数理解析研究所講究録
第 1769 巻 2011 年 101-110

A short history of repetition-free words

YUJI KOBAYASHI

Department of Information Science, Toho University,
Funabashi 274–8510, Japan

1 Introduction

The word "repetition" contains a 2-repetition (square) titi = $(\mathrm{ti})^2$ and a 3/2-repetition epe = $(\mathrm{ep})^{3/2}$. The word "homomorphism" contains a 5/2-repetition omomo = $(\mathrm{om})^{5/2}$, and "peeped" contains a 5/3-repetition peepe = $(\mathrm{pee})^{5/3}$.

It is Thue [43, 44] who first studied systematically repetition-free infinite words, but his pioneering works had been forgotten for a long time. Morse and Hedlund [28] developed the theory of symbolic dynamics without knowing his results. It was 60 years later when Hedland reported Thue's works in [19] (see Berstel [4]). The subject has become popular since the book by Laitare [24] was published. Berstel [5] gave a survey on the subject.

The words Thue constructed found important applications, for example, to the solution of the Burnside problem by Novikov [30], [31] (he used a result by Arson [2] without noticing Thue's works, see Adjan [1]) and the study of semigroup varieties (Burrie & Nelson [9], Sapir [39]).

2 X-free words

Let Σ be an alphabet (a finite set of letters), and let Σ^* be the free monoid generated by Σ. Σ^* is the set of finite words over Σ including the empty word 1. Furthermore, we consider the set Σ^ω of words of ω-words (one-sided infinite words). Set $\Sigma^\# = \Sigma^* \cup \Sigma^\omega$. For $x \in \Sigma^*$, x is a *subword* (or *factor)* if $y \in \Sigma^\#$ if $y = uxv$ ($u \in \Sigma^*, v \in \Sigma^\#$). Here if $u = 1$ (resp. $v = 1$), x is a *prefix* (resp. *suffix*) of y.

Define a distance δ on $\Sigma^\#$ as

$$\delta(x,y) = 2^{-\min\{n \mid a_n \neq b_n\}}.$$

for $x = a_1 \cdots a_n \cdots$ and $y = b_1 \cdots b_n \cdots$. It satisfies

$$\delta(x,y) \leq \max\{\delta(x,z),\ \delta(z,y)\}.$$

Proposition 2.1. *(Σ^ω, δ) and $(\Sigma^\#, \delta)$ are compact totally disconnected metric spaces.*

For $X \subset \Sigma^*$, $x \in \Sigma^\#$ is *X-free* (or *avoids* X), if any subword of x is not in X. A language L is X-free if any word in L is X-free. Let $L(\Sigma, X)$ be the language of X-free words and $L^\omega(\Sigma, X)$ be the set of X-free ω-words over Σ. Set $L^\#(\Sigma, X) = L(\Sigma, X) \cup L^\omega(\Sigma, X)$.

Proposition 2.2. *$L^\#(\Sigma, X)$ is the closure of $L(\Sigma, X)$ in $\Sigma^\#$, and $L^\omega(\Sigma, X)$ is the set of limit points of $L(\Sigma, X)$.*

Corollary 2.3. *$L^\omega(\Sigma, X)$ is nonempty if and only if $L(\Sigma, X)$ is infinite.*

As easily seen, $L^\omega(\Sigma, X)$ is perfect (there is no isolated points) if and only if any prefix of an ω-word $x \in L^\omega(\Sigma, X)$ is a prefix of two distinct ω-words in $L^\omega(\Sigma, X)$. If $L^\omega(\Sigma, X)$ is perfect, then it is homeomorphic to the Cantor ternary set and is uncountable.

We define

$$L(n) = L(\Sigma, X; n) = L(\Sigma, X) \cap \Sigma^n$$

and

$$d(n) = d(\Sigma, X; n) = |L(\Sigma, X; n)|.$$

Lemma 2.4. *We have*

$$d(n+m) \leq d(n) \cdot d(m)$$

for $m, n \in \mathbb{N}$.

