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AUTOMATICITY AND PRESENTATIONS OF SEMIGROUPS *

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In this article, we make a survey of automaticity and presentations of semigroups.

1. Presentations of semigroups

Definition 1 (1) *Let X be finite alphabets and R a subset of $X^{+}\times X^{+}$. Then R is called a string-rewriting system.*

(2) *For $u, v\in X^{+}$, $(w_1, w_2)\in R$, $uw_1v\Rightarrow_R uw_2v$.*

The congruence μ_R on X^{+} generated by $\Rightarrow_R$ is called the Thue congruence defined by R.

(3) *A semigroup S is (finitely)presented if there exists a (finite) set of X, there exists a surjective homomorphism ϕ of X^{+} to S and there exists a (finie) string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2)\in X^{*}\times X^{+}\mid\phi(w_1)=\phi(w_2)\}$.*

In this case, we say that S has a presentation by X and R denoted by $S=<X:R>$.

Definition 2 *A semigroup S has a presentation with finite [resp. regular, context-free] congruence classes if there exists a finite set X and there exists a surjective homomorphism ϕ of X^{+} to M such that for each word $w\in X^{+}$, $\phi^{-1}(\phi(w))$ is a finite [resp. regular, context-free] language.*

Definition 3 *A semigroup S is called residually finite if for each pair of elements $m, m'\in S$, there exists a conguence on S such that the factor monoid S/μ is finite and $(m, m')\notin\mu$.*

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*This is an absrtact and the paper will appear elsewhere.

Definition 4 *A semigroup S is called residually finite if for each pair of elements $m, m' \in S$, there exists a conguence on S such that the factor monoid S/μ is finite and $(m, m') \notin \mu$.*

Result 1 ([9]). *If a finitely generated semigroup S has a presentation with regular congruence classes, then S is residually finite.*

Definition 5 *Let M be a monoid and m an element of M. The syntactic congruence σ_m on M is defined by $s\sigma_m t$ $(s, t \in M)$ if and only if $\{(x, y) \in M \times M \mid xsy = m\} = \{(x, y) \in M \times M \mid xty = m\}$.*

The factor monoid M/σ_m is called the syntactic monoid of M at m.

Result 2 ([9]) *Let S be a finitely generated semigroup.*

Then S has a presentation with finite congruence classes if and only if the following are satisfied :

(1) *S has no idempotent.*

(2) *For any $s \in S$, S/σ_s is a finite nilpotent semigroup with a zero element 0.*

2. Automatic semigroups

Definition 6 *Let $X(2, \$) = (X \cup \{\$\}) \times (X \cup \{\$\}) - \{(\$, \$)\}$, where \$ is* **padding symbol.**

$\nu : X^* \times X^* \longrightarrow X(2, \$)^*$ $((u, v) \mapsto (u\$^*, v\$^*))$

where $max(|u|, |v|) = max(|u\$^*|, |v\$^*|)$! \$ $\nu(\epsilon, \epsilon) = \epsilon$.

$$e.g.: (abba, bbabab) \mapsto (a, b)(b, b)(b, a)(a, b)(\$, a)(\$, b)$$

Definition 7 *A semigroup S is called automatic if the following conditions hold ;*

(1) *There exists a regular language $L (\subseteq X^*)$ and a surjective map $: L \to S$ $(w \mapsto \overline{w})$. (X, L) is called a rational structure of S.*

(2) *$A_a = \nu(\{(w, w') \in L \times L \mid \overline{wa} = \overline{w'}\})$ is a regular language over $X(2, \$)$ for each $a \in X \cup \{\epsilon\}$.*

In this case, $(L, A_a(a \in X \cup \{\epsilon\}))$ *is called an automatic structure.*

Result 3 ([1]) *An automatic semigroup with a rational structure* (X, L) *has a rational structure* (X, L') *with uniqueness.*

That is, there exists a regular language $L'(\subseteq L \subseteq X^*)$ *and a bijective map* : $L' \to S$ $(w \mapsto \overline{w})$.

