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THE ORBIT DECOMPOSITION AND ORBIT TYPE OF THE AUTOMORPHISM GROUP OF CERTAIN EXCEPTIONAL JORDAN ALGEBRA AND ITS APPLICATIONS

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ABSTRACT. Let $\mathcal{J}^1$ be the real form of complex simple Jordan algebra with the automorphism group $F_{4(-20)}$. The classification of $F_{4(-20)}$-orbits and the stabilizer groups of $F_{4(-20)}$-orbit on $\mathcal{J}^1$ are determined. As applications, for $F_{4(-20)}$, the Bruhat and Gauss decomposition, the Iwasawa decomposition and also the Iwasawa decomposition with respect to K_ϵ in sense of T. Oshima and J. Sekiguchi are given concretely.

1. THE EXCEPTIONAL JORDAN ALGEBRA $\mathcal{J}^1$ AND THE AUTOMORPHISM GROUP $F_{4(-20)}$.

Denote the cartesian n-power of a set X as $X^n := X \times \cdots \times X$ (n times). For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let V be a $\mathbb{F}$-linear space, $\mathrm{GL}_{\mathbb{F}}(V)$ the group of $\mathbb{F}$-linear automorphism of V, and $\mathrm{End}_{\mathbb{F}}(V)$ the linear space of $\mathbb{F}$-linear endomorphisms on V. A subset C is said to be a *cone* if $x \in V$ and $\lambda > 0$ imply that $\lambda x \in C$. For a mapping $f : V \to V$ and $c \in \mathbb{F}$, put $V_{f,c} := \{v \in V \mid f(v) = cv\}$ and $V_f := V_{f,1}$. Let G be a subgroup of $\mathrm{GL}_{\mathbb{F}}(V)$, ϕ an automorphism on G and $v, v_i \in V$. Then denote the subgroups $G^\phi := \{g \in G | \ \phi g = g\}$, the *stabilizer* of v as $G_v := \{g \in G \mid gv = v\}$ and $G_{v_1,\cdots,v_n} := \cap_{i=1}^n G_{v_i}$. And denote the *G-orbit* of v as $Orb_G(v) := \{gv \mid g \in G\}$.

For $\mathbf{H}$ (Quaternions), the $\mathbf{O}$ (*Octonions*) is defined as $\mathbf{O} := \mathbf{H} \oplus \mathbf{H}e = \{m + ae | \ m, a \in \mathbf{H}\}$, the *conjugation*, the *multiplication*, the *inner product* and the *quadratic form* as $\overline{m + ae} := \overline{m} - ae$, $(m+ae)(n+be) := (mn - \bar{b}a) + (a\overline{n} + bm)e$ (especially, $e^2 = -1$), $(m + ae|n + be) := (m|n) + (a|b)$ and $\mathrm{n}(x) := (x|x)$, respectively. For $x \in \mathbf{O}$, the *scalar part* and the *vector part* of x and the set $\mathrm{Im}\mathbf{O}$ are defined by $\mathrm{Re}(x) := \frac{1}{2}(x + \overline{x})$, $\mathrm{Im}(x) := \frac{1}{2}(x - \overline{x})$ and $\mathrm{Im}\mathbf{O} := \{x \in \mathbf{O} \mid \overline{x} = -x\}$, respectively.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbf{O}^3$, denote

$$h^1(\xi; x) := \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}\overline{x_2} \\ \sqrt{-1}\overline{x_3} & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

and

$$\mathcal{J}^1 := \{h^1(\xi;x) \mid \xi \in \mathbb{R}^3,\ x \in \mathbf{O}^3\}.$$

The *Jordan product* is defined by

$$X \circ Y := \frac{1}{2}(XY + YX) \quad \text{for } X, Y \in \mathcal{J}^1.$$

Then the identity element of the Jordan product is $E := diag(1,1,1)$. For $X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)$ and $Y = h^1(\eta_1, \eta_2, \eta_3; y_1, y_2, y_3) \in \mathcal{J}^1$, the *trace* and the *inner product* are defined as

$$\begin{aligned} &\mathrm{tr}(X) = \xi_1 + \xi_2 + \xi_3, \\ &(X|Y) = \mathrm{tr}(X \circ Y) = (\sum_{k=1}^{3} \xi_k \eta_k) + 2(x_1|y_1) - 2(x_2|y_2) - 2(x_3|y_3), \end{aligned}$$

respectively, the *cross product* and the *determinant* as

$$\begin{aligned} X \times Y &:= \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y))E), \\ \det(X) &:= \frac{1}{3}(X|X \times X) \\ &= \xi_1\xi_2\xi_3 + 2\mathrm{Re}((x_1x_2)x_3) - \xi_1(x_1|x_1) + \xi_2(x_2|x_2) + \xi_3(x_3|x_3) \end{aligned}$$

respectively. Hereafter we denote $X \times X := X^{\times 2}$. The *characteristic polynomial* $\Phi_X(\lambda)$ of $X \in \mathcal{J}^1$ is defined by

$$\begin{aligned} \Phi_X(\lambda) :=& \det(\lambda E - X) = \frac{1}{3}(\lambda E - X|(\lambda E - X)^{\times 2}) \\ =& \lambda^3 - \mathrm{tr}(X)\lambda^2 + \mathrm{tr}(X^{\times 2})\lambda - \det(X). \end{aligned}$$

For $i \in \{1,2,3\}$ and $x \in \mathbf{O}$, denote

$$\begin{aligned} E_i &:= h^1(\delta_{i1}, \delta_{i2}, \delta_{i3}; 0,0,0), & F_i^1(x) &:= h^1(0,0,0; \delta_{i1}x, \delta_{i2}x, \delta_{i3}x), \\ P^+ &:= h^1(1,-1,0;0,0,1), & P^- &:= h^1(-1,1,0;0,0,1), \\ Q^+(x) &:= h^1(0,0,0;x,\overline{x},0), & Q^-(x) &:= h^1(0,0,0;x,-\overline{x},0) \end{aligned}$$

where δ_{ij} is the Kronecker's delta. Then $X \in \mathcal{J}^1$ can be expressed by

$$X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))$$

for some $\xi_i \in \mathbb{R}$ and $x_i \in \mathbf{O}$, and denote

$$(X)_{E_i} := \xi_i = (X|E_i), \quad (X)_{F_i^1} := x_i.$$

Lemma 1.1. (cf. [25, Lemma 1.6 with $\mathcal{J}^1 \subset \mathcal{J}^{\mathbb{C}}$]) *For all* $X \in \mathcal{J}^1$,

$$(X^{\times 2})^{\times 2} = \det(X)X.$$

The linear Lie group $F_{4(-20)}$ is defined by

$$F_{4(-20)} := Aut(\mathcal{J}^1) = \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \circ Y) = gX \circ gY\}.$$

The following result is proved after [34, 35], [39, Lemma 2.1.2, Proposition 2.1.3] and [33, p.159, Proposition 5.9.4, §5.10].