Proposition 2.5 (see Kobayashi [22]). *The limit $\mu = \mu(\Sigma, X) = \lim_{n\to\infty} d(n)^{1/n}$ exists, and it equals $\inf_n d(n)^{1/n}$. Either $\mu = 0$ or $1 \leq \mu \leq |\Sigma|$ holds.*

We say $L(\Sigma, X)$ *grows exponentially* if there is $C > 1$ such that $d(n) \geq C^n$, and $L(\Sigma, X)$ *grows polynomially* if there is a polynomial p such that $d(n) \leq p(n)$. X is *avoidable* on Σ, if $L(\Sigma, X)$ is infinite, otherwise it is *unavoidable*.

Proposition 2.6. *(1) $\mu = 0$ if and only if $L(\Sigma, X)$ is finite.*
(2) $\mu > 1$ if and only if $L(\Sigma, X)$ grows exponentially.

We call μ the *growth rate*, *complexity* or *entropy* of $L(\Sigma, X)$.

3 Avoiding a finite set of words

Let X be a finite subset of Σ^*. Let $L = L(\Sigma, X)$, $\ell = \max\{|x| \mid x \in X\}$ and $V = L \cap \Sigma^{\ell-1} = \{v_1, ..., v_s\}$. Define the *characteristic matrix* $M = (m_{ij})$ of X by

$$m_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is a suffix of } v_i a \in L \text{ for some } a \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1. *For $\ell \geq 0$, $d(n + \ell - 1)$ is the number of paths of length n in the graph with adjacent matrix M.*

Theorem 3.2. *The growth rate $\mu = \mu(\Sigma, X)$ is equal to the Frobenius root (the largest real eigenvalue) of M, and*

(1) if $\mu = 1$, L grows polynomially and L^ω is finite,

(2) if $\mu > 1$, L grows exponentially and L^ω is perfect.

Corollary 3.3. *For a finite set X, it is decidable whether X is unavoidable, L grows polynomially, or L grows exponentially.*

Example 3.4. Let $\Sigma = \{a, b\}$. $X_1 = \{aa, ab, bb\}$, $X_2 = \{aa, ab\}$ and $X_3 = \{aa\}$. Then, X_1 is unavoidable, $L(X_2, \Sigma)$ grows polynomially, $L(X_3, \Sigma)$ grows exponentially. The graphs associated with them are shown as follows respectively:

$$(1)\ a \leftarrow b \qquad (2)\ a \leftarrow b\ \circlearrowright \qquad (3)\ a \leftrightarrows b\ \circlearrowright$$

The following gives a way to give a good upper bound of $\mu(\Sigma, X)$ for an infinite X (see Shur 2008 [40]).

Theorem 3.5. *Let X be an infinite subset of Σ^*. Let $X_n = X \cap \Sigma^{\leq n} = \{x \in X \mid |x| \leq n\}$ and $\mu_n = \mu(\Sigma, X_n)$. Then, the sequence $\{\mu_n\}$ is decreasing and converges to $\mu(\Sigma, X)$.*

4 Unavoidable Patterns

Let V be an alphabet disjoint with Σ. A word p in V^* is called a *pattern*. An *instance* of p is a word in Σ^* obtained by substituting every variable in p by a nonempty word in Σ^* (see Bean, Ehrenfeught & McNulty 1979 [3]). For a set P of patterns, $x \in \Sigma^*$ is *P-free* (or *avoids P*), if x is free from any instance of a pattern in P. Let $L(\Sigma, P)$ denote the set of P-free words over Σ and $L^\omega(\Sigma, P)$ be the set of P-free ω-words over Σ.

Example 4.1. (1) For $P_n = \{u^n\}$, $u \in V$, a P_n-free word is *n-power free* (*square-free* if $n = 2$, *cube-free* if $n = 3$).

(2) For $Q = \{u^3, uvuvu\}, u, v \in V$, a Q-free word is nothing but a *overlap-free* word.

P is *unavoidable* on Σ, if the set of the instances of patterns in P is unavoidable, that is, $L(\Sigma, P)$ is finite. P is *absolutely unavoidable* if it is unavoidable on any (finite) alphabet Σ.

