z}$  for  $g \in \text{Sp}_2(\mathbb{R})$  and  $Z \in \mathfrak{H}_2$ .

Let  $L := \mathbb{Z}^5, L^* := Q^{-1}L$  and  $L^*_{\text{prim}} := \{\lambda \in L^* \mid n^{-1}\lambda \notin L^* \text{ for any integer } n > 1\}$ . For an integer  $d \in \mathbb{Z}$ , let

$$\mathcal{H}_d := \sum_{X \in \mathcal{L}_d} \left\{ Z \in \mathfrak{H}_2 \mid Q(X, \tilde{Z}) = 0 \right\},$$

where  $\mathcal{L}_d := \{X \in L^*_{\text{prim}} \mid Q[X] = -d/2\}$ . Note that  $\mathcal{H}_d = \emptyset$  unless  $d > 0$  and  $d \equiv 0$  or  $1 \pmod{4}$ . Since  $L^*_d$  is  $\iota(\Gamma_2)$ -invariant,  $\mathcal{H}_d$  is  $\Gamma_2$ -invariant. Denote by  $H_d$  the image of  $\mathcal{H}_d$  in  $\Gamma_2 \backslash \mathfrak{H}_2$  by the natural projection  $\mathfrak{H}_2 \rightarrow \Gamma_2 \backslash \mathfrak{H}_2$ . The divisor  $H_d$  of  $\Gamma_2 \backslash \mathfrak{H}_2$  is called the *Humbert surface* of discriminant  $d$ . It is known that  $H_d$  is nonzero and irreducible if  $d \equiv 0$  or  $1 \pmod{4}$  (see [Ge2], page 212, Theorem 2.4; see also [GH], Section 3). Note that

$$\mathcal{H}_1 = \bigcup_{\gamma \in \Gamma_2} \gamma \{(\tau_1, 0, \tau_2) \mid \tau_1, \tau_2 \in \mathfrak{H}\}$$

$$\mathcal{H}_4 = \bigcup_{\gamma \in \Gamma_2} \gamma \{(\tau, z, \tau) \mid \tau \in \mathfrak{H}, z \in \mathbb{C}\}.$$

### 4.2 Properties of Borcherds lifts and examples

Recently [HM] we found an explicit description of the Borcherds lifts related to single Heegner divisors. As a by-product one can see that the character is only related to the divisors  $H_1$  and  $H_4$ .

**Theorem 4.2.**

- (i) For each positive integer  $d$  with  $d \equiv 0$  or  $1 \pmod{4}$ , there exists an  $F_d \in M_{k_d}(\Gamma_2, \nu^{\alpha_d})$  with  $\alpha_d \in \{0, 1\}$  satisfying  $\text{div}(F_d) = H_d$ .
- (ii) We have  $F_1 \in S_5(\Gamma_2, \nu), F_4 \in S_{30}(\Gamma_2, \nu)$  and  $F_d \in M_{k_d}(\Gamma_2)$  if  $d > 4$ .
- (iii) A Borcherds lift  $F \in M_k(\Gamma_2, \nu^\alpha)$  ( $\alpha \in \{0, 1\}$ ) is a constant multiple of  $\prod_d F_d^{A(d)}$ , where  $d$  runs over the positive integers with  $d \equiv 0$  or  $1 \pmod{4}$ , and  $A(d)$  is a nonnegative integer ( $A(d) = 0$  except for a finite number of  $d$ ) satisfying  $A(1) + A(4) \equiv \alpha \pmod{2}$ .

Here  $S_k(\Gamma_2, \nu)$  denotes the cuspidal subspace of  $M_k(\Gamma_2, \nu)$

It is well-known that  $\dim S_{10}(\Gamma_2) = 1$  (see also [Kl]). Hence  $\chi_{10}$  is proportional to  $F_1^2$ .

**Remark 4.3.** The Borcherds lifts in  $M_k(\Gamma_2)$  with  $k \leq 60$  are listed as follows:

Borcherds lift	weight	divisor
$F_1^{2a}$ ( $1 \leq a \leq 6$ )	$10a$	$2aH_1$
$F_1^{2a+1}F_4$ ( $1 \leq a \leq 2$ )	$10a + 35$	$(2a + 1)H_1 + H_4$
$F_1^{2a}F_5$ ( $1 \leq a \leq 3$ )	$10a + 24$	$2aH_1 + H_5$
$F_4^2$	$60$	$2H_4$
$F_5^2$	$48$	$2H_5$
$F_8$	$60$	$H_8$

The table shows that every Borcherds lift of weight less than or equal to 60 is a monomial of  $F_1, F_4, F_5$  and  $F_8$ . We also see that there is no holomorphic Borcherds lift of weight 12.

Assume that  $F \in M_k(\Gamma_2)$  is a Borcherds lift. Then  $\Phi(F)$  is proportional to a power  $\Delta^r$  of the modular discriminant  $\Delta$  with  $r \geq 0$ .

