数理解析研究所講究録
第 1769 巻 2011 年 131-139

An Extension of Automorphisms of a Petri Net

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Abstract

A Petri net is a mathematical model which is applied to descriptions of parallel processing systems. So far, a some types of morphisms related to Petri nets (or condition/event net) in terms of the category theory, in order to simplify the behavior of more complicated Petri nets and understand the concurrency in other computation models [2][8].

Studying how the structure of Petri nets have an effect on Petri net languages and codes, we often realize that the ratio between the number of tokens in a place and the weights of edges connected to the place is important and essential. So we give our definition of morphims between Petri nets focusing on the connection state/level of edges which come in or go out a place. This is an extension of an automorphism which we used to introduce to a net in [3][4].

We introduce a morphims between two Petri nets. The set of all morphisms of a Petri net forms a monoid expressed by a semi-direct product. Especially, the set of all automorphisms of a Petri net forms a group. We investigate the inclusion relations among such monoids and groups. Next, we deals with a pre-order induced by a surjective morphism. Two diamond properties is proved.

1. Preliminaries

Here we give our definition of morphisms of a Petri net and state the properties of some monoids composed of these morphisms.

1.1 Petri Nets and Morphisms

In this section, we give definitions and fundamental properties related to Petri nets. We denote the set of all nonnegative integers by $\boldsymbol{N}_0$, that is, $\boldsymbol{N}_0 = \{0, 1, 2, \dots\}$.

First of all, a Petri net is viewed as a particular kind of directed graph, together with an initial state μ_0, called the *initial marking*. The underlying graph N of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called *places* and *transitions*, where arcs are either from a place to a transition or from a transition to a place.

DEFINITION 1.1 (Petri net) A *Petri net* is a 4-tuple (P, T, W, μ_0) where
(1) $P = \{p_1, p_2, \dots, p_m\}$ is a finite set of places,
(2) $T = \{t_1, t_2, \dots, t_n\}$ is a finite set of transitions,
(3) $W : \boldsymbol{E}(P, T) \rightarrow \{0, 1, 2, 3, \dots\}$, i.e.,$W \in {\boldsymbol{N}_0}^{\boldsymbol{E}(P,T)}$, is a *weight function*, where $\boldsymbol{E}(P, T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \rightarrow \{0, 1, 2, 3, \dots\}$, i.e., $\mu_0 \in {\boldsymbol{N}_0}^P$, is the initial marking,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by N, a Petri net with a given initial marking μ_0 is denoted by (N, μ_0). □

In the graphical representation, the places are drawn as circles and the transitions are drawn as bars or boxes. Arcs are labeled with their weights(positive integers), where a k-weighted arc can be interpreted as the set of k parallel arcs. Labels for unity weights are usually omitted. A marking (state) assigns a nonnegative integer k to each place. If a marking assigns a nonnegative integer k to a place p, we say that p is *marked with* k tokens. Pictorially, we put k black dots (tokens) in place p. A marking is denoted by μ, an n-dimensional row vector, where n is the total number of places. The p-th component of μ, denoted by $\mu(p)$, is the number of tokens in place p.

EXAMPLE 1.1 Figure 1 shows a graphical representation of a Petri net. This Petri net $\mathcal{P} = (P, T, W, \mu_0)$ represents a process that a bicycle is assembled from one body and two wheels. The places are $P = \{\textbf{body}, \textbf{wheel}, \textbf{bicycle}\}$ and the transitions are $T = \{\textbf{assembly}\}$. Arcs $f_1 = (\textbf{body}, \textbf{assembly})$, $f_2 = (\textbf{wheel}, \textbf{assembly})$ and $f_3 = (\textbf{assembly}, \textbf{bicycle})$ have the weights of 1, 2 and 1, respectively. The other arcs have the weights of 0, and they are not usually drawn in the picture. Note that the weights of f_1 and f_3 is omitted since they are unity. That is, $W(f_1) = W(f_3) = 1, W(f_2) = 2$, $W(f) = 0$ for each $f \in (P \times T) \cup (T \times P) \setminus \{f_1, f_2, f_3\}$.

The initial marking μ_0 is often denoted by a vector $\mu_0 = (4, 3, 0)$. The place **body** is marked with three tokens. Then we usually put the number of tokens in a place, instead of black dots(tokens). □

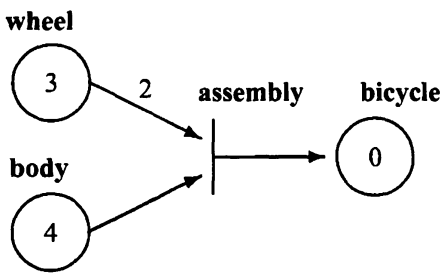

Figure 1. Graphical representation of a Petri net

Now we introduce a Petri net morphism based on place connectivity. We denote the set of all positive rational numbers by $\boldsymbol{Q}_+$.

DEFINITION 1.2 Let $\mathcal{P}_1 = (P_1, T_1, W_1, \mu_1)$ and $\mathcal{P}_2 = (P_2, T_2, W_2, \mu_2)$ be Petri nets. Then a triple $(f, (\alpha, \beta))$ of maps is called a *morphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if the maps $f : P_1 \to \boldsymbol{Q}_+$, $\alpha : P_1 \to P_2$ and $\beta : T_1 \to T_2$ satisfy the condition that for any $p \in P_1$ and $t \in T_1$,

$$\begin{aligned} W_2(\alpha(p), \beta(t)) &= f(p)W_1(p, t), \\ W_2(\beta(t), \alpha(p)) &= f(p)W_1(t, p), \\ \mu_2(\alpha(p)) &= f(p)\mu_1(p). \end{aligned} \tag{1.1}$$

In this case we write $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$. Moreover, a morphism $(f, (\alpha, \beta))$ is said to be *strong* if $f(p) = 1$ for any $p \in P$. □

The morphism $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ is called *injective* (resp. *surjective*) if both α and β are injective (resp. surjective). Especially, it is called an *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if it is injective and surjective. Then $\mathcal{P}_1$ is said to be *isomorphic* to $\mathcal{P}_2$ and we write $\mathcal{P}_1 \simeq \mathcal{P}_2$. Moreover, in case of $\mathcal{P}_1 = \mathcal{P}_2$, an isomorphism is called an *automorphism* of $\mathcal{P}_1$.

Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets, $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be morphisms. Then, since

$$\begin{aligned} W_3(\gamma(\alpha(p)), \delta(\beta(t))) &= g(\alpha(p))W_2(\alpha(p), \beta(t)) \\ &= g(\alpha(p))f(p)W_1(p, t), \\ W_3(\delta(\beta(t)), \gamma(\alpha(p))) &= g(\alpha(p))W_2(\beta(t), \alpha(p)) \\ &= g(\alpha(p))f(p)W_1(t, p), \\ \mu_3(\gamma(\alpha(p))) = g(\alpha(p))\mu_2(\alpha(p)) &= g(\alpha(p))f(p)\mu_1(p). \end{aligned}$$

hold, $(f \otimes_{P_1} (\alpha g), (\alpha\gamma, \beta\delta))$ is a morphism from $\mathcal{P}_1$ to $\mathcal{P}_3$, which is called the *composition* of morphisms $(f, (\alpha, \beta))$ and $(g, (\gamma, \delta))$. In this manuscript compositions of maps like $g \circ \alpha$, $\gamma \circ \alpha$ and $\delta \circ \beta$ are written in the form of multiplications like αg, $\alpha\gamma$ and $\beta\delta$. $f \otimes_{P_1} (\alpha g)$ is the map from P_1 to $\boldsymbol{Q}_+$ sending a place $p \in P_1$ to $f(p)g(\alpha(p)) \in \boldsymbol{Q}_+$.

2. Binary Relation $\sqsupseteq$ on Petri nets

For Petri nets $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if there exists a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. We show that this relation forms a pre-order and satisfies two diamond properties.

2.1 Basic Properties of the Relation $\sqsupseteq$

The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as an order by identifying isomorphisms.

PROPOSITION 2.1 Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be Petri nets. Then,
(1) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_1$.
(2) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2$.
(3) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$ imply $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$.

Proof) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ through the proof. The proof complete in the order (1), (3), (2).
(1) Trivial.
(3) There exist surjective morphisms $(f_i, (\alpha_i, \beta_i)) : \mathcal{P}_i \to \mathcal{P}_{i+1}$ $(i = 1, 2)$. We define a map $f : P_1 \to \boldsymbol{Q}_+$ by $f(p) = f_1(p) \cdot f_2(\alpha(p))$. Then $(f, (\alpha_1\alpha_2, \beta_1\beta_2))$ is a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$.
(2) $(\Rightarrow)$ There exist surjective morphisms $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g, (\alpha', \beta')) : \mathcal{P}_2 \to \mathcal{P}_1$. Since $\alpha\alpha'$ is surjective by (3) above and P_1 is finite, both α and α' are bijections. β and β' are also. Therefore $\mathcal{P}_1 \simeq \mathcal{P}_3$.
$(\Leftarrow)$ If $(f, (\alpha, \beta))$ be a isomorphism from $\mathcal{P}_1$ to $\mathcal{P}_2$, then it is easily shown that $(\alpha^{-1}f^{-1}, (\alpha^{-1}, \beta^{-1}))$ is a isomorphism from $\mathcal{P}_2$ to $\mathcal{P}_1$, where $f^{-1} : P_2 \to \boldsymbol{Q}_+, p \mapsto 1/f(p)$. □

EXAMPLE 2.1 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(1 \le i \le 3)$ be Petri nets shown in Figure 2. The four morphisms $x_i = (f_i, (\alpha_i, \beta_i))$ $(0 \le i \le 3)$ are from $\mathcal{P}_1$ to $\mathcal{P}_2$, where

$$f_0 = \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix},$$
$$f_1 = \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1/3 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} p_1 & p_2 \\ q_2 & q_1 \end{pmatrix},$$
$$f_2 = \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1/3 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_1 \end{pmatrix},$$
$$f_3 = \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} p_1 & p_2 \\ q_2 & q_2 \end{pmatrix},$$

and $\beta_0 = \beta_1 = \beta_2 = \beta_3 : T_1 \to T_2, t_1 \mapsto s, t_2 \mapsto s$. Especially only x_0 and x_1 are surjective morphisms.
Only one morphism $y = (g, (\gamma, \delta))$ exists from $\mathcal{P}_2$ to $\mathcal{P}_3$, where

$$g : P_2 \to \boldsymbol{Q}_+, q_1 \mapsto 1,\ q_2 \mapsto 1/3,$$
$$\gamma : P_2 \to P_3, q_1 \mapsto r, q_2 \mapsto r,$$
$$\delta : T_2 \to T_3, s \mapsto u.$$

This is a surjective morphism. The composition of morphisms x_i $(0 \le i \le 3)$ and y is the surjective morphism $(h, (\sigma, \tau))$ from $\mathcal{P}_1$ to $\mathcal{P}_3$, where

$$h : P_1 \to \boldsymbol{Q}_+, p_1 \mapsto 1/2,\ p_2 \mapsto 1/3,$$
$$\sigma = \alpha_i\gamma : P_1 \to P_3, p_1 \mapsto r, p_2 \mapsto r,$$
$$\tau = \beta_i\delta : T_1 \to T_3, t_1 \mapsto u, t_2 \mapsto u.$$

for any $i = 1, 2, 3, 4$. Note that h is expressed as $h = f_i \otimes (\alpha_i g)$. □