Example 4.2. (1) The square u^2 is unavoidable on the two-letter alphabet, but it is not absolutely unavoidable.

(2) The pattern uvu is absolutely unavoidable.

The *adjacency graph* $\mathcal{A}(p)$ of $p \in V^*$ is the bipartite graph $(V^\ell \cup V^r, E)$, where $V^\ell \cup V^r$ is the union of two copies of V and $(u^\ell, v^r) \in E$ if uv appears in p. A *free set* F is a subset of V such that there is no path in $\mathcal{A}(p)$ from u^ℓ to v^r for any $u, v \in F$. A pattern p *reduces* to a pattern q (denoted as $p \Rightarrow q$), if q is obtained from p removing all letters in some free set.

Theorem 4.3 (Zimin 1982 [45], see Lothaire 2003 [25]). *A pattern p is absolutely unavoidable if and only if it is reduced to 1 using a finite number of reductions.*

Example 4.4. The pattern *uvuwuvu* is absolutely unavoidable because

$$uvuwuvu \Rightarrow vwv \Rightarrow w \Rightarrow 1.$$

Corollary 4.5. *It is decidable whether a given pattern is absolutely unavoidable.*

5 Repetition-free words and morphisms

Let x be a nonempty word in Σ^*, y a prefix of x, and $s \in \mathbb{N}$. In this situation the word $x^s y$ is called a *t-repetition* of x, where

$$t = s + |y|/|x|.$$

For $t \in \mathbb{R}$, $x \in \Sigma^{\#}$ is *t-repetition-free* if x contains no s-repetition with $s \geq t$, and x is *weakly t-repetition-free* if x contains no s-repetition with $s > t$.

To treat these two kinds of repetition-freeness commonly we introduce the ordered set

$$\overline{\mathbb{R}} := \mathbb{R} \cup \mathbb{Q}^+,\ \mathbb{Q}^+ = \{t^+ \mid t \in \mathbb{Q}\},$$

in which $a < a^+ < b$ for $a \in \mathbb{Q}$, $b \in \mathbb{R}$ with $a < b$. For $\alpha \in \overline{\mathbb{R}}$, x is *α-repetition-free* if it has no t-repetition with $t \geq \alpha$ as subword. A 2^+-repetition is an *overlap*.

Let $L(\Sigma, \alpha) = L(k, \alpha)$ denote the set of all α-repetition-free words over Σ with $|\Sigma| = k$. Define

$$d(\Sigma, \alpha; n) = d(k, \alpha; n) = |L(k, \alpha) \cap \Sigma^n|,$$

and

$$\mu(\Sigma, \alpha) = \mu(k, \alpha) = \lim_{n \to \infty} d(k, \alpha; n)^{1/n}.$$

Let Δ be another alphabet. A morphism $\Phi : \Sigma^* \to \Delta^*$ of monoids is *growing* if $|\Phi(a)| \geq 1$ for all $a \in \Sigma$, and $|\Phi(a)| \geq 2$ for some $a \in \Sigma$. Φ is *strictly growing* if $|\Phi(a)| \geq 2$ for all $a \in \Sigma$. Φ is *uniformly growing* if there is $p \geq 2$ such that $|\Phi(a)| = p$ for all $a \in \Sigma$. Φ is *α-repetition preserving* if $x \in \Sigma^*$ is an α-repetition, then so is $\Phi(x)$. Φ is *α-repetition-free* if $x \in \Sigma^*$ is α-repetition-free, then so is $\Phi(x)$. A nontrivial uniform morphism is α-repetition preserving for any α.

The existing of morphisms with above properties gives information about repetition-free words.