Proposition 1.2. (cf. [24, Theorem 1.4], [25, Proposition 0.1(1)])

$$\begin{aligned} F_{4(-20)} &= \{g \in F_{4(-20)} \mid \mathrm{tr}(gX) = \mathrm{tr}(X)\} \\ &= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X),\ gE = E\} \\ &= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} \\ &= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X),\ (gX|gY) = (X|Y)\} \\ &= \{g \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \times Y) = gX \times gY\}. \end{aligned}$$

A *characteristic root* of $X \in \mathcal{J}^1$ is said to be a solution of $\Phi_X(\lambda) = 0$ over $\mathbb{C}$. By Proposition 1.2, the trace, the inner product, the determinant, the identity element, the cross product and the characteristic polynomial are invariant under the action of $F_{4(-20)}$. Moreover the set of all characteristic roots and those multiplicities are invariant under the action of $F_{4(-20)}$.

Proposition 1.3. ([39]) *$F_{4(-20)}$ is a connected and simply connected non-compact simple real Lie group of type $\mathbf{F}_{4(-20)}$.*

2. The orbit decomposition of $F_{4(-20)}$-orbits on $\mathcal{J}^1$.

The subset $\mathcal{H} \subset \mathcal{J}^1$ and the *Cayley hyperbolic planes* $\mathcal{H}(\mathbf{O})$ and $\mathcal{H}'(\mathbf{O})$ of $\mathcal{J}^1$ are defined as

$$\begin{aligned} \mathcal{H} &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0,\ \mathrm{tr}(X) = 1\}, \\ \mathcal{H}(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0,\ \mathrm{tr}(X) = 1,\ (X|E_1) \geq 1\}, \\ \mathcal{H}'(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0,\ \mathrm{tr}(X) = 1,\ (X|E_1) \leq 0\} \end{aligned}$$

respectively.

Proposition 2.1. (cf. [24, Propositions 1.6(1) and 2.10])

(1) $\mathcal{H} = \mathcal{H}(\mathbf{O}) \coprod \mathcal{H}'(\mathbf{O})$.

(2) $\mathcal{H}(\mathbf{O}) = Orb_{F_{4(-20)}}(E_1)$.

(3) $\mathcal{H}'(\mathbf{O}) = Orb_{F_{4(-20)}}(E_2) = Orb_{F_{4(-20)}}(E_3)$.

The cone $\mathcal{N}$ of $\mathcal{J}^1$ is defined by

$$\mathcal{N} = \{X \in \mathcal{J}^1 | \ \mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0\}.$$

Then using Lemma 1.1, $\mathcal{N}$ contains the following cones:

$$\mathcal{N}_1(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ X \neq 0\},$$
$$\mathcal{N}_1^+(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ (X|E_1) > 0\},$$
$$\mathcal{N}_1^-(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 0,\ (X|E_1) < 0\},$$
$$\mathcal{N}_2(\mathbf{O}) := \{X \in \mathcal{J}^1 |\ \mathrm{tr}(X) = \mathrm{tr}(X^{\times 2}) = \det(X) = 0,\ X^{\times 2} \neq 0\},$$
$$\mathcal{N}_0(\mathbf{O}) := \{0\}.$$

Proposition 2.2. (cf . [24, Propositions 1.6(2), 2.10(2) and 4.3(4)])
(1) $\mathcal{N}_1(\mathbf{O}) = \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O})$.
(2) $\mathcal{N} = \mathcal{N}_0(\mathbf{O}) \coprod \mathcal{N}_1^+(\mathbf{O}) \coprod \mathcal{N}_1^-(\mathbf{O}) \coprod \mathcal{N}_2(\mathbf{O})$.
(3) $\mathcal{N}_1^+(\mathbf{O}) = Orb_{F_{4(-20)}}(P^+)$.
(4) $\mathcal{N}_1^-(\mathbf{O}) = Orb_{F_{4(-20)}}(P^-)$.
(5) $\mathcal{N}_2(\mathbf{O}) = Orb_{F_{4(-20)}}(Q^+(1))$.

For $X \in \mathcal{J}^1$, denote $L^\times(X) \in \mathrm{End}_{\mathbf{R}}(\mathcal{J}^1)$ as

$$L^\times(X)Y := X \times Y \quad \text{for } Y \in \mathcal{J}^1$$

and the *minimal space* of X as

$$V_X := \{aX^{\times 2} + bX + cE \mid a, b, c \in \mathbb{R}\}.$$

Then V_X is closed under the cross product ([25, Lemma 1.6(3)]). And for $\lambda_0 \in \mathbb{R}$, denote the elements $p(X),\ E_{X,\lambda_0},\ W_{X,\lambda_0} \in V_X$ as

$$\begin{aligned} p(X) &:= X - \frac{1}{3}\mathrm{tr}(X)E, \\ E_{X,\lambda_0} &:= \frac{1}{\mathrm{tr}((\lambda_0 E - X)^{\times 2})}(\lambda_0 E - X)^{\times 2}, \\ W_{X,\lambda_0} &:= X - (\lambda_0 E_{X,\lambda_0} + \frac{\mathrm{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0})) \end{aligned}$$

respectively. If E_{X,λ_1} is well-defined (*ie*, $\mathrm{tr}((\lambda_1 E - X)^{\times 2}) \neq 0$), then

$$X = \lambda_0 E_{X,\lambda_0} + \frac{\mathrm{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0}) + W_{X,\lambda_0}.$$

For $r \in \mathbb{R}$, consider the eigenspace $\mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),r}$. Then we have the following two lemmas (cf. [24]):

Lemma 2.3. *Let $X \in \mathcal{J}^1$. Then for all $g \in F_{4(-20)}$,*

$$g(V_X) = V_{gX},\ gE_{X,\lambda_1} = E_{gX,\lambda_1},\ gW_{X,\lambda_1} = W_{gX,\lambda_1},\ gp(X) = p(gX).$$

Lemma 2.4. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1.
(1) E_{X,λ_1} *is well-defined (ie,* $\mathrm{tr}((\lambda_1 E - X)^{\times 2}) \neq 0$*), and* $E_{X,\lambda_1} \in \mathcal{H} \cap V_X$.

