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Maximal Centralizing Monoids and Minimal Clones

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Abstract

Let A be a non-empty set. For a set F of (multi-variable) functions on A the centralizer of F is the set of functions which commute with every member of F. A centralizing monoid on A is the set of unary functions of some centralizer. Equivalently, it is the set of endomorphisms of some algebra. Even for a small base set A it is known to be hard to determine explicitly such centralizing monoids .

In this paper we focus on maximal centralizing monoids and report all maximal centralizing monoids on a three-element set A. The result suggests that, in general, maximal centralizing monoids are strongly related to special kinds of minimal clones.

Keywords: clone; centralizer; centralizing monoid

1 Introduction

Commutation is defined for multi-variable functions as a generalization of commutation for unary functions. For a set F of multi-variable functions the centralizer F^* of F is the set of functions which commute with all members in F.

The main object of this paper is a centralizing monoid[1] which is the unary part of a centralizer. A set of functions which we call a witness determines a centralizing monoid. We focus on maximal centralizing monoids and present relationship between maximal centralizing monoids and special kinds of minimal clones on a three-element set A. Unary constant functions, which necessarily generate minimal clones, and majority functions generating minimal clones are shown to have strong connection with maximal centralizing monoids on a three-element set.

In Section 2 basic definitions and properties concerning commutation and centralizers are given. In Section 3 our fundamental result connecting maximal centralizing monoids and minimal clones is presented. It is followed by Section 4 where the complete list of maximal centralizing monoids on a three-element set is given. Finally, in Section 5, we give a general remark that all constant functions correspond, as witnesses, to maximal centralizing monoids.

This paper is a report from the author's joint work with Ivo G. Rosenberg, mainly from [MR 11] as well as from [MR 09] and [MR 10].

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[1] In our previous papers a centralizing monoid was called an endoprimal monoid. In order to avoid confusion we use the term *centralizing monoid* in this paper.

2 Basic Definitions and Properties

Let A be a non-empty finite set. For an integer n (> 0) denote by $\mathcal{O}_A^{(n)}$ the set of all n-ary functions defined over A, i.e., maps from A^n into A. Let $\mathcal{O}_A$ be the set of all functions defined over A, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. A function $e_i^n \in \mathcal{O}_A^{(n)}$ for $1 \le i \le n$ is the i-th n-ary *projection* which is defined by $e_i^n(a_1, \ldots, a_i, \ldots, a_n) = a_i$ for every $(a_1, \ldots, a_n) \in A^n$. Denote by $\mathcal{J}_A$ the set of all projections defined on A.

2.1 Commutation and Centralizer

For functions $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that f *commutes* with g, or f and g *commute*, if

$$f(g({}^t\boldsymbol{c}_1), \ldots, g({}^t\boldsymbol{c}_n)) \;=\; g(f(\boldsymbol{r}_1), \ldots, f(\boldsymbol{r}_m))$$

holds for every $m \times n$ matrix M over A with rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$ and columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$. We write $f \perp g$ when f commutes with g. The binary relation $\perp$ on $\mathcal{O}_A$ is symmetric.

For $m = n = 1$, $f \perp g$ means that $f(g(x)) = g(f(x))$ for every $x \in A$, which is an ordinary commutation for unary functions.

Definition 2.1 *For $F \subseteq \mathcal{O}_A$, the* centralizer F^* *of F is the set of all functions which commute with every member of F, that is,*

$$F^* \;=\; \{\, g \in \mathcal{O}_A \mid g \perp f \ \text{ for all } \ f \in F \,\}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is a clone. When $F = \{f\}$ we often write f^* for F^*. Also, we write F^{**} for $(F^*)^*$. The map $F \mapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

Lemma 2.1 *For any F, $G \subseteq \mathcal{O}_A$ the following holds:*

(1) $F \subseteq F^{**}$

(2) $F \subseteq G \implies F^* \supseteq G^*$

(3) $F^{***} = F^*$

2.2 Centralizing Monoid

A subset M of $\mathcal{O}_A^{(1)}$ is a *monoid* (*transformation monoid*) if it is closed with respect to composition and contains the identity. A centralizing monoid is a monoid satisfying the following property.

Definition 2.2 *For $M \subseteq \mathcal{O}_A^{(1)}$, M is a* centralizing monoid *if M satisfies*

$$M \;=\; M^{**} \cap \mathcal{O}_A^{(1)}.$$

In other words, a centralizing monoid is the unary part of some centralizer as the next lemma shows.