Theorem 5.1 (Thue 1906 [43], Kobayashi 1986 [22]). *Let $\alpha \in \overline{\mathbb{R}}$ and $\Phi : \Sigma^* \to \Sigma^*$ be a uniformly growing α-repetition-free morphism. Then, $L^\omega(\Sigma, \alpha)$ contains a nonempty perfect subset, in particular, $L^\omega(\Sigma, \alpha)$ is uncountable.*

Theorem 5.2 (Brandebberg 1983 [7]). *Suppose that $|\Sigma| < |\Delta|$ and $\Phi : \Delta^* \to \Sigma^*$ is a uniformly growing injective α-free morphism. If $L(\Sigma, \alpha) \neq \{1\}$, Then $L(\Sigma, \alpha)$ grows exponentially.*

Theorem 5.3 (Restivo & Salemi 1985 [35], Kobayashi 1986 [22]). *Let* $\Phi : \Sigma^* \to \Sigma^*$ *be a strictly growing* α*-repetition preserving morphism. If* $\exists N > 0$ *s.t.* $\forall x \in L(\Sigma, \alpha)$, $\exists u, v, y \in \Sigma^*$ *s.t.* $|u|, |v| \leq N$, $x = u\Phi(y)v$. *Then,* $L(\Sigma, \alpha)$ *grows polynomially.*

6 Binary words

The Thue *morphism* $\Theta : \{a, b\}^* \to \{a, b\}^*$ is defined by

$$\Theta(a) = ab, \ \Theta(b) = ba.$$

It produces the Thue *words*

$$a, \ \Theta(a) = ab, \ \Theta^2(a) = abba, \ \Theta^3(a) = abbabaab, \ \ldots.$$

Theorem 6.1 (Thue 1906 [43]). Θ *is overlap-free. So, the Thue words are overlap-free.*

Corollary 6.2. $L^\omega(2, 2^+)$ *contains a nonempty perfect set and uncountable.*

More strongly, we have

Theorem 6.3 (Fife 1983 [18]). $L^\omega(2, 2^+)$ *is perfect.*

Though $L^\omega(2, 2^+)$ is uncountable, $L(2, 2^+)$ grows very slowly.

Lemma 6.4. *For any* $x \in L(2, 2^+)$, $\exists\, u, v, y \in \Sigma^*$ *s.t.*

$$x = u\Theta(y)v, \ |v| \leq 2, |v| \leq 2.$$

Theorem 6.5 (Restivo & Salemi 1985 [35]). $L(2, 2^+)$ *grows polynomially.*

Though $L(2, 2^+)$ grows polynomially, $d(n) = |L(2, 2^+) \cap \Sigma^n|$ cannot be approximated by a single polynomial (see (3) below). The estimation of $d(n)$ has been impoved as follows.

(1) Restivo & Salemi 1985 [35]: $d(n) \leq C \cdot n^{3.906\ldots}$.

(2) Kobayashi 1988 [23]: $C_1 \cdot n^{1.155} < d(n) < C_2 \cdot n^{1.587}$.

(3) Cassaigne 1993 [11]: $\sigma^- < 1.276 < 1.332 < \sigma^+$, where

$$\sigma^- = \underline{\lim} \log d(n)/\log n, \ \sigma^+ = \overline{\lim} \log d(n)/\log n.$$

(4) Jungers, Protasov & Blondel 2009 [20]:

$$1.2690 < \sigma^- < 1.2736 < 1.3322 < \sigma^+ < 1.3326,$$

and $\log d(n)/\log n \to \sigma$ on a set of density 1 of n with $1.3005 < \sigma < 1.3098$.

Define a morphism $\beta : \{a, b, c\}^* \to \{a, b\}^*$ by

$$\beta(a) = aababb, \ \beta(b) = aabbab, \ \beta(c) = abbaab.$$

Theorem 6.6 (Brandenburg 1983 [7]). *β is cube-free, and $L(2,3)$ grows exponentially.*

The estimation of $\mu(2,3)$ has been improved as follws.

(1) Brandenberg 1983 [7]: $1.08 < \mu(2,3) < 1.522$.

(2) Edlin 1999 [16]: $\mu < 1.4576$.

(3) Shur 2008 [40] 2009 [41] 2010 [42]: $1.45757131 < \mu(2,3) < 1.457577286$.

(4) Shur 2009 [41]: $1.82109999323 < \mu(2,4) < 1.8210999324$.