(2) $E_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),0}$, $E - E_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),1} \cap V_X$ *and* $W_{X,\lambda_1} \in \mathcal{J}^1_{L^\times(2E_{X,\lambda_1}),-1} \cap V_X$.

Main Theorem 1. (**$F_{4(-20)}$-orbits on $\mathcal{J}^1$** [24, Main Theorem]) *$F_{4(-20)}$-orbits on $\mathcal{J}^1$ are classified as follows.*

(I) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then there exists the unique $i \in \{1,2,3\}$ such that $\mathcal{H}(\mathbf{O}) \cap V_X = \{E_{X,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_X = \{E_{X,\lambda_{i+1}}, E_{X,\lambda_{i+2}}\}$ where i, $i+1$, $i+2$ are counted modulo 3. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.*

Cases	*The canonical forms of X*
1. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$
2. $E_{X,\lambda_2} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_3, \lambda_1)$
3. $E_{X,\lambda_3} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_3, \lambda_1, \lambda_2)$

(II) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then X can be transformed to the following canonical form by $F_{4(-20)}$.*

the characteristic roots of X	*The canonical form of X*
4. $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$	$\mathrm{diag}(p, p, \lambda_1) + F^1_3(q)$

(III) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then $W_{X,\lambda_1} \in \mathcal{N}_1(\mathbf{O}) \coprod \{0\}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.*

Cases	*The canonical form of X*
5. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$
6. $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$, $W_{X,\lambda_1} = 0$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1)$
7. $W_{X,\lambda_1} \in \mathcal{N}_1^+(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$
8. $W_{X,\lambda_1} \in \mathcal{N}_1^-(\mathbf{O})$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$

(IV) *Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then $p(X) \in \mathcal{N}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.*

Cases	*The canonical form of X*
9. $p(X) = 0$	$\frac{1}{3}\mathrm{tr}(X)E$
10. $p(X) \in \mathcal{N}_1^+(\mathbf{O})$	$\frac{1}{3}\mathrm{tr}(X)E + P^+$
11. $p(X) \in \mathcal{N}_1^-(\mathbf{O})$	$\frac{1}{3}\mathrm{tr}(X)E + P^-$
12. $p(X) \in \mathcal{N}_2(\mathbf{O})$	$\frac{1}{3}\mathrm{tr}(X)E + Q^+(1)$

(V) *By $F_{4(-20)}$, the above canonical forms cannot be transformed from each other.*

3. The stabilizer groups of Spin group type.

Let G be a topological group with identity element 1. Then G^0 denotes the identity connected component. Denote the quadratic form $Q_{p,q}$ on $\mathbb{R}^{p+q}$ as $Q_{p,q}(x) := -(x_1^2 + \cdots + x_p^2) + (x_{p+1}^2 + \cdots + x_{p+q}^2)$ for $x = (x_1, \cdots, x_{p+q})$, the quadratic space as $(\mathbb{R}^{p,q}, Q_{p,q})$, the set of all orthogonal transformations as $\mathrm{O}(\mathbb{R}^{p,q}, Q_{p,q})$ and $\mathrm{SO}(\mathbb{R}^{p,q}, Q_{p,q}) := \{g \in \mathrm{O}(\mathbb{R}^{p,q}, Q_{p,q}) | \det(g) = 1\}$ where $\det(g)$ is the determinant of $g \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{p,q})$. Then $\mathrm{O}(\mathbb{R}^{p,q}, Q_{p,q})$ and $\mathrm{SO}(\mathbb{R}^{p,q}, Q_{p,q})$ are linear Lie groups. Denote the quadratic form Q on $\mathcal{J}^1$ as $Q(X) := -\mathrm{tr}(X^{\times 2})$ for $X \in \mathcal{J}^1$ and consider the subspace $\mathcal{J}^1_{0,9}$, $\mathcal{J}^1_{8,1}$ and $\mathcal{J}^1_{7,1}$ of eigenspace of $L^\times(2E_i)$ with eigenvalue -1 as

$$\mathcal{J}^1_{0,9} := \mathcal{J}^1_{L^\times(2E_1),-1}, \quad \mathcal{J}^1_{8,1} := \mathcal{J}^1_{L^\times(2E_3),-1},$$
$$\mathcal{J}^1_{7,1} := \{X \in \mathcal{J}^1_{8,1} | \ (F_3^1(1)|X) = 0\}.$$

Then $\mathcal{J}^1_{0,9} = \{\xi(E_2 - E_3) + F_1^1(x) | \xi \in \mathbb{R}, x \in \mathbf{O}\}$, $\mathcal{J}^1_{8,1} = \{\xi(E_1 - E_2) + F_3^1(x) | \xi \in \mathbb{R}, x \in \mathbf{O}\}$ and $\mathcal{J}^1_{7,1} = \{\xi(E_1 - E_2) + F_3^1(x) | \xi \in \mathbb{R}, x \in \mathrm{Im}\mathbf{O}\}$. Since $Q(\xi(E_2 - E_3) + F_1^1(x)) = \xi^2 + \mathrm{n}(x)$ and $Q(\xi(E_1 - E_2) + F_3^1(x)) = \xi^2 - \mathrm{n}(x)$, we see that $(\mathcal{J}^1_{0,9}, Q)$, $(\mathcal{J}^1_{8,1}, Q)$ and $(\mathcal{J}^1_{7,1}, Q)$ are isomorphic to $(\mathbb{R}^9, |\cdot|^9)$, $(\mathbb{R}^{8,1}, Q_{8,1})$ and $(\mathbb{R}^{7,1}, Q_{7,1})$, respectively. Moreover, denote

$$\begin{aligned} S^8 &:= \{X \in \mathcal{J}^1_{0,9} \mid Q(X) = 1\}, \\ S^{8,1}_+ &:= \{X \in \mathcal{J}^1_{8,1} \mid Q(X) = 1, \ (E_3|X) > 0\}, \\ S^{7,1}_+ &:= \{X \in \mathcal{J}^1_{7,1} \mid Q(X) = 1, \ (E_3|X) > 0\}. \end{aligned}$$