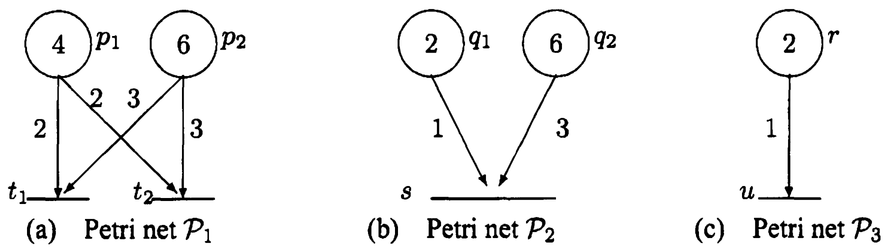

Figure 2. Petri nets $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2 \sqsupseteq \mathcal{P}_3$.

2.2 Diamond Properties of the Relation $\sqsupseteq$

Here we show the diamond property of the relation $\sqsupseteq$. The following notation of some equivalence relation is used in the manuscript.

Let P be a set and f, g maps whose domain is P. The relation $\sim_f$ on P defined by $(\forall x, y \in P)\{x \sim_f y \overset{\text{def}}{\Longleftrightarrow} f(x) = f(y)\}$. Then $(\sim_f \cup \sim_g)^*$ is the smallest equivalence relation on P which includes both $\sim_f$ and $\sim_g$, where $(\sim_f \cup \sim_g)^*$ is the reflexive and transitive closure of $\sim_f \cup \sim_g$.

PROPOSITION 2.2 (Diamond Property I) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$.

Proof) Let $(f_i, (\alpha_i, \beta_i)) : \mathcal{P}_0 \to \mathcal{P}_i$ $(i = 1, 2)$ be surjective morphisms. To prove the claim, we construct the Petri net $\mathcal{P}_3$ satisfying the condition above. Next set

$$P_3 = P_0/(\sim_{\alpha_1} \cup \sim_{\alpha_2})^*, \quad T_3 = T_0/(\sim_{\beta_1} \cup \sim_{\beta_2})^*,$$

and let α be a canonical surjection from P_0 onto P_3, β a canonical surjection from T_0 onto T_3, and $f : P_0 \to \boldsymbol{Q}_+$ the map defined as follows: If all of $\mu_0(p)$, $W_0(p, t_1)$, ... $W_0(p, t_n)$, $W_0(t_1, p)$, ..., $W_0(t_n, p)$ are 0's (in this case we say that p is *0-isolated*), then $f(p) = 1$. Otherwise,

$$f(p) = 1/gcd(\mu_0(p),\ W_0(p, t_1),\ \ldots,\ W_0(p, t_n),\ W_0(t_1, p),\ \ldots,\ W_0(t_n, p)),$$

where $T_0 = \{t_1, t_2, \ldots, t_n\}$ and the function *gcd* returns the greatest common divisor of its arguments.

Before showing that $(f, (\alpha, \beta))$ is a surjective morphism from $\mathcal{P}_0$ to $\mathcal{P}_3$, we show the following lemma.

LEMMA 2.1 Let $i \in \{1, 2\}, p, p' \in P_0$ with $\alpha_i(p) = \alpha_i(p')$ and $t, t' \in T_0$ with $\beta_i(t) = \beta_i(t')$.
(1) If neither p nor p' is 0-isolated, then $f(p)f_i(p') = f(p')f_i(p)$.
(2) $f(p)\mu_0(p) = f(p')\mu_0(p')$.
(3) $f(p)W_0(p, t) = f(p')W(p', t')$ and $f(p)W_0(t, p) = f(p')W(t', p')$.

Proof) (1) Since p and p' are not 0-isolated, the greatest common divisors give the following equations.

$$\begin{aligned} f(p)f_i(p') &= f(p')\{f(p)f_i(p')\}f^{-1}(p') = f(p')f(p) \times f_i(p')f^{-1}(p') \\ &= f(p')f(p) \times gcd(f_i(p')\mu_0(p'),\ f_i(p')W_0(p', t_1),\ \ldots,\ f_i(p')W_0(p', t_n), \\ &\qquad f_i(p')W_0(t_1, p'),\ \ldots,\ f_i(p')W_0(t_n, p')) \\ &= f(p')f(p) \times gcd(f_i(p)\mu_0(p),\ f_i(p)W_0(p, t_1),\ \ldots,\ f_i(p)W_0(p, t_n), \\ &\qquad f_i(p)W_0(t_1, p),\ \ldots,\ f_i(p)W_0(t_n, p)) \\ &= f(p')f(p) \times f_i(p)f^{-1}(p) = f(p')f_i(p)\{f(p)f^{-1}(p)\} = f(p')f_i(p) \end{aligned}$$

(2) $f_i(p)\mu_0(p) = \mu_i(\alpha_i(p)) = \mu_i(\alpha_i(p')) = f_i(p')\mu_0(p')$ implies that $\mu_0(p) = 0 \iff \mu_0(p') = 0$. Noting this, we may consider the two cases of $\mu_0(p) = 0$ and $\mu_0(p) \neq 0$. Since it is trivial in case of

$\mu_0(p) = 0$, we may assume that $\mu_0(p) \neq 0$.

$$\begin{aligned} f(p)\mu_0(p) &= f(p)f_i(p)^{-1}f_i(p)\mu_0(p) = f(p)f_i(p)^{-1}f_i(p')\mu_0(p') \\ &= f(p')f_i(p)^{-1}f_i(p)\mu_0(p') = f(p')\mu_0(p'). \end{aligned}$$

Note that the third equation is due to (1).