7 Repetition threshold

Define the *repetition threshold* $\mathrm{RT}(r)$ and the *exponential repetition threshold* $\mathrm{ERT}(r)$ for $r \geq 2$ by

$$\mathrm{RT}(r) = \sup\{\alpha \in \overline{\mathbb{R}} \mid L^{\omega}(r,\alpha) = \emptyset\},$$

and

$$\mathrm{ERT}(r) = \inf\{\alpha \in \overline{\mathbb{R}} \mid L(r,\alpha) \text{ grows exponentially}\}.$$

By Corollary 6.2, Corollary 6.5 and Theorem 6.6, we see

$$2 = \mathrm{RT}(2) < \mathrm{ERT}(2) \leq 3.$$

Theorem 7.1 (Karhmäki & Shallit 2004 [21]). $\mathrm{ERT}(2) = 7/3 = 2.333\ldots$. *Moreover, $d(2,7/3) \leq C \cdot n^{4.644}$ and $C_1 \cdot 1.011^n \leq d(2,7^+/3) \leq C_2 \cdot 1.23^n$.*

The estimation of $d(2,7/3)$ has been impoved as follows, where

$$\sigma^- = \underline{\lim} \log d(2,7/3;n)/\log n, \ \sigma^+ = \overline{\lim} \log d(2,7/3;n)/\log n.$$

(1) Karhmäki & Shallit 2004 [21]: $\sigma^+ < 4.644$.

(2) Blondel, Cassaigne & Jungers 2009 [6]:

$$1.2690 < \sigma^- < 2.0035 < 2.0121 < \sigma^+ < 2.1050.$$

The estinmation of $\mu(2,7^+/3)$ has been developed as

(1) Karhmäki & Shallit 2004 [21]: $1.011 < \mu(2,7^+/3) < 1.23$.

(2) Shur 2008 [40] 2009 [41]: $1.22062539 < \mu(2,7^+/3) < 1.22064486$.

Define a morphism $\beta' : \{a,b,c,d\}^* \to \{a,b,c\}^*$ by

$$\beta'(a) = abacabcacbabcbacbc,$$

$$\beta'(b) = abacabcacbacabacbc,$$

$$\beta'(c) = abacabcacbcabcbabc,$$

$$\beta'(d) = abacabcbacabacbabc.$$

Theorem 7.2 (Brandenburg [7]). *β' is square-free, and $L(3,2)$ grows exponentially.*

The estimation of $\mu(3,2)$ has been improved as follows.

(1) Brinkhuis 1983 [8]: $1.0293 < \mu(3,2) < 1.316$.
(2) Brandebberg 1983 [7]: $1.032 < \mu(3,2) < 1.38$.
(3) Richard & Grimm 2004 [37]: $\mu(3,2) < 1.301762$.
(4) Shur 2008 [40] 2009 [41]: $1.30175824 < \mu(3,2) < 1.3017619138$.

About the perfectness of $L^\omega(r,\alpha)$ we have

Theorem 7.3 (Shelton 1981, 1982 [38]). *$L^\omega(3,2)$ is perfect.*

Theorem 7.4. *(1)* (Currie & Shelton 1996 [14]) *$L^\omega(r,\alpha)$ is perfect, if $1 < \alpha < 2$ and r is sufficiently large.*

(2) (Mignosi, Restivo & Salemi 1995 [26]) *$L^\omega(r,\alpha)$ is perfect, if $\alpha \geq 2$ and $r > (5+\sqrt{5})/2 = 3.618\ldots$.*

By Theorem 7.2 we see $1 < \mathrm{RT}(3) \leq \mathrm{ERT}(3) \leq 2$.
Define a morphism $\delta : \{a,b,c\}^* \to \{a,b,c\}^*$ by

$$\delta(a) = abcacbcabcbacbcacba,$$

$$\delta(b) = bcabacabcacbacabacb,$$

$$\delta(c) = cabcbabcabacbabcbac.$$

Theorem 7.5 (Déjean 1972 [15]). *δ is $7^+/4$ repetition-free, and*

$$\mathrm{RT}(3) = 7/4 = 1.75.$$

Conjecture 7.6 (Déjean). $\mathrm{RT}(4) = 7/5$, *and* $\mathrm{RT}(r) = r/(r-1)$ *for* $r \geq 5$.