From now on, the groups $\mathrm{SO}(8)$ and $\mathrm{SO}(7)$ are identified with the groups $\mathrm{SO}(8) = \{g \in GL_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y), \ \det(g) = 1\}$ and $\mathrm{SO}(7) = \{g \in \mathrm{SO}(8) \mid g1 = 1\}$, respectively. The subgroup $T(\mathbf{O})$ of $\mathrm{SO}(8)^3$ is defined as

$$T(\mathbf{O}) := \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 | (g_1x)(g_2y) = g_3(xy) \text{ for all } x, y \in \mathbf{O}\}$$

(cf. [2], [9, (2.4.6)], [22], [33], [43]), and the subgroup $\tilde{D}_4$ of $\mathrm{SO}(8)^3$ as

$$\tilde{D}_4 := \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 | (g_1x)(g_2y) = \overline{g_3(\overline{xy})} \text{ for all } x, y \in \mathbf{O}\}.$$

For $i \in \{1, 2, 3\}$, the homomorphism $p_i : \tilde{D}_4 \to \mathrm{SO}(8)$ is defined by

$$p_i(g_1, g_2, g_3) := g_i \quad \text{for } (g_1, g_2, g_3) \in \tilde{D}_4.$$

The subgroup $\tilde{B}_3$ of $\tilde{D}_4$ is defined as

$$\tilde{B}_3 := \{(g_1, g_2, g_3) \in \tilde{D}_4 \mid g_3 1 = 1\}$$

and the homomorphism $q : \tilde{B}_3 \to \mathrm{SO}(7)$ as $q := p_3|\tilde{B}_3$. Denote $\epsilon_i(j) := (-1)^{1+\delta_{ij}}$ where δ_{ij} is the Kronecker delta. Thus if $i = j$, then $\epsilon_i(j) = 1$, else $\epsilon_i(j) = -1$.

Lemma 3.1.
(1) ([43, Theorems 1.15.1 and 1.16.1]) $\tilde{D}_4$ *and* $\tilde{B}_3$ *are connected.*
(2) (The principle of triality: [2], [9, (2.4.6)], cf. [43, Theorem 1.14.2]) *The following sequence is exact:*

$$1 \to \{(1,1,1), (\epsilon_i(1), \epsilon_i(2), \epsilon_i(3))\} \to \tilde{D}_4 \xrightarrow{p_i} \mathrm{SO}(8) \to 1.$$

(3) ([43, Theorem 1.15.2]) *The following sequence is exact:*

$$1 \to \{(1,1,1), (-1,-1,1)\} \to \tilde{B}_3 \xrightarrow{q} \mathrm{SO}(7) \to 1.$$

By Lemma 3.1, we see that $\tilde{D}_4$ is connected and a two-hold covering group of SO(8), and $\tilde{B}_3$ is connected and a two-hold covering group of SO(7). So denote

$$\mathrm{Spin}(8) := \tilde{D}_4, \quad \mathrm{Spin}(7) := \tilde{B}_3.$$

Lemma 3.2. ([22], cf. [43, Theorem 2.7.1], [26, lemma 3.2]) *The following homomorphisms are group isomorphisms:*

(1) $\varphi_0 : \mathrm{Spin}(8) \to (F_{4(-20)})_{E_1,E_2,E_3}$;

$$\varphi_0(g_1, g_2, g_3)(\sum(\xi_i E_i + F_i^1(x_i))) = \sum(\xi_i E_i + F_i^1(g_i x_i)),$$

(2) $\varphi_0 : \mathrm{Spin}(7) \to (F_{4(-20)})_{E_1,E_2,F_3^1(1)}$; $\quad \varphi_0 = \varphi_0|\mathrm{Spin}(7)$.

Hereafter Spin(8) and Spin(7) are identified with $(F_{4(-20)})_{E_1,E_2,E_3}$ and $(F_{4(-20)})_{E_1,E_2,F_3^1(1)}$ via φ_0, respectively.

Lemma 3.3. ([38], [39], cf. [26, Lemmas 3.9 and 3.12])

(1) $(F_{4(-20)})_{E_1}/\mathrm{Spin}(8) \simeq S^8$, $\qquad$ (2) $(F_{4(-20)})_{E_3}/\mathrm{Spin}(8) \simeq S_+^{8,1}$,

(3) $(F_{4(-20)})_{F_3^1(1)}/\mathrm{Spin}(7) \simeq S_+^{7,1}$.

Furthermore, $(F_{4(-20)})_{E_1}$, $(F_{4(-20)})_{E_3}$ *and* $(F_{4(-20)})_{F_3^1(1)}$ *are connected.*

Lemma 3.4. ([38], [39], cf. [26, Lemmas 3.10 and 3.13])
(1) *The following sequence is exact.*

$$1 \to \mathbb{Z}_2 \to (F_{4(-20)})_{F_3^1(1)} \xrightarrow{f} \mathrm{O}^0(\mathcal{J}_{7,1}^1, Q) \to 1$$

where $f(g) = g|\mathcal{J}_{7,1}^1$.
(2) *The following sequence is exact.*

$$1 \to \mathbb{Z}_2 \to (F_{4(-20)})_{E_1} \xrightarrow{f} \mathrm{SO}(\mathcal{J}_{0,9}^1, Q) \to 1$$

where $f(g) = g|\mathcal{J}_{0,9}^1$.
(3) *The following sequence is exact.*

$$1 \to \mathbb{Z}_2 \to (F_{4(-20)})_{E_3} \xrightarrow{f} \mathrm{O}^0(\mathcal{J}_{8,1}^1, Q) \to 1$$

where $f(g) = g|\mathcal{J}_{8,1}^1$.

Since Lemmas 3.3, 3.4 and $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2 = \pi_1(\mathrm{O}^0(n,1))$ $(n \geq 3)$, we can put

$$\mathrm{Spin}^0(7,1) := (F_{4(-20)})_{F_3^1(1)}, \qquad \mathrm{Spin}(9) := (F_{4(-20)})_{E_1},$$
$$\mathrm{Spin}^0(8,1) := (F_{4(-20)})_{E_3} \cong (F_{4(-20)})_{E_2}.$$

The element $\sigma_i \in F_{4(-20)}$ is defined by

$$\sigma_i(\sum_{j=1}^{3}(\xi_j E_j + F_j^1(x_j)) := \sum_{j=1}^{3}(\xi_j E_j + \epsilon_i(j) F_j^1(x_j))$$

[38] (cf. [39]) where indices are counted modulo 3. The involutive automorphism $\tilde{\sigma}_i$ of $F_{4(-20)}$ is defined as

$$\tilde{\sigma}_i(g) := \sigma_i g \sigma_i \qquad \text{for } g \in F_{4(-20)},$$

and the subgroup K of $F_{4(-20)}$ as

$$K := (F_{4(-20)})^{\tilde{\sigma}_1} = \{g \in F_{4(-20)} \mid \sigma_1 g = g \sigma_1\}.$$

Proposition 3.5. ([38, Theorem 8],[39, Theorem 2.4.4], cf. [26, Proposition 3.16]).