(3)

$$f_i(p)W_0(p,t) = W_i(\alpha_i(p),\beta_i(t)) = W_i(\alpha_i(p'),\beta_i(t')) = f_i(p')W_0(p',t')$$

implies that $W_0(p,t) = 0 \iff W_0(p',t') = 0$. Since it is trivial in case of $W_0(p,t) = 0$, we may assume that $W_0(p,t) \neq 0$ and thus p is not 0-isolated.

$$\begin{aligned} f(p)W_0(p,t) &= f(p)f_i(p)^{-1}f_i(p)W_0(p,t) = f(p)f_i(p)^{-1}f_i(p')W_0(p',t') \\ &= f(p')f_i(p)^{-1}f_i(p)W_0(p',t') = f(p')W_0(p',t') \end{aligned}$$

Note that the third equation is due to (1). Similarly we can show the equation $f(p)W_0(t,p) = f(p')W_0(t',p')$. □

Continue the proof of PROPOSITION 2.2. Let $p, p' \in P_0$ with $p(\sim_{\alpha_1} \cup \sim_{\alpha_2})^* p'$ and $t, t' \in T_0$ with $t(\sim_{\beta_1} \cup \sim_{\beta_2})^* t'$. Then we may assume that

$$\begin{aligned} &p \sim_{\alpha_{i_1}} p_1 \sim_{\alpha_{i_2}} p_2 \sim_{\alpha_{i_3}} \cdots \sim_{\alpha_{i_n}} p' \\ &t \sim_{\beta_{j_1}} t_1 \sim_{\beta_{j_2}} t_2 \sim_{\beta_{j_3}} \cdots \sim_{\beta_{j_m}} t' \end{aligned}$$

where n and m are positive integers and $i_1, \ldots, i_n, j_1, \ldots, j_m \in \{1, 2\}$. By LEMMA 2.1 (2) and (3),

$$\begin{aligned} &f(p)\mu_0(p) = f(p_1)\mu_0(p_1) = \cdots = f(p')\mu_0(p'), \\ &f(p)W_0(p,t) = f(p_1)W_0(p_1,t) = \cdots = f(p')W_0(p',t) \\ &\quad = f(p')W(p',t_1) = \cdots = f(p')W_0(p',t'), \\ &f(p)W_0(t,p) = f(p_1)W_0(t,p_1) = \cdots = f(p')W_0(t,p') \\ &\quad = f(p')W(t_1,p') = \cdots = f(p')W_0(t',p'). \end{aligned}$$

So $\mu_3(\alpha(p))$, $W_3(\alpha(p),\beta(t))$ and $W_3(\beta(t),\alpha(p))$ can be defined and

$$\begin{aligned} &\mu_3(\alpha(p)) = f(p)\mu_0(p), \\ &W_3(\alpha(p),\beta(t)) = f(p)W_0(p,t), \\ &W_3(\beta(t),\alpha(p)) = f(p)W_0(t,p). \end{aligned}$$

Thus $(f, (\alpha, \beta))$ is well-defined and it is a morphism from $\mathcal{P}_0$ to $\mathcal{P}_3$. Since both α and β are canonical surjections, we have $\mathcal{P}_0 \sqsupseteq \mathcal{P}_3$.

Finally we show that $\mathcal{P}_i \sqsupseteq \mathcal{P}_3$ $(i = 1, 2)$ hold. By LEMMA 2.1 (2) and (3), the following maps are well-defined.

$$\begin{aligned} &\alpha_i' : P_i \to P_3, q \mapsto \alpha(p) &&\text{where } \alpha_i(p) = q, \\ &\beta_i' : T_i \to T_3, s \mapsto \beta(t) &&\text{where } \beta_i(t) = s, \\ &f_i' : P_i \to \boldsymbol{Q}_+, q \mapsto f(p)f_i(p)^{-1} &&\text{where } \alpha_i(p) = q. \end{aligned}$$

Let $i \in \{1, 2\}$. For any $q \in P_i$ and $s \in T_i$, there exist $p \in P_0$ and $t \in T_0$ such that $\alpha_i(p) = q$ and $\beta_i(t) = s$, and thus we have

$$\begin{aligned} &\mu_3(\alpha_i'(q)) = \mu_3(\alpha(p)) = f(p)\mu_0(p) = f(p)f_i(p)^{-1}\mu_i(\alpha_i(p)) = f_i'(q)\mu_i(q), \\ &W_3(\alpha_i'(q),\beta_i'(s)) = W_3(\alpha(p),\beta(t)) = f(p)W_0(p,t) \\ &\qquad = f(p)f_i(p)^{-1}W_i(\alpha_i(p),\beta_i(t)) = f_i'(q)W_i(q,s), \\ &W_3(\beta_i'(s),\alpha_i'(q)) = W_3(\beta(t),\alpha(p)) = f(p)W_0(t,p) \\ &\qquad = f(p)f_i(p)^{-1}W_i(\beta_i(t),\alpha_i(p)) = f_i'(q)W_i(s,q). \end{aligned}$$

Therefore $(f_i', (\alpha_i', \beta_i'))$ is a morphism from $\mathcal{P}_i$ to $\mathcal{P}_3$. We can easily show that α_i' and β_i' are surjective. Thus $\mathcal{P}_i \sqsupseteq \mathcal{P}_3$ $(i = 1, 2)$. □

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$.

DEFINITION 2.1 A Petri net $\mathcal{P}$ is called a *$\sqsupseteq$-irreducible* if $\mathcal{P} \sqsupseteq \mathcal{P}'$ implies $\mathcal{P} \simeq \mathcal{P}'$ for any Petri net $\mathcal{P}'$. □

COROLLARY 2.1 Let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. Then one has: If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$.