The conjecture has been finally proved to be true. The following is its history.

$r = 3$: Déjean 1972 [15],
$r = 4$: Pansiot 1984 [34],
$5 \leq k \leq 11$: Moulin-Ollagnier 1992 [29],
$12 \leq k \leq 14$: Mohammad-Moori & Currie 2007 [27],
$33 \leq k$: Carpi 2007 [10],
$27 \leq k$: Currie & Rampersad 2009 [12].
$8 \leq k \leq 38$: Rao 2011 [36], Currie & Rampersad 2011 [13].

Theorem 7.7 (Ochen 2006 [32]). *$L(3,7^+/4)$ and $L(4,7^+/5)$ grow exponentially, that is,* $\mathrm{RT}(3) = \mathrm{ERT}(3)$, $\mathrm{RT}(4) = \mathrm{ERT}(4)$.

Conjecture 7.8 (Ochen). $\mathrm{RT}(r) = \mathrm{ERT}(r)$ *for all* $r \geq 3$.

If this conjecture is true, the case $r = 2$ is very exceptional.

References

[1] S.I. Adyan The Burnside Problem and Identities in Groups (Springer, Berlin 1979).

[2] S. Arson, Proof of the existence of asymmetric infinite sequences, Mat. Sb. **2** (1937) 769–779.

[3] D.R. Bean, A. Ehrenfeught and G.F. McNulty, Avoidable patterns in strings of symbols, Pacific J. Math. **85** (1979), 261- 294.

[4] J. Berstel, Axel Thue's work on repetitions in words, Proc. 4th FPSAC, Montreal (1995), 65-80.

[5] J. Berstel, Growth of repetition-free words - a review, Theoret. Comp. Sci. **340** (2005), 280–290.

[6] V.D. Bondel, J.C. Cassaigne and R.M. Jungers, On the number of α-power-free binary words for $2 < \alpha < 7/3$, Theoret. Comp. Sci. **410** (2009) 2823–2834.

[7] F.-J. Brandenberg, Uniformly growing kth power free homomorphisms, Theoret. Comp. Sci. **23** (1983) 69–82.

[8] J. Brinkhuis, Non-repetitive sequences on three symbols, Quart. J. Math. Oxford **34** (1983) 145–149.

[9] S. Burrie and E. Nelson, Embedding the dual of π_∞ in the lattice of equational classes of semigroups, Algebra Univ. **1** (1971/72) 248–253.

[10] A. Carpi, On Dejean's conjecture over large alphabets, Theoret. Comp. Sci. **385** (2007) 137–151.

[11] J. Cassaigne, Counting overlap-free binary words, *Theoretical Aspects of Computer Science*, Lecture Notes in Comp. Sci. **665** (Springer, 1993) 216–225.

[12] J. Currie and N. Rampersad, Dejean's conjecture holds for $n \geq 27$, RAIRO - Theoret. Inform. Appl. **43** (2009) 775–778.

[13] J. Currie and N. Rampersad, A proof of Dejean's conjecture, Mathematics of Computation **80** (2011) 1063–1070.

[14] J. Currie and R. Shelton, Cantor sets and Dejean's conjecture, J. Automata, Lang. Comb. **1** (1996) 113-127.

[15] F. Déjean, Sur un théorème de Thue, J. Comb. Th. A **13** (1972), 90–99.

[16] A. Edlin, The number of binary cube-free words of length at to 47 and their numerical analysis, J. Diffrence Equ. Appl. **5** (1999) 153–154.

[17] E.D. Fife, Binary sequences which contains no BBb, Trans Amer. Math. Soc. **261** (1980) 115–136.

[18] E.D. Fife, Irreducible binary sequences, *Combinatorics on words* (Academic Press, New York, 1983) 91–99.

[19] G.A. Hedlund, Remarks on the work of Axel Thue on sequences, Nordisk Mat. Tidskr. **15** (1967) 147–150.