(1) $(F_{4(-20)})^{\tilde{\sigma}_i} = (F_{4(-20)})_{E_i}$.

(2) $K = (F_{4(-20)})_{E_1} = \mathrm{Spin}(9)$.

(3) $(F_{4(-20)})^{\tilde{\sigma}_2} = (F_{4(-20)})_{E_2} \cong \mathrm{Spin}^0(8,1)$.

4. The stabilizer groups of semidirect product group type.

Denote the Lie algebras $\mathfrak{o}(8) = Lie(\mathrm{O}(8))$ and $\mathfrak{f}_{4(-20)} = Lie(F_{4(-20)})$. Since $\varphi_0 : D_4 \to (F_{4(-20)})_{E_1,E_2,E_3}$ is an isomorphism by Lemma 3.2, the Lie subalgebra $\mathfrak{d}_4$ of $\mathfrak{f}_{4(-20)}$ is defined by

$$\mathfrak{d}_4 := \left\{ d\varphi_0(D_1, D_2, D_3) \;\middle|\; \begin{array}{c} (D_1, D_2, D_3) \in \mathfrak{o}(8)^3, \\ (D_1 x)y + x(D_2 y) = \overline{D_3(\overline{xy})} \\ \text{for all } x, y \in \mathbf{O} \end{array} \right\}.$$

Then

$$d\varphi_0(D_1, D_2, D_3)(\sum(\xi_i E_i + F_i^1(x_i))) = \sum F_i^1(D_i x_i).$$

For $a \in \mathbf{O}$, denote

$$A_1^1(a) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}, \quad A_2^1(a) := \begin{pmatrix} 0 & 0 & \sqrt{-1}\overline{a} \\ 0 & 0 & 0 \\ -\sqrt{-1}a & 0 & 0 \end{pmatrix},$$
$$A_3^1(a) := \begin{pmatrix} 0 & \sqrt{-1}a & 0 \\ -\sqrt{-1}\overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\tilde{A}_i^1(a) \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ is defined as

$$\tilde{A}_i^1(a) := [A_i^1(a), X] \quad \text{for } X \in \mathcal{J}^1$$

and the subspaces $\mathfrak{u}_i^1$ of $\mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ as $\mathfrak{u}_i^1 := \{\tilde{A}_i^1(a) \mid a \in \mathbf{O}\}$. The differential $d\tilde{\sigma}_i$ of the involutive automorphism $\tilde{\sigma}_i$ is written by same letter $\tilde{\sigma}_i$. Then $\tilde{\sigma}_i(\phi) = \sigma_i \phi \sigma_i$ for $\phi \in \mathfrak{f}_{4(-20)}$.

Lemma 4.1.
(1) ([9], cf. [24, Proposition 2.1]) $\mathfrak{f}_{4(-20)} = \mathfrak{d}_4 \oplus \mathfrak{u}_1^1 \oplus \mathfrak{u}_2^1 \oplus \mathfrak{u}_3^1$.
(2) ([43], cf.[26, Lemma 4.2]) *$\tilde{\sigma}_1$ is a Cartan involution.*
(3) *If $\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition with respect to $\tilde{\sigma}_1$, then*

$$\mathfrak{k} = \mathfrak{d}_4 \oplus \mathfrak{u}_1^1, \quad \mathfrak{p} = \mathfrak{u}_2^1 \oplus \mathfrak{u}_3^1.$$

Now $\tilde{A}_3^1(1) \in \mathfrak{p}$. Let us define the abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, the 1-parameter subgroup A, and $\alpha \in \mathfrak{a}^*$ as

$$\mathfrak{a} := \{t\tilde{A}_3^1(1) |\ t \in \mathbb{R}\}, \quad A := \{\exp(t\tilde{A}_3^1(1)) |\ t \in \mathbb{R}\}, \quad \alpha(\tilde{A}_3^1(1)) := 1$$

respectively. Denote

$$\begin{aligned} \mathfrak{g}_\lambda &:= \{\phi \in \mathfrak{f}_{4(-20)} \mid [H, \phi] = \lambda(H)\phi \quad \text{for all } H \in \mathfrak{a}\}, \\ \Sigma &:= \{\lambda \in \mathfrak{a}^* |\ \lambda \neq 0,\ \mathfrak{g}_\lambda \neq \{0\}\}, \end{aligned}$$

and the centralizer $\mathfrak{a}$ of the group K and its Lie algebra as

$$\begin{aligned} M &:= Z_K(\mathfrak{a}) = \{k \in K |\ k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)\}, \\ \mathfrak{m} &:= Z_{\mathfrak{k}}(\mathfrak{a}) = \{\phi \in \mathfrak{k} \mid [\phi, \tilde{A}_3^1(1)] = 0\} \end{aligned}$$

respectively. For $p \in \mathrm{Im}\mathbf{O}$, $l_p, r_p, t_p \in \mathrm{End}_{\mathbb{R}}(\mathbf{O})$ are defined by

$$l(p)x := px, \quad r(p)(x) := xp, \quad t(p)x := px + xp \quad \text{for } x \in \mathbf{O}$$

respectively. Then we see that

$$\delta(p) := d\varphi_0(l_p, r_p, t_{-p}) \in \mathfrak{d}_4.$$

For $p \in \mathrm{Im}\mathbf{O}$ and $x \in \mathbf{O}$, denote

$$\begin{aligned} \mathcal{G}_1(x) &:= \tilde{A}_1^1(x) + \tilde{A}_2^1(-\overline{x}), & \mathcal{G}_2(p) &:= -\tilde{A}_3^1(p) - \delta(p), \\ \mathcal{G}_{-1}(x) &:= \tilde{A}_1^1(x) + \tilde{A}_2^1(\overline{x}), & \mathcal{G}_{-2}(p) &:= \tilde{A}_3^1(p) - \delta(p) \end{aligned}$$

For $i = \pm 1$ and $j = \pm 2$, denote the subspaces $\mathfrak{g}_i$ and $\mathfrak{g}_j$ as $\mathfrak{f}_{4(-20)}$

$$\mathfrak{g}_i := \{\mathcal{G}_i(p) \mid p \in \mathrm{Im}\mathbf{O}\}, \quad \mathfrak{g}_j := \{\mathcal{G}_j(x) \mid x \in \mathbf{O}\}$$

respectively.