Lemma 2.2 *For any $M \subseteq \mathcal{O}_A^{(1)}$, M is s centralizing monoid if and only if*

$$M \;=\; F^* \cap \mathcal{O}_A^{(1)}$$

for some $F \subseteq \mathcal{O}_A$.

Proof. *Only-if*-part is trivial. In order to prove *if*-part, let M satisfy $M = F^* \cap \mathcal{O}_A^{(1)}$ for some $F \subseteq \mathcal{O}_A$. It follows that $M \subseteq F^*$. Applying $*$-operator twice to both sides and appealing to Lemma 2.1 (2), (3) we have $M^{**} \subseteq F^*$, from which we get $M^{**} \cap \mathcal{O}_A^{(1)} \subseteq F^* \cap \mathcal{O}_A^{(1)} = M$. On the other hand, we have $M \subseteq M^{**}$ by (1) of Lemma 2.1. Taking the intersection with $\mathcal{O}_A^{(1)}$ we obtain $M = M \cap \mathcal{O}_A^{(1)} \subseteq M^{**} \cap \mathcal{O}_A^{(1)}$. Hence, we have $M = M^{**} \cap \mathcal{O}_A^{(1)}$ as desired. □

A particular case where $m = 1$ in the preceding definition gives the commutation in a familiar form, that is, $f \perp g$ for $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(1)}$ means that

$$f(g(a_1), \ldots, g(a_n)) = g(f(a_1, \ldots, a_n))$$

holds for every $(a_1, \ldots, a_n) \in A^n$.

For an algebra $\mathcal{A} = (A; F)$ and a map $\varphi : A \longrightarrow A$, i.e., $\varphi \in \mathcal{O}_A^{(1)}$, φ is an *endomorphism* of $\mathcal{A}$ if

$$f(\varphi(x_1), \ldots, \varphi(x_n)) = \varphi(f(x_1, \ldots, x_n))$$

holds for every $f \in F$ and all $(x_1, \ldots, x_n) \in A^n$. In other words, φ is an endomorphism of $\mathcal{A} = (A; F)$ if and only if $\varphi \perp f$ for all $f \in F$, i.e., $\varphi \in F^*$. This means that a centralizing monoid is the set of all endomorphisms of some algebra.

2.3 Witness Lemma

The *if*-part of Lemma 2.2 may be expressed in the following way. We call it the *Witness Lemma.*

Lemma 2.3 *(Witness Lemma) Let S be a non-empty subset of $\mathcal{O}_A$. For a monoid $M \subseteq \mathcal{O}_A^{(1)}$, suppose the following conditions (i) and (ii) hold:*

(i) *For any $f \in M$ and any $u \in S$, f and u commute, i.e., $f \perp u$.*

(ii) *For any $g \in \mathcal{O}_A^{(1)} \setminus M$ there exists $w \in S$ such that g does not commute with w, i.e., $g \not\perp w$.*

Then M is a centralizing monoid.

A subset S in the lemma will be called a *witness* for a centralizing monoid M. We denote by $M(S)$ the centralizing monoid M with S as its witness. (i.e., $M(S) = S^* \cap \mathcal{O}_A^{(1)}$.) In particular, when $S = \{f\}$ we write $M(f)$ instead of $M(\{f\})$.

The following basic properties are easy to prove. (Recall that we assumed that the basic set A is finite.)

Proposition 2.4 *(1) Every centralizing monoid M has a witness.*

(2) Moreover, every centralizing monoid M has a finite *witness, that is, for every centralizing monoid M there exists a finite subset of $\mathcal{O}_A$ which is a witness of M.*

3 Maximal Centralizing Monoid and Minimal Clone

The purpose of this and the following sections is to study the "top" part of the lattice of centralizing monoids.

Definition 3.1 *A centralizing monoid M is* maximal *if $\mathcal{O}_A^{(1)}$ is the only centralizing monoid properly containing M.*

Proposition 3.1 *Every maximal centralizing monoid has a* singleton *witness, that is, for every maximal centralizing monoid M there exists $u\,(\in \mathcal{O}_A)$ such that $M = M(u)$.*

For the proof the reader is referred to [MR 11].

Now, maximal centralizing monoids are strongly related to minimal functions.