[20] R.M. Jungers, V. Protasov and V.D. Blondel, Overlap-free words and spectra of matirices, Theoret. Comp. Sci. **410** (2009) 3670–3684.

[21] J. Karhumäki and J. Shallit, Polynomial versus exponential growth in repetition-free words, J. Comb. Th. A **105** (2004) 335–347.

[22] Y. Kobayashi, Repetition-free words, Theoret. Comp. Sci. **44** (1986) 175–197.

[23] Y. Kobayashi, Enumeration of irreducible binary words, Discrete Appl. Math. **20** (1988) 221–232.

[24] M. Lothaire, *Combinatorics on Words* (Addison-Wesley, MA, 1983).

[25] M. Lothaire, *Algebraic Combinatorics on Words* (Cambridge Univ. Press, Cambridge, 2002).

[26] Mignosi, Restivo and Salemi, A periodicity theorem on words and applications, MFCS'95, Lect. Not. Comp. Sci. **969** (1995) 227-348.

[27] M. Mohammad-Noori and J.D. Currie, Dejean's conjecture and Sturmian words, Europian J. Comb. **28** (2007) 876–890.

[28] M. Morse and G.A. Hedlund, Symbolic dynamics I, II, Amer. J. Math. **60** (1938) 815–866, **62** (1940) 1–42.

[29] J. Moulin Ollagnier, Proof of Dejean's conjecture for alphabets with 5, 6, 7, 8, 9, 10 and 11 letters, Theoret. Comp. Sci. **95** (1992) 187–205.

[30] P.S. Novikov, On periodic groups, Dokl. Akad. Nauk SSSR **127** (1959) 749–752.

[31] P.S. Novikov and Adjan, Infinite periodic groups I, II, III, Math USSR Izv. **2** (1968) 209-236, 241-479, 665–685.

[32] P. Ochem, A generator of morphisms for infinite words, RAIRO - Theoret. Inform. Appl. **40** (2006) 427–441.

[33] J.M. Ollagnier, Proof of Dejean's conjecture for alphabets with 5,6,7,8,9,10 and 11 letters, Theoret. Comp. Sci. **95** (1992) 187–205.

[34] J.-J. Pansiot, A propos de'une conjecture de F. Déjean sur les répétitions dans les mots, Discrete Appl. Math. **7** (1984) 279–311.

[35] A. Restivo and S. Salemi, Overlap free words on two symbols, *Automata and Infinite words*, Lecture Notes Comp. Sci. **192** (Springer, Heiderberg, 1985) 198–206.

[36] M. Rao, Last cases of Dejean's conjecture, Theoret. Comp. Sci. (2011) to appear.

[37] C. Richard and U. Grimm, On the entropy and letter frequences of ternary square-free words, Electron. J. Comb. **11** (2004) #R14.

[38] R.O. Shelton (and R.P. Soni), Aperiodic words on three symbols, I, II, III, J. Reine Angew. Math. **321** (1981) 195-209, **327** (1981) 1–11, **330** (1982) 44–52.

[39] M. Sapir, Inherently nonfinitely based finite semigroups, Mat. Sb. **133** (1987) 154–166.

[40] A.M. Shur, Combinatorial complexity of regular languages, CSR2008, Lect. Not. Comp. Sci. **5010** (Springer, 2008) 289–301.

[41] A.M. Shur, Two-sided bounds for the growth rates of power-free languages, DLT2009, Lecture Notes in Comp. Sci. **5583** (Springer, 2009) 466–477.

[42] A.M. Shur, Growth rates of complexity of power-free languages, Theoret. Comp. Sci. **411** (2010) 3209–3232.

[43] A. Thue, Über unendliche Zeichenrehe, Norske Vid. Selesk. Skr. I, Math. Nat. Kl. Christiana **7** (1906) 1–22.

[44] A. Thue, Über die gegenseitige lage gleicher Teile gewisser Zeichrenreihen, ibid. **12** (1912) 1–67.

[45] A.I. Zimin, Blocking sets of terms, Mat. Sb. (NS), **119** (1982) 363–375.