Proposition 4.2. (cf. [26, Proposition 4.4])

$$\begin{aligned} M &= (F_{4(-20)})_{E_1, F_3^1(1)} = (F_{4(-20)})_{E_2, F_3^1(1)} = (F_{4(-20)})_{E_1, E_2, E_3, F_3^1(1)} \\ &= \varphi_0(\mathrm{Spin}(7)). \end{aligned}$$

Lemma 4.3. (cf. [26, Lemma 4.5])
$\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$,

$$\mathfrak{g}_{\pm\alpha} = \mathfrak{g}_{\pm 1}, \quad \mathfrak{g}_{\pm 2\alpha} = \mathfrak{g}_{\pm 2} \quad (resp),$$

and $(\mathfrak{f}_{4(-20)}, \mathfrak{a})$-root space decomposition of $\mathfrak{f}_{4(-20)}$ is given by

$$\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{2}.$$

So the nilpotent subalgebras $\mathfrak{n}^{\pm}$ are defined as

$$\mathfrak{n}^{+} := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{\alpha} = \{\mathcal{G}_2(p) + \mathcal{G}_1(x) | \ p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\},$$
$$\mathfrak{n}^{-} := \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} = \{\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x) | \ p \in \mathrm{Im}\mathbf{O}; x \in \mathbf{O}\} \quad (resp).$$

Then

$$[\mathfrak{n}^{+}, [\mathfrak{n}^{+}, \mathfrak{n}^{+}]] = [\mathfrak{n}^{-}, [\mathfrak{n}^{-}, \mathfrak{n}^{-}]] = 0.$$

And the nilpotent subgroups $N^{\pm}$ of $F_{4(-20)}$ are defined as

$$N^{+} := \exp \mathfrak{n}^{+} = \{\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \mid p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\},$$
$$N^{-} := \exp \mathfrak{n}^{-} = \{\exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \mid p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\} \quad (resp).$$

Lemma 4.4.
(1) $\exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p)$.
(2) $\tilde{\sigma}_1 \mathfrak{n}^{+} = \mathfrak{n}^{-}$ *and* $\tilde{\sigma}_1 \mathfrak{n}^{-} = \mathfrak{n}^{+}$*. Furthermore,*

$$\tilde{\sigma}_1(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(x)) = \mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(x) \quad (resp).$$

(3) $\tilde{\sigma}_1(N^{+}) = N^{-}$ *and* $\tilde{\sigma}_1(N^{-}) = N^{+}$*. Furthermore,*

$$\tilde{\sigma}_1(\exp(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(p))) = \exp(\mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(p)) \quad (resp).$$

Lemma 4.5. ([26, Lemma 5.3])
Let $g = (g_1, g_2, g_3), h \in \mathrm{Spin}(7)$, $p, q \in \mathrm{Im}\mathbf{O}$, $x, y \in \mathbf{O}$.

$$\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y))\varphi_0(h)$$
$$= \exp(\mathcal{G}_2(p + g_3 q + \mathrm{Im}(x\overline{(g_1 y)}) + \mathcal{G}_1(x + g_1 y))\varphi_0(gh).$$

Let us consider $G := \mathrm{Spin}(7) \times \mathrm{Im}\mathbf{O} \times \mathbf{O}$ in which multiplication is defined by

$$(g, p, x)(h, q, y) := (gh, p + g_3 q + \mathrm{Im}(x\overline{(g_1 y)}), x + g_1 y)$$

where $p, q \in \mathrm{Im}\mathbf{O}$, $x, y \in \mathbf{O}$ and $g = (g_1, g_2, g_3), h \in \mathrm{Spin}(7)$. Denote

$H := \{(g, 0, 0) | \ g \in \mathrm{Spin}(7)\}$,
$N := \{(1, p, x) | \ p \in \mathrm{Im}\mathbf{O}, x \in \mathbf{O}\}$,
$G' := \{(g, p, 0) | \ g \in \mathrm{Spin}(7), \ p \in \mathrm{Im}\mathbf{O}\}$, $N_1 := \{(1, p, 0) | \ p \in \mathrm{Im}\mathbf{O}\}$,
$G'' := \{(g, p, q) | \ g \in G_2, \ p, q \in \mathrm{Im}\mathbf{O}\}$,
$H'' := \{(g, 0, 0) | \ g \in G_2\}$, $N_2 := \{(1, p, q) | \ p, q \in \mathrm{Im}\mathbf{O}\}$.

Lemma 4.6. ([26, Lemma 5.2])

(1) *G is a group with respect to the multiplication.*

(2) *H, N, G', N_1, G'', H'' and N_2 are subgroups of G.*

(3) *We have*

$$G = H \ltimes N, \quad G' = H \ltimes N_1, \quad G'' = H'' \ltimes N_2.$$

Denote $\mathrm{Spin}(7) := H$, $\mathrm{Im}\mathbf{O} \times \mathbf{O} := N$, $\mathrm{Im}\mathbf{O} = N_1$, $G_2 := H''$ and $\mathrm{Im}\mathbf{O} \times \mathrm{Im}\mathbf{O} := N''$ so that

$$\mathrm{Spin}(7) \ltimes (\mathrm{Im}\mathbf{O} \times \mathbf{O}) = G, \quad \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O} = G'$$
$$G_2 \ltimes (\mathrm{Im}\mathbf{O} \times \mathrm{Im}\mathbf{O}) = G''.$$

The homomorphisms $\varphi : \mathrm{Spin}(7) \ltimes (\mathrm{Im}\mathbf{O} \times \mathbf{O}) \to (F_{4(-20)})_{P^-}$, $\varphi_1 : \mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O} \to (F_{4(-20)})_{E_3,P^-}$ and $\varphi_2 : G_2 \ltimes (\mathrm{Im}\mathbf{O} \times \mathrm{Im}\mathbf{O}) \to (F_{4(-20)})_Q$ are defined as

$$\varphi(g,p,x) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi(g), \quad \varphi_1(g,p) = \exp(\mathcal{G}_2(p))\varphi(g)$$
$$\varphi_2(g,p,q) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(q))\varphi(g) \quad \text{for } p, q \in \mathrm{Im}\mathbf{O} \text{ and } x \in \mathbf{O}$$

respectively.