Proof) Trivial by PROPOSITION 2.2 and the definition of $\sqsupseteq$-irreducibility. □

PROPOSITION 2.3 (Diamond Property II) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. Then there exists a Petri net $\mathcal{P}_0$ such that $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$.

Proof) Let $i \in \{1,2\}$ and $(f_i, (\alpha_i, \beta_i)) : \mathcal{P}_i \longrightarrow \mathcal{P}_3$ be surjective morphisms. We have

$$\begin{aligned}\mu_3(q) &= f_i(p_i)\mu_i(p_i),\\ W_3(q,s) &= f_i(p_i)W_i(p_i,t_i),\\ W_3(s,q) &= f_i(p_i)W_i(t_i,q_i),\end{aligned}$$

where $p_i \in P_i$, $t_i \in T_i$, $\alpha_i(p_i) = q$, $\beta_i(t_i) = s$. We construct the Petri net $\mathcal{P}_0 = (P_0, T_0, W_0, \mu_0)$ in the following way.

$$\begin{aligned}&P_0 = \{(p_1,p_2) \,|\, \alpha_1(p_1) = \alpha_2(p_2)\} \subset P_1 \times P_2,\\ &T_0 = \{(t_1,t_2) \,|\, \beta_1(t_1) = \beta_2(t_2)\} \subset T_1 \times T_2,\\ &W_0((p_1,p_2),(t_1,t_2)) = W_3(q,s),\\ &W_0((t_1,t_2),(p_1,p_2)) = W_3(s,q),\\ &\mu_0((p_1,p_2)) = \mu_3(q),\end{aligned}$$

where $\alpha_i(p_i) = q$, $\beta_i(t_i) = s$. Then it is enough to show that $(g_i, (\gamma_i, \delta_i)) : \mathcal{P}_0 \longrightarrow \mathcal{P}_i$ $(i = 1, 2)$, defined by equation (2.1), is a surjective morphism.

$$\begin{aligned}&g_i : P_0 \longrightarrow \boldsymbol{Q}_+, (p_1,p_2) \mapsto f_i(p_i)^{-1},\\ &\gamma_i : P_0 \longrightarrow P_i, (p_1,p_2) \mapsto p_i,\\ &\delta_i : T_0 \longrightarrow T_i, (t_1,t_2) \mapsto t_i.\end{aligned} \tag{2.1}$$

Indeed, setting $q = \alpha_i(p_i)$, $s = \beta_i(t_i)$,

$$\begin{aligned}\mu_i(\gamma_i((p_1,p_2))) &= \mu_i(p_i) = f_i(p_i)^{-1}\mu_3(q) = g_i((p_1,p_2))\mu_0((p_1,p_2)),\\ W_i(\gamma_i((p_1,p_2)),\delta_i((t_1,t_2))) &= W_i(p_i,t_i) = f_i(p_i)^{-1}W_3(q,s)\\ &= g_i((p_1,p_2))W_0((p_1,p_2),(t_1,t_2)),\\ W_i(\delta_i((t_1,t_2)),\gamma_i((p_1,p_2))) &= W_i(t_i,p_i) = f_i(p_i)^{-1}W_3(s,q)\\ &= g_i((p_1,p_2))W_0((t_1,t_2),(p_1,p_2)).\end{aligned}$$

Thus we have $\mathcal{P}_0 \sqsupseteq \mathcal{P}_i$. □

3 Monoids of Morphisms of a Petri Net

Here a finite set P of places and a finite set T of transitions are fixed. And we deal with monoids which consist of morphisms of a Petri net and investigate some properties of such monoids.

An algebraic system $(\boldsymbol{Q}_+{}^P, \otimes_P)$ forms a commutative group under the operation $\otimes_P$ defined by $f \otimes_P g : p \mapsto f(p)g(p)$. $\mathbf{1}_{\otimes_P} : P \longrightarrow \boldsymbol{Q}_+ : p \mapsto 1$ is the identity and $f^{-1} : P \longrightarrow \boldsymbol{Q}_+ : p \mapsto 1/f(p)$ is the inverse of a $f \in \boldsymbol{Q}_+{}^P$. Whenever it does not cause confusion, we write $\otimes$ instead of $\otimes_P$. Then we obtain the following lemma.

LEMMA 3.1 Let α and β be arbitrary maps on P and $f, g : P \longrightarrow \boldsymbol{Q}_+$. Then the following equations are true.

(1) $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T) \simeq (\boldsymbol{Q}_+{}^P \rtimes P^P) \times T^T$.

(2) The subset $\boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$ of $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ forms a group with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P,\ \mathbf{1}_T))$.

(3) $\mathbf{Mor}_+(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$.
(4) $\mathbf{Mor}_+(\mathcal{P})$ is a submonoid of $\mathbf{Mor}_+(\mathcal{P}_0)$.
(5) $\mathbf{Aut}_+(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$.
(6) $\mathbf{Aut}_+(\mathcal{P})$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P}_0)$.