Definition 3.2 *A function f $(\in \mathcal{O}_A)$ is called a* minimal function *if*

(i) *f generates a minimal clone C, and*

(ii) *f has the minimum arity among functions generating C.*

Theorem 3.2 *Every maximal centralizing monoid has a witness which is a* minimal *function, that is, for any maximal centralizing monoid M there exists a minimal function $f\,(\in \mathcal{O}_A)$ such that*

$$M = M(f).$$

Proof By Proposition 3.1, a maximal centralizing monoid has a singleton witness, that is, there exists $g \in \mathcal{O}_A$ such that $M = M(g)$. It is well-known that over a finite base set A every non-trivial clone C (i.e., $C \neq \mathcal{J}_A$) contains a minimal clone. Hence, there exists $f \in \mathcal{O}_A$ which satisfies the following.

(i) $\langle f \rangle$ is a minimal clone.

(ii) $\langle f \rangle \subseteq \langle g \rangle \quad (\Leftrightarrow\ f \in \langle g \rangle)$

It is easy to see by the definition of commutation that, in general, two conditions $u \in \langle v \rangle$ and $v \perp w$ imply $u \perp w$ for any u, v, $w \in \mathcal{O}_A$. Equivalently, $u \in \langle v \rangle$ implies $v^* \subseteq u^*$ for any u, $v \in \mathcal{O}_A$. Hence, for f and g given above, it holds that $g^* \subseteq f^*$. Taking the unary part we get

$$M(g) = g^* \cap \mathcal{O}_A^{(1)} \subseteq f^* \cap \mathcal{O}_A^{(1)} = M(f)$$

Since $M(g)$ is a maximal centralizing monoid, by assumption, it holds either

$$M(g) = M(f) \quad (\text{i.e., } M = M(f))$$

or

$$M(f) = \mathcal{O}_A^{(1)}.$$

However, in [MR 05], we proved $(S_A \cup \mathrm{CONST})^* = \mathcal{J}_A$ for any A with $|A| > 2$, where S_A is the symmetric group on A and CONST is the set of unary constant functions on A. (Note: $\mathcal{J}_A$ is the clone of projections.) This implies $(\mathcal{O}_A^{(1)})^* = \mathcal{J}_A$. Therefore, $M(f) = \mathcal{O}_A^{(1)}$ cannot happen for a minimal function f, and so we get

$$M = M(f).$$

This completes the proof. □

4 The Case on a Three-Element Set

In this section we present all maximal centralizing monoids on a three-element set A, namely, on $A = \{0, 1, 2\}$. In the sequel, we write $E_3 = \{0, 1, 2\}$.

4.1 Minimal Clones on E_3

The complete list of minimal clones on E_3 was given by B. Csákány (1983).

Proposition 4.1 ([Cs 83]) *On E_3 there are 84 minimal clones. The number of minimal clones generated by each of five types of minimal functions is as follows:*

Unary functions	:	13	(4)
Binary idempotent functions	:	48	(12)
Ternary majority functions	:	7	(3)
Ternary semiprojections	:	16	(5)

The numbers in the parentheses indicate the number of conjugate classes. (Note that the fifth type of minimal functions does not appear on E_3.)

For each minimal function $f \in \mathcal{O}_3^{(1)}$, let $\{f\}$ be a witness and construct a centralizing monoid $M(f)$. Then some of such centralizing monoids are maximal while some are not.

4.2 Minimal Functions corresponding to Maximal Centralizing Monoids

We have explicitly determined all centralizing monoids on E_3 which have minimal functions as their witnesses. We shall not present the full list of them here due to the lack of space. The complete list will appear elsewhere.

By inspecting all centralizing monoids having minimal functions as their witnesses, we have determined all maximal centralizing monoids on E_3.

Proposition 4.2 *On the three-element set E_3, there are 10 maximal centralizing monoids. More precisely, there are 3 maximal centralizing monoids having unary constant functions as their witnesses and 7 maximal centralizing monoids having ternary majority functions which generate minimal clones as their witnesses.*

Recall that there are exactly 7 minimal clones generated by ternary majority functions. Hence every minimal clone generated by a ternary majority function corresponds to a maximal centralizing monoid.

Furthermore, it should be noted that some pairs of minimal functions serve as witnesses of the same maximal centralizing monoid.