Proposition 4.7. ([26, Proposition 5.6])

(1) *φ_1 is an isomorphism onto $(F_{4(-20)})_{E_3,P^-}$.*

(2) *φ is an isomorphism onto $(F_{4(-20)})_{P^-}$.*

(3) *φ_2 is an isomorphism onto $(F_{4(-20)})_Q$.*

The key of proof of (2): By direct calculation,

$$Orb_{N^+}(E_3) = \{X \in \mathcal{J}^1 |\ P^- \times X = -\frac{1}{2}P^-,\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 1\}.$$

Then this equation deduces $Orb_{N^+}(E_3) = Orb_{(F_{4(-20)})_{P^-}}(E_3)$. □

The mappings $\psi_1 : F_{4(-20)} \to \mathbf{O}$, $\psi_2 : F_{4(-20)} \to \mathrm{Im}\mathbf{O}$ and $\psi_3 : F_{4(-20)} \to F_{4(-20)}$ are defined as for $g \in F_{4(-20)}$,

$$\psi_1(g) := \frac{1}{2}((gE_3)_{F_1^1} + \overline{(gE_3)_{F_2^1}}),$$
$$\psi_2(g) := -\frac{1}{2}\mathrm{Im}\left((g(-E_1 + E_2))_{F_3^1}\right),$$
$$\psi_3(g) := \exp\left(-\mathcal{G}_1(\psi_1(g)) - \mathcal{G}_2(\psi_2(g))\right) g$$

respectively.

Proposition 4.8. ([26, Proposition 5.7])

(1) *Let $g \in (F_{4(-20)})_{P^-}$. Then $\psi_3(g) \in M$ and*

$$g = \exp\left(\mathcal{G}_1(\psi_1(g)) + \mathcal{G}_2(\psi_2(g))\right) \psi_3(g) \in N^+M.$$

(2) *We have*

$$(F_{4(-20)})_{P^-} = N^+M = MN^+.$$

5. The orbit types of $F_{4(-20)}$-orbits on $\mathcal{J}^1$.

Main Theorem 2. (The orbit types of $F_{4(-20)}$-orbits on $\mathcal{J}^1$ [26, Main Theorem 1])
The orbit types of $F_{4(-20)}$-orbits on $\mathcal{J}^1$ are given as follows.

(1) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following type of stabilizer group.*

The canonical forms of X	*The type of stabilizer group*
1. $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$	$\mathrm{Spin}(8)$
2. $\mathrm{diag}(\lambda_2, \lambda_3, \lambda_1)$	$\mathrm{Spin}(8)$
3. $\mathrm{diag}(\lambda_3, \lambda_1, \lambda_2)$	$\mathrm{Spin}(8)$

(2) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then X can be transformed to the following canonical form by $F_{4(-20)}$ with the following type of stabilizer group.*

The canonical forms of X	*The type of stabilizer group*
4. $\mathrm{diag}(p, p, \lambda_1) + F_3^1(q)$	$\mathrm{Spin}^0(7,1)$

(3) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity* 1 *and λ_2 of multiplicity* 2*. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.*

The canonical forms of X	*The type of stabilizer group*
5. $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$	$\mathrm{Spin}(9)$
6. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1)$	$\mathrm{Spin}^0(8,1)$
7. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$	$\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$
8. $\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$	$\mathrm{Spin}(7) \ltimes \mathrm{Im}\mathbf{O}$

(4) *Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity* 3*. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.*

The canonical forms of X	*The type of stabilizer group*
9. $\frac{1}{3}\mathrm{tr}(X)E$	$F_{4(-20)}$
10. $\frac{1}{3}\mathrm{tr}(X)E + P^+$	$\mathrm{Spin}(7) \ltimes (\mathrm{Im}\mathbf{O} \times \mathbf{O})$
11. $\frac{1}{3}\mathrm{tr}(X)E + P^-$	$\mathrm{Spin}(7) \ltimes (\mathrm{Im}\mathbf{O} \times \mathbf{O})$
12. $\frac{1}{3}\mathrm{tr}(X)E + Q^+(1)$	$G_2 \ltimes (\mathrm{Im}\mathbf{O} \times \mathrm{Im}\mathbf{O})$

6. The three decompositions of a linear connected semisimple Noncompact Lie groups.

Let G be a linear connected semisimple Lie group with its Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Let θ be a Cartan involution of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. $\mathfrak{a}^*$

denotes the dual space of $\mathfrak{a}$. For any element $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X$ for all $H \in \mathfrak{a}\}$. λ is called a *root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. The set of roots of $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ follows. Denote by Σ^+ a set of positive root of $(\mathfrak{g}, \mathfrak{a})$ with respect to the some ordering in $\mathfrak{a}^*$, $\Sigma^- := \{-\lambda \mid \lambda \in \Sigma^+\}$, $\mathfrak{n}^+ := \sum_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$ and $\mathfrak{n}^- := \sum_{\lambda\in\Sigma^-} \mathfrak{g}_\lambda$. Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras, $\theta\, \mathfrak{n}^\pm = \mathfrak{n}^\mp$ (*resp*), and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$ follow. Suppose that there exists an involutive automorphism Θ on G such that the differential $d\Theta = \theta$, and the center $Z(G)$ of G is finite. Denote the subgroup $K := G^\Theta$ of G. Then $Lie(K) = \mathfrak{k}$ and K is connected, closed, and K is a maximal compact subgroup of G since $Z(G)$ of G is finite. Set $A := \exp \mathfrak{a}$, $M := Z_K(\mathfrak{a}) = \{k \in K \mid kXk^{-1} = X$ for all $X \in \mathfrak{a}\}$ and $N^\pm := \exp \mathfrak{n}^\pm$ (*resp*). Then the identity connected component M^0 of M is the analytic subgroup corresponding to $\mathfrak{m}$, and $\Theta N^\pm = N^\mp$ (*resp*). denote the normalizer of $\mathfrak{a}$ of the group K as $M^* := N_K(\mathfrak{a}) = \{k \in K \mid k\mathfrak{a}k^{-1} \subset \mathfrak{a}\}$ and the finite factor group $W := M^*/M$.