## 5 Proof of the Theorem

In the following we give a sketch of the proof of the main theorem. The complete proof will appear elsewhere. Let  $F \in M_k(\Gamma_2)$ ,  $F \neq 0$ . Let  $F$  be a Borcherds lift (BL) and Saito-Kurokawa lift (SKL). First of all we can assume that the weight is even (SKL). This implies that  $k \geq 4$ . The structure theorem (BL) leads to

$$(5.1) \quad F \sim \prod_{d \in \mathcal{D}} F_d^{n_d}.$$

The product is finite,  $n_1 + n_4 \equiv 0 \pmod{2}$  and  $n_d \in \mathbb{N}_0$ . The symbol  $\sim$  indicates that two functions are equal up to a non-zero constant.

**Remark 5.1.** A refined analysis of the modular forms  $F_d$  shows that

$$\text{ord}(F_1) = \frac{1}{2}, \text{ord}(F_4) = \frac{3}{2}.$$

If  $d \geq 5$  then  $\text{ord}(F_d) \geq 2$  iff  $d$  is a square and  $\text{ord}(F_d) = 0$  otherwise.

Since  $F$  is also a SKL we have  $\text{ord}(F) \leq 1$ . This leads to

$$(5.2) \quad F \sim F_1^\alpha \cdot \prod_{d \geq 5, d \text{ not a square}} F_d^{n_d} \quad (\alpha = 0, 2).$$

Put  $G := F/F_1^\alpha$ . Since  $G$  is a BL and not a cusp form we have [HM]

$$\Phi(G) \sim \Delta^r \quad \left(r = \frac{k - 5\alpha}{12} \in \mathbb{N}\right).$$

Then it is easy to see that

$$\mathcal{W}(G) \sim \Delta^r \otimes E_k + E_k \otimes \Delta^r + \text{cuspidal}.$$

This shows that, if  $\alpha = 0$ , then  $G = F$  is not a SKL, a contradiction. Thus we have  $\alpha = 2$ . Finally the case  $\alpha = 2$  remains. We show that  $r \geq 1$  is not possible (then the theorem is proven).

Let in the following  $F \sim F_1^2 \dot{G}$ , with  $\Phi(G) = \Delta^r$  ( $r \geq 1$ ). Then  $\Psi_0^F$  is identically 0. Since  $F$  is a SKL and not identically zero, we can assume that  $\Psi_2^F \neq 0$  and that

$$(5.3) \quad \Psi_2^F \in \text{Sym}^2(S_{k+2}(\Gamma))^D.$$

Since the second Taylor coefficient of  $F_1^2$  is proportional to  $\Delta \otimes \Delta$  we obtain

$$(5.4) \quad \Psi_2^F \sim (\Delta \otimes \Delta) \cdot \mathcal{W}(G).$$

On the other hand  $\mathcal{W}(G)$  can be expressed in terms of the modular function  $j$  and the primitive modular polynomial. This can be directly proven by comparing the weights and the divisors on  $\mathfrak{H} \times \mathfrak{H}$ .

For  $m \in \mathbb{Z}_{>0}$ , let  $\mathcal{M}_m^*$  be the set of primitive matrices in  $M_2(\mathbb{Z})$  of determinant  $m$ . As is well-known, there exists a polynomial  $\Phi_m^*$  in  $\mathbb{Z}[X, Y]$ , called the primitive modular polynomial of degree  $m$ , such that

$$\prod_{M \in \text{SL}_2(\mathbb{Z}) \setminus \mathcal{M}_m^*} (X - j(M\langle\tau\rangle)) = \Phi_m^*(X, j(\tau)).$$

Here  $\tau \mapsto M\langle\tau\rangle$  denotes the action on  $\mathfrak{H}$ . The degree of  $\Phi_m^*(X, Y)$  in  $X$  is larger than  $m$  for  $m > 1$ . Then

$$(5.5) \quad \mathcal{W}(G)(\tau_1, \tau_2) \sim (\Delta^r(\tau_1) \otimes \Delta^r(\tau_2)) \prod_{n>0} \Phi_n^*(j(\tau_1), j(\tau_2))^{a(n)},$$

where  $a(n) \in \mathbb{N}_0$ . Hence we obtain

$$(5.6) \quad \Psi_2^F(\tau_1, \tau_2) \sim (\Delta^{r+1}(\tau_1) \otimes \Delta^{r+1}(\tau_2)) \prod_{n>0} \Phi_n^*(j(\tau_1), j(\tau_2))^{a(n)}.$$

Combining this property with (5.3) leads to a contradiction by employing well-known properties of the modular polynomial, multiplicative properties of the Fourier coefficients of primitive Hecke eigenforms and the explicit Fourier expansion of the  $\Delta$ -function.

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