The following is the complete list of minimal functions on E_3 which give maximal centralizing monoids as their witnesses.

(I) **Unary constant functions**

$$c_i(x) = i \quad \text{for any } x \in E_3 \qquad (i = 0, 1, 2)$$

(II) **Binary idempotent function**

$$b_{624}(x, y) = \begin{cases} x & \text{if } x = y \\ z & \text{if } |\{x, y, z\}| = 3 \end{cases}$$

(III) **Ternary majority functions** (showing the values only for mutually distinct x, y and z.)

$$\begin{aligned} m_0\,(x, y, z) &= 0 \quad \text{if } |\{x, y, z\}| = 3 \\ m_{364}(x, y, z) &= 1 \quad \text{if } |\{x, y, z\}| = 3 \\ m_{728}(x, y, z) &= 2 \quad \text{if } |\{x, y, z\}| = 3 \end{aligned}$$

$$m_{109}(x,y,z) = \begin{cases} 0 & \text{if } (x,y,z) \in \sigma \\ 1 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{473}(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z) \in \sigma \\ 2 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{510}(x,y,z) = \begin{cases} 2 & \text{if } (x,y,z) \in \sigma \\ 0 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{624}(x,y,z) = y \quad \text{if } |\{x,y,z\}| = 3$$

Here the sets σ and τ of triples are defined as follows:

$$\begin{aligned} \sigma &= \{(0,1,2),(1,2,0),(2,0,1)\} \\ \tau &= \{(0,2,1),(1,0,2),(2,1,0)\} \end{aligned}$$

(IV) **Ternary semiprojections**

$$p_{76}(x,y,z) = \begin{cases} z & \text{if } |\{x,y,z\}| \le 2 \\ 0 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 0 \\ 2 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 1 \\ 1 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 2 \end{cases}$$

$$p_{332}(x,y,z) = \begin{cases} z & \text{if } |\{x,y,z\}| \le 2 \\ 1 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 0 \\ 0 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 1 \\ 2 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 2 \end{cases}$$

$$p_{684}(x,y,z) = \begin{cases} z & \text{if } |\{x,y,z\}| \le 2 \\ 2 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 0 \\ 1 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 1 \\ 0 & \text{if } |\{x,y,z\}| = 3 \text{ and } z = 2 \end{cases}$$

$$p_{424}(x,y,z) = \begin{cases} z & \text{if } |\{x,y,z\}| \le 2 \\ x & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

We present the list of all maximal centralizing monoids on E_3. Recall that $M(f)$ means the centralizing monoid having f as its witness. The set of unary functions is given in Table 1.

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5
0	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1
1	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2
2	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

	s_1	s_2	s_3	s_4	s_5	s_6
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	2	1	2	0	1	0

Table 1: Unary Functions in $\mathcal{O}_3^{(1)}$

Proposition 4.3 *There are* 10 *maximal centralizing monoids on* E_3*, which are explicitly given as follows:*

$$\begin{aligned}
M\,(c_0) &= \{s_1, s_2\} \cup \{j_1, j_2, j_5, u_1, u_2, u_5\} \cup \{c_0\} \\
M\,(c_1) &= \{s_1, s_6\} \cup \{j_1, j_3, j_5, v_0, v_2, v_4\} \cup \{c_1\} \\
M\,(c_2) &= \{s_1, s_3\} \cup \{u_2, u_4, u_5, v_2, v_4, v_5\} \cup \{c_2\} \\
M\,(m_0) &= \{s_1, s_2\} \cup \{j_1, j_2, j_3, j_4, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\} \cup \{c_0, c_1, c_2\} \\
M(m_{364}) &= \{s_1, s_6\} \cup \{j_0, j_2, j_3, j_5, u_0, u_2, u_3, u_5, v_0, v_2, v_3, v_5\} \cup \{c_0, c_1, c_2\} \\
M(m_{728}) &= \{s_1, s_3\} \cup \{j_0, j_1, j_4, j_5, u_0, u_1, u_4, u_5, v_0, v_1, v_4, v_5\} \cup \{c_0, c_1, c_2\} \\
M(m_{109}) &= \{s_1, s_3\} \cup \{j_2, j_3, u_2, u_3, v_2, v_3\} \cup \{c_0, c_1, c_2\} \\
M(m_{473}) &= \{s_1, s_2\} \cup \{j_0, j_5, u_0, u_5, v_0, v_5\} \cup \{c_0, c_1, c_2\} \\
M(m_{510}) &= \{s_1, s_6\} \cup \{j_1, j_4, u_1, u_4, v_1, v_4\} \cup \{c_0, c_1, c_2\} \\
M(m_{624}) &= \{s_1, s_2, s_3, s_4, s_5, s_6\} \cup \{c_0, c_1, c_2\}
\end{aligned}$$