Proof) For each $p \in P$, the following equations hold.
(1) $((\alpha\beta)f)(p) = f(\beta(\alpha(p))) = (\beta f)(\alpha(p)) = (\alpha(\beta f))(p)$.
(2) $(\alpha(f \otimes g))(p) = f(\alpha(p)) \cdot g(\alpha(p)) = (\alpha f)(p) \cdot (\alpha g)(p) = ((\alpha f) \otimes (\alpha g))(p)$.
(3) $(\alpha \mathbf{1}_\otimes)(p) = \mathbf{1}_\otimes(\alpha(p)) = \mathbf{1}_\otimes(p)$.
(4) By (2) and (3) above, $(\alpha f) \otimes (\alpha f^{-1}) = \alpha(f \otimes f^{-1}) = \alpha \mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(5) $(\alpha f)^{-1}(p) = 1/f(\alpha(p)) = f^{-1}(\alpha(p)) = (\alpha f^{-1})(p)$. □

Let $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ be the semi-direct product of the group $\boldsymbol{Q}_+{}^P$ and the monoid $P^P \times T^T$, equipped with the multiplication defined by

$$(f, (\alpha, \beta))(g, (\alpha', \beta')) \stackrel{\text{def}}{=} (f \otimes \alpha\, g, (\alpha\alpha', \beta\beta')), \tag{3.1}$$

where P^P is the set of all maps from P to P and T^T is the set of all maps from T to T. $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ forms a monoid with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$, where $\mathbf{1}_\otimes$ is the identity of the group $\boldsymbol{Q}_+{}^P$, $\mathbf{1}_P$ and $\mathbf{1}_T$ are the identity maps on P and T respectively.

Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Now we consider the following monoids and groups related to the Petri net. Note that $\mathbf{Mor_1}(\mathcal{P})$ (resp. $\mathbf{Aut_1}(\mathcal{P})$) is the set of all strong monoids (resp. automorphism) of $(\mathcal{P})$.

$$\begin{aligned}
\mathbf{Mor}_+(\mathcal{P}) &: \text{ the set of all the morphisms of } \mathcal{P} = (P, T, W, \mu) \\
\mathbf{Mor_1}(\mathcal{P}) &\stackrel{\text{def}}{=} \{(f, (\alpha, \beta)) \in \mathbf{Mor}_+(\mathcal{P}) \mid f = \mathbf{1}_\otimes\}, \\
\mathbf{Aut}_+(\mathcal{P}) &: \text{ the set of all the automorphisms of } \mathcal{P} = (P, T, W, \mu) \\
\mathbf{Aut_1}(\mathcal{P}) &\stackrel{\text{def}}{=} \{(f, (\alpha, \beta)) \in \mathbf{Aut}_+(\mathcal{P}) \mid f = \mathbf{1}_\otimes\}.
\end{aligned}$$

By 0^P we denote the marking with $0^P : P \to \boldsymbol{N}_0, p \mapsto 0$ and By $0^{\boldsymbol{E}(P,T)}$ we denote the weight function with $0^{\boldsymbol{E}(P,T)} : \boldsymbol{E}(P,T) \to \boldsymbol{N}_0, e \in \boldsymbol{E}(P,T) \mapsto 0$.

For give two Petri nets $\mathcal{P} = (P, T, W, \mu)$ and $\mathcal{P}_0 = (P, T, 0^{\boldsymbol{E}(P,T)}, 0^P)$, Figure 3 shows (not necessarily proper) inclusion relations among monoids and groups related to these Petri nets. We show these relations below.

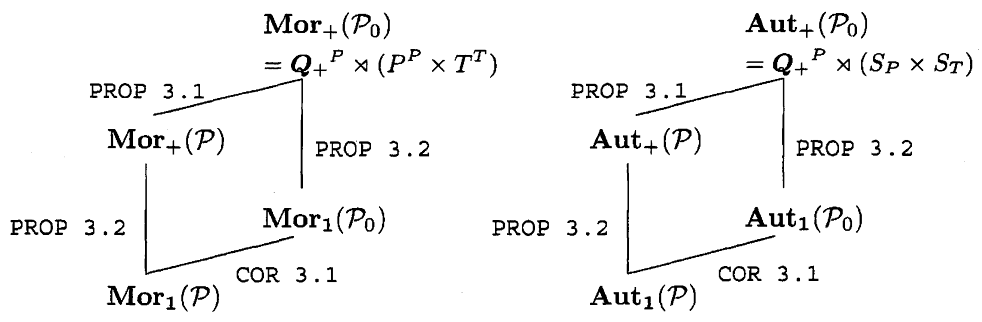

Figure 3. Inclusion relations among monoids of morphisms and groups of automorphisms related to the Petri nets $\mathcal{P}$ and $\mathcal{P}_0$

PROPOSITION 3.1 Let $\mathcal{P} = (P, T, W, \mu)$ and $\mathcal{P}_0 = (P, T, 0^{\boldsymbol{E}(P,T)}, 0^P)$ be Petri nets. And let S_P and S_T be the symmetric groups of P and T, respectively.

(1) The subset $Q_+{}^P \rtimes (S_P \times S_T)$ of $Q_+{}^P \rtimes (P^P \times T^T)$ forms a group with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$.
(2) $\mathbf{Mor}_+(\mathcal{P}_0) = Q_+{}^P \rtimes (P^P \times T^T)$.
(3) $\mathbf{Mor}_+(\mathcal{P})$ is a submonoid of $\mathbf{Mor}_+(\mathcal{P}_0)$.
(4) $\mathbf{Aut}_+(\mathcal{P}_0) = Q_+{}^P \rtimes (S_P \times S_T)$.
(5) $\mathbf{Aut}_+(\mathcal{P})$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P}_0)$.