For all $w \in W$, we fix a representative $\bar{w} \in M^*$. Then the following decompositions:

$$(1)\ G = \coprod_{w\in W} MAN^+\bar{w}N^- \qquad \text{(Bruhat decomposition)},$$

$$(1)'\ G = \overline{MAN^+N^-} \qquad \text{(Gauss decomposition)},$$

$$(2)\ G = KAN^+ \qquad \text{(Iwasawa decomposition)}.$$

(cf. [15],[18], [27],[23]). In (1)', the set MAN^+N^- is open dense in G, and so almost any $g \in G$ can be expressed by

$$g = m_G(g)a_G(g)n_G(g)\overline{n}_G(g)$$

for some $m_G(g) \in M$, $a_G(g) \in A$, $n_G(g) \in N^+$ and $\overline{n}_G(g) \in N^-$ with uniquely determined factors. In (2), any $g \in G$ can be uniquely expressed by

$$g = k(g)(\exp H(g))n(g)$$

for some $k(g) \in K, H(g) \in \mathfrak{a}$ and $n(g) \in N$.

A *signature of roots* is defined by the mapping ϵ of Σ to $\{-1, 1\}$ such that ϵ satisfies the conditions:

$$\text{(i) } \epsilon(\lambda) = \epsilon(-\lambda) \qquad \text{for all } \lambda \in \Sigma,$$

$$\text{(ii) } \epsilon(\lambda + \mu) = \epsilon(\lambda)\epsilon(\mu) \qquad \text{if } \lambda, \mu, \lambda + \mu \in \Sigma$$

[27, Definition 1.1]. For the Cartan involution θ and any signature ϵ of roots, let us define an involutive automorphism θ_ϵ of $\mathfrak{g}$ such that

$$\text{(i) } \theta_\epsilon(X) := \epsilon(\lambda)\theta(X) \qquad \text{for all } \lambda \in \Sigma \text{ and } X \in \mathfrak{g}_\lambda,$$

$$\text{(ii) } \theta_\epsilon(X) := \theta(X) \qquad \text{for all } X \in \mathfrak{a} \oplus \mathfrak{m}$$

[27, Definition 1.2]. θ_ϵ is called the $(\theta, \epsilon)-$*involution* of $\mathfrak{g}$. Set

$$\mathfrak{k}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = X\}, \quad \mathfrak{p}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = -X\}.$$

Then $\mathfrak{g} = \mathfrak{k}_\epsilon \oplus \mathfrak{p}_\epsilon$. Let $(K_\epsilon)_0$ be the analytic subgroup of G with the Lie algebra $\mathfrak{k}_\epsilon$ and the subgroup K_ϵ of G as $K_\epsilon := (K_\epsilon)_0 M$. In fact, since all elements of M normalize $(K_\epsilon)_0$ by [27, Lemma 1.4(i)], K_ϵ is a subgroup of G. Denote $M^*_\epsilon := K_\epsilon \cap M^*$ and $W_\epsilon := M^*_\epsilon / M$.

Proposition 6.1. (Iwasawa decomposition with respect to K_ϵ in sense of T. Oshima and J. Sekiguchi [27, Proposition 1.10]) *Let the factor set $W_\epsilon \backslash W = \{w_1 = 1, w_2, \cdots, w_r\}$ where $r = [W : W_\epsilon]$. Fix representatives $\bar{w}_1 = 1, \bar{w}_2, \cdots, \bar{w}_r \in M^*_\epsilon = K_\epsilon \cap M^*$ for $w_1 = 1, w_2, \cdots, w_r$. Then the decomposition*

$$G \supset \cup_{i=1}^{r} K_\epsilon \bar{w}_i A N^+$$

has the following properties.

(1) *If $k\bar{w}_i a n = k'\bar{w}_j a' n'$ with $k, k' \in K_\epsilon$, $a, a' \in A$ and $n, n' \in N^+$, then $k = k'$, $i = j$, $a = a'$ and $n = n'$.*

(2) *The map $(k, a, n) \mapsto k\bar{w}_i a n$ defines an analytic diffeomorphism of the product manifold $K_\epsilon \times A \times N^+$ onto the open submanifold $K_\epsilon \bar{w}_i A N^+$ of G $(i = 1, \cdots r)$.*

(3) *The submanifolds $\cup_{i=1}^{r} K_\epsilon \bar{w}_i A N^+$ is open dense in G.*

7. The Gauss decomposition of $F_{4(-20)}$.

We have

$$\mathcal{N}_1^-(\mathbf{O}) = Orb_{F_{4(-20)}}(P^-) \simeq F_{4(-20)}/(F_{4(-20)})_{P^-} = F_{4(-20)}/N^+M.$$

So considering AN^--orbits on $\mathcal{N}_1^-(\mathbf{O})$, we obtain:

Main Theorem 3. (The Bruhat and Gauss decomposition of $F_{4(-20)}$ [26, Main Theorem 2])

(1) *Assume that $g \in F_{4(-20)}$ and $(gP^+|P^-) \neq 0$. Let*

$$\begin{aligned}
t &:= -\frac{1}{2}\log\left(-\frac{(gP^+|P^-)}{4}\right) \in \mathbb{R},\\
a_G(g) &:= \exp(t\tilde{A}_3^1(1)) \in A,\\
\overline{n}_G(g) &= \tilde{\sigma}_1(\exp(-\mathcal{G}_1\left(\frac{(\sigma_1 g^{-1}P^-)_{F_1^1} - \overline{(\sigma_1 g^{-1}P^-)_{F_2^1}}}{(gP^+|P^-)}\right)\\
&\quad - \mathcal{G}_2\left(\frac{\mathrm{Im}((\sigma_1 g^{-1}P^-)_{F_3^1})}{(gP^+|P^-)}\right))) \in N^-,\\
n_G(g) &:= \exp(t(\mathcal{G}_1(\psi_1(a_G(g)\overline{n}_G(g)g^{-1})\\
&\qquad + 2\mathcal{G}_2(\psi_2(a_G(g)\overline{n}_G(g)g^{-1}))) \in N^+,\\
m_G(g) &:= \psi_3(a_G(g)\overline{n}_G(g)g^{-1})^{-1}.
\end{aligned}$$

Then

(i) $(gP^+|P^-) < 0$, *and* $a_G(g)$, $\overline{n}_G(g)$, $n_G(g)$, $m_G(g)$ *are well-defined,*

(ii) $m_G(g) \in M$ *and*

$$g = m_G(g)a_G(g)n_G(g)\overline{n}_G(g) \in MAN^+N^-.$$

(2) *Assume* $g \in F_{4(-20)}$ *and* $(gP^+|P^-) = 0$. *Let*

$$\begin{aligned} t &:= -\frac{1}{2}\log(-(gE_1|P^-)) \in \mathbb{R}, \\ a'(g) &= \exp(t\tilde{A}_