Remark. Four of the maximal centralizing monoids given above, namely, $M(m_{109})$, $M(m_{473})$, $M(m_{510})$ and $M(m_{624})$, have other minimal functions as their witnesses.

$$\begin{aligned}
M(m_{109}) &= M(p_{332}) \\
M(m_{473}) &= M(p_{76}) \\
M(m_{510}) &= M(p_{684}) \\
M(m_{624}) &= M(b_{624}) = M(p_{424})
\end{aligned}$$

To summarize, (i) there are 10 maximal centralizing monoids on E_3, and (ii) there are 15 minimal clones on E_3 whose generators (minimal functions) serve as witnesses of maximal centralizing monoids.

5 Constant Witness

In the previous section we have observed that, on a three-element set, every unary constant function, which is necessarily a minimal function, and every ternary majority *and* minimal function serves as a witness for a maximal centralizing monoid. For unary constant functions this phenomenon can be extended to the general case, i.e., the case for any finite set A with $|A| > 2$.

Theorem 5.1 *Let* $k > 2$ *be an integer. For any constant function* c *on* E_k*,* $M(c)$ *is a maximal centralizing monoid.*

Here we present only a rough sketch of the proof. For the details of the proof the reader is referred to [MR 11].

Sketch of the Proof. Without loss of generality, we assume $c = c_0$ (= unary constant function taking value 0). The proof will come after a series of lemmas.

Lemma 1 $M(c_0) = (\mathrm{Pol}(0))^{(1)}$

Lemma 2 $(\mathrm{CONST})^* = \mathrm{IDEMP}$

Lemma 3 For $f \in \mathcal{O}_k$, if $f \in (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}$ then f is conservative.

Note: f is *conservative* if $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for all $a_1, \ldots, a_n \in E_k$.

Lemma 4 $(\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP} = \mathcal{J}_k$

Lemmas 1 and 2 are trivial. Lemma 4 is the main part of the whole proof. It is worth mentioning that for the proof of Lemma 4 we need three or more elements in the base set E_k.

The proof of Theorem 5.1 proceeds as follows.
For any $u \in \mathcal{O}_k^{(1)} \setminus M(c_0)$ let M be a monoid such that

$$M \supsetneq M(c_0) \cup \{u\}.$$

Since $M(c_0) = \mathrm{Pol}(0)^{(1)}$, the function u maps 0 to some element in E_k other than 0. Then it is easy to see that M must contain all constant functions. Hence we have

$$M \supset M(c_0) \cup \mathrm{CONST}.$$

It follows that

$$M^* \subseteq M(c_0)^* \cap \mathrm{CONST}^* .$$

which implies, by Lemmas 1 and 2, that

$$M^* \subseteq (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}.$$

Since M^* is a clone and contains $\mathcal{J}_k$ it follows by Lemma 4 that

$$M^* = \mathcal{J}_k .$$

By applying $*$-operator to both sides, we obtain

$$M^{**} = \mathcal{J}_k^* \,(= \mathcal{O}_k).$$

It follows that

$$M^{**} \cap \mathcal{O}_k^{(1)} = \mathcal{O}_k^{(1)}.$$

Therefore, if M is a centralizing monoid then, by definition,

$$M \,(= M^{**} \cap \mathcal{O}_k^{(1)}) = \mathcal{O}_k^{(1)}.$$

This concludes that $M(c_0)$ is a maximal centralizing monoid. □

6 To conclude

In the previous section we asserted that every constant function serves as a witness of some maximal centralizing monoid. We present two other general open problems, both of which were verified to hold for the case of the three-element set E_3.

Q1 Let $k \geq 3$ be arbitrary. Does every ternary majority minimal function on E_k serve as a witness of some maximal centralizing monoid ?

Q2 Let $k \geq 3$ be arbitrary. Does every maximal centralizing monoid on E_k have either a constant function or a ternary majority function as its witness ?

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