Proof) (1) Set $\mathcal{S} = Q_+{}^P \rtimes (P^P \times T^T)$ and $\mathcal{T} = (Q_+{}^P \rtimes P^P) \times T^T$. We consider the map $\phi : \mathcal{S} \to \mathcal{T}, (f, (\alpha, \beta)) \mapsto ((f, \alpha), \beta)$. It is easy to check that ϕ is a bijection and a monoid morphism.
(2) Obviously $Q_+{}^P \rtimes (S_P \times S_T)$ is closed under the multiplication defined in the equation (3.1) and $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)) \in Q_+{}^P \rtimes (S_P \times S_T)$. Let $(f, (\alpha, \beta))$ be an arbitrary element of $Q_+{}^P \rtimes (S_P \times S_T)$. Then $(\alpha^{-1} f^{-1}, (\alpha^{-1}, \beta^{-1}))$ is in $Q_+{}^P \rtimes (S_P \times S_T)$ and satisfies

$$\begin{aligned}
&(f, (\alpha, \beta))(\alpha^{-1} f^{-1}, (\alpha^{-1}, \beta^{-1}))\\
&= (f \otimes \alpha\alpha^{-1} f^{-1}, (\alpha\alpha^{-1}, \beta\beta^{-1}))\\
&= (\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)), \quad \because \text{LEMMA 3.1 (1)}\\
&(\alpha^{-1} f^{-1}, (\alpha^{-1}, \beta^{-1}))(f, (\alpha, \beta))\\
&= (\alpha^{-1} f^{-1} \otimes \alpha^{-1} f, (\alpha^{-1}\alpha, \beta^{-1}\beta))\\
&= (\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)) \quad \because \text{LEMMA 3.1 (4)}.
\end{aligned}$$

This is an inverse of $(f, (\alpha, \beta))$. Therefore $Q_+{}^P \rtimes (S_P \times S_T)$ forms a group.
(3) By the definition, each morphism in $\mathbf{Mor}_+(\mathcal{P}_0)$ is obviously an element of $Q_+{}^P \rtimes (P^P \times T^T)$. Conversely, let $(f, (\alpha, \beta))$, p and t be any elements in $Q_+{}^P \rtimes (P^P \times T^T)$, P and T, respectively. Then, $0^P(p) = 0 = f(p) \cdot 0^P(\alpha(p))$, $0^{\boldsymbol{E}(P,T)}(\alpha(p), \beta(t)) = 0 = f(p) \cdot 0^{\boldsymbol{E}(P,T)}(p, t)$, and $0^{\boldsymbol{E}(P,T)}(\beta(t), \alpha(p)) = 0 = f(p) \cdot 0^{\boldsymbol{E}(P,T)}(t, p)$. Thus, $(f, (\alpha, \beta))$ is a morphism of $\mathcal{P}_0$. Since the composition of $\mathbf{Mor}_+(\mathcal{P}_0)$ is identical with the multiplication of $Q_+{}^P \rtimes (P^P \times T^T)$ by the definition (3.1), thus $\mathbf{Mor}_+(\mathcal{P}_0)$ and $Q_+{}^P \rtimes (P^P \times T^T)$ are equal as a monoid.
(4) Let $(f, (\alpha, \beta)) \in \mathbf{Mor}_+(\mathcal{P})$. $0^P(\alpha(p)) = 0 = f(p)0^P(p)$ for any $p \in P$. $0^{\boldsymbol{E}(P,T)}(\alpha(p), \beta(t)) = 0 = f(p)0^{\boldsymbol{E}(P,T)}(p, t)$ and $0^{\boldsymbol{E}(P,T)}(\beta(t), \alpha(p)) = 0 = f(p)0^{\boldsymbol{E}(P,T)}(t, p)$ for any $p \in P$ and $t \in T$. Therefore $(f, (\alpha, \beta)) \in \mathbf{Mor}_+(\mathcal{P}_0)$. Since $\mathbf{Mor}_+(\mathcal{P})$ is closed under the composition of morphisms and has $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$ as the identity element, thus $\mathbf{Mor}_+(\mathcal{P})$ is a submonoid of $\mathbf{Mor}_+(\mathcal{P}_0)$.
(5) In a similar manner to (3), we can show that $\mathbf{Aut}_+(\mathcal{P}_0)$ and $Q_+{}^P \rtimes (S_P \times S_T)$ are equal as a group.
(6) Obviously $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)) \in \mathbf{Aut}_+(\mathcal{P}) \subset \mathbf{Aut}_+(\mathcal{P}_0)$. $\mathbf{Aut}_+(\mathcal{P})$ is closed under the composition of morphisms. For an arbitrary $(f, (\alpha, \beta)) \in \mathbf{Aut}_+(\mathcal{P})$, we must show $(\alpha^{-1} f^{-1}, (\alpha^{-1}, \beta^{-1})) \in \mathbf{Aut}_+(\mathcal{P})$. Due to $\mu(p) = \mu(\alpha(\alpha^{-1}(p))) = f(\alpha^{-1}(p))\mu(\alpha^{-1}(p))$ and LEMMA 3.1 (5),

$$\mu(\alpha^{-1}(p)) = (\alpha^{-1} f)^{-1}(p)\mu(p) = (\alpha^{-1} f^{-1})(p)\mu(p).$$

Similarily, we have

$$\begin{aligned}
W(\alpha^{-1}(p), \beta^{-1}(t)) &= (\alpha^{-1} f^{-1})(p)W(p, t),\\
W(\beta^{-1}(t), \alpha^{-1}(p)) &= (\alpha^{-1} f^{-1})(p)W(t, p).
\end{aligned}$$

Therefore the inverse of $(f, (\alpha, \beta))$ is in $\mathbf{Aut}_+(\mathcal{P})$.

□

PROPOSITION 3.2 Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Then,
(1) $\mathbf{Mor_1}(\mathcal{P})$ is a submonoid of $\mathbf{Mor}_+(\mathcal{P})$,
(2) $\mathbf{Aut_1}(\mathcal{P})$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P})$,
(3) $\mathbf{Aut_1}(\mathcal{P})$ is a normal * subgroup of $\mathbf{Aut}_+(\mathcal{P})$ if and only if $\gamma f = f$ for any $(f, (\alpha, \beta)) \in \mathbf{Aut}_+(\mathcal{P})$ and $(\mathbf{1}_\otimes, (\gamma, \delta)) \in \mathbf{Aut_1}(\mathcal{P})$.