Result 4 [1] *The followings hold ;*

(1) *Automatic groups are finitely presented.*

(2) *The semigroup* $< x, y : xy^i x = xyx (i > 2) >$ *is*

automatic but is not finitely presented.

Result 5 ([5]) *Automaticity of monoids is preserved by taking any change of generators.*

However, this is not the case for semigroups.

For $w = a_1 \cdots a_n \in X^+$,

we denote $w(t) = a_1 \cdots a_t$ if $t \leq n$, $\quad w(t) = a_1 \cdots a_n$ if $n < t$.

Definition 8 *A semigroup(group) S with a rational structure* (X, L) *has the fellow traveller property if there exists a constant k such that the Cayley graph* Γ *of S with generators* X *, whenever* $d(w(t), w'(t)) < k$ *for all* $t \geq 1$ *if* $d(w, w') \leq 1$.

(*the distance function* $d(w, w') = min\{|z| \mid z \in X^* \text{ with } \overline{wz} = \overline{w'} \text{ or } \overline{w} = \overline{w'z}\}$)

Result 6 (1) *A group G with a rational structure* (X, L) *is automatic if and if G has the fellow traveller property.* (See [6])

(2) *If A semigroup S with a rational structure* (X, L) *is automatic, then S has the fellow traveller property.* (See [1])

(3) *The semigroup* S^0 *(an non-automatic semigroup S with an adjoined zero* 0*) has the fellow traveller property, but is not automatic.* (See [1])

3. Exsamples of automatic semigroups and non-automatic semigroups

Example 1 *Finite groups, finite semigroups, Hyperbolic groups, finitely generated commutative groups.*

Example 2 *Finitely generated commutative semigroups, hyperbolic semigroups are not always automatic.*

Result 7 ([6]) *For the Baumslag-Solitar group* $BSG(m,n) =< x, y : yx^m = xy^n >$*,*

we have

(1) $BSG(m,n)$ *is not automatic if* $m \neq n$.

(2) $BSG(m,n)$ *is automatic if* $m = n$.

Result 8 ([2]) *For the Baumslag-Solitar semigroup* $BS(m,n) =< x, y : yx^m = xy^n >$*,*

we have

(1) $BS(m,n)$ *is automatic if* $m > n$.

(2) $BS(m,n)$ *is left automatic if* $m > n$.

(3) $BS(m,n)$ *is non-automatic if* $m = n$.

Result 9 ([3]) *The monogenic free semigroup* FA_x *does not have any rational structure with uniqueness.*

4. Automaticity and presentations with context-free congruence classes

Result 10 ([3]) *Let* S *be a finitely generated subsemigroup of vertually free group* G*. Then* S *is a semigroup having a presentation with context-free congruence classes.*

Result 11 ([3]) *Finitely generated subsemigroups of virtually free groups are automatic.*

Result 12 *Bicyclic monoid* $< a, b : ba = \epsilon > = < a, b, e : ba = e; ae = ea = a, be = eb = b >$ *is automatic and has a presetation with context-free congruence classes.*

Result 13 ([1]) *The foundamental four-spiral semigroup* $SP_4 =< a, b, c, d : a^2 = a, b^2 = b, c^2 = c, d^2 = d,$

$ba = a, ab = b, bc = b, cb = c, dc = c, cd = d, da = d >$ *is automatic.*

Moreover, every finitely generated subsemigroups of SP_4 *is automatic.*

5. Problems on commutative automatic semigroups

Result 14 ([7]) *The finitely generated commutative semigroup* $< a, b, x, y : aax = bx, bby = ay,\ ab = ba, ax = xa, ay = ya, bx = xb, by = yb,\ xy = yx >$ *is not automatic.*

Question. *If a finitely generated commutative semigroup* S *has a presentation with finite congruence classes, then is* S *automatic?*

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