Proof) (1) $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)) \in \mathbf{Mor_1}(\mathcal{P}) \subset \mathbf{Mor}_+(\mathcal{P})$. For any $(\mathbf{1}_\otimes, (\alpha, \beta))$ and $(\mathbf{1}_\otimes, (\gamma, \delta)) \in \mathbf{Mor_1}(\mathcal{P})$, $(\mathbf{1}_\otimes, (\alpha, \beta))(\mathbf{1}_\otimes, (\gamma, \delta)) = (\mathbf{1}_\otimes, (\alpha\gamma, \beta\delta)) \in \mathbf{Mor_1}(\mathcal{P})$. Thus $\mathbf{Mor_1}(\mathcal{P})$ is a submonoid of $\mathbf{Mor}_+(\mathcal{P})$.
(2) $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T)) \in \mathbf{Aut_1}(\mathcal{P}) \subset \mathbf{Aut}_+(\mathcal{P})$. Let $(\mathbf{1}_\otimes, (\alpha, \beta))$ and $(\mathbf{1}_\otimes, (\gamma, \delta))$ be arbitrary elements in $\mathbf{Aut_1}(\mathcal{P})$. Then since $\alpha\mathbf{1}_\otimes \otimes \mathbf{1}_\otimes = \mathbf{1}_\otimes$, $(\mathbf{1}_\otimes, (\alpha, \beta))^{-1} (\mathbf{1}_\otimes, (\gamma, \delta)) = (\mathbf{1}_\otimes, (\alpha^{-1}\gamma, \beta^{-1}\delta)) \in \mathbf{Aut_1}(\mathcal{P})$. Therefore $\mathbf{Aut_1}(\mathcal{P})$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P})$.

*Generally a subgroup H of a group G is said to be normal if $xH = Hx$ for any $x \in G$.

(3) Let $(f,(\alpha,\beta)) \in \mathbf{Aut}_+(\mathcal{P})$ and $(\mathbf{1}_\otimes,(\gamma,\delta)) \in \mathbf{Aut_1}(\mathcal{P})$. Then by the definition of the operation of the semi-direct product and LEMMA 3.1, the following equations hold

$$\begin{aligned}
&(f,(\alpha,\beta))^{-1}(\mathbf{1}_\otimes,(\gamma,\delta))(f,(\alpha,\beta))\\
&=(\alpha^{-1}f^{-1},(\alpha^{-1},\beta^{-1}))(\mathbf{1}_\otimes,(\gamma,\delta))(f,(\alpha,\beta))\\
&=(\alpha^{-1}f^{-1}\otimes\alpha^{-1}\mathbf{1}_\otimes,(\alpha^{-1}\gamma,\beta^{-1}\delta))(f,(\alpha,\beta))\\
&=(\alpha^{-1}f^{-1}\otimes\alpha^{-1}\mathbf{1}_\otimes\otimes\alpha^{-1}\gamma f,(\alpha^{-1}\gamma\alpha,\beta^{-1}\delta\beta))\\
&=(\alpha^{-1}(f^{-1}\otimes\gamma f),(\alpha^{-1}\gamma\alpha,\beta^{-1}\delta\beta))
\end{aligned}$$

(Sufficiency). By the condition $\gamma f = f$, $\alpha^{-1}(f^{-1}\otimes\gamma f) = \alpha^{-1}(f^{-1}\otimes f) = \mathbf{1}_\otimes$.($\because$ LEMMA 3.1 (3)) Therefore, since $(f,(\alpha,\beta))^{-1}(\mathbf{1}_\otimes,(\gamma,\delta))(f,(\alpha,\beta)) \in \mathbf{Aut_1}(\mathcal{P})$, the subgroup $\mathbf{Aut_1}(\mathcal{P})$ is normal.

(Necessity). Since $\mathbf{Aut_1}(\mathcal{P})$ is a normal subgroup, $\alpha^{-1}(f^{-1}\otimes\gamma f) = \mathbf{1}_\otimes$. Multiplying α and then f to both sides from the left, We have $\gamma f = f$. □

COROLLARY 3.1 Let $\mathcal{P} = (P,T,W,\mu)$ and $\mathcal{P}_0 = (P,T,0^{E(P,T)},0^P)$ be Petri nets.
(1) $\mathbf{Mor_1}(\mathcal{P})$ is a submonoid of $\mathbf{Mor_1}(\mathcal{P}_0)$.
(2) $\mathbf{Aut_1}(\mathcal{P})$ is a subgroup of $\mathbf{Aut_1}(\mathcal{P}_0)$. □

Remark For a given Petri net $\mathcal{P} = (P,T,W,\mu)$, we called $N = (P,T,W)$ a net and defined the automorphism group of the net N, denoted by $\mathbf{Aut}(N)$ in [3]. It is obvious that $\mathbf{Aut}(N)$ coincides with $\mathbf{Aut_1}(P,T,W,0^P)$.

4. Conclusions

In this paper we introduce Petri net morphisms/automorphism based on place connectivity and investigate the properties related to them. We first investigate some inclusion relation among monoids of morphisms and groups of automorphisms of given Petri nets and next show that the pre-order induced by surjective morphisms satisfies the two diamond properties. Finally we show that for two Petri nets ordered by a surjective morphism, the languages generated by them and their reachability sets have close correspondence.

The correspondence between the structure of a Petri net and the structure of the group of of Petri net automorphims still remains. We wonder whether the Petri nets with a same irreducible form constitute a lattice with respect to the order or not. In addition to these problems, we will apply this idea to the code theory, the language theorey and computation theory and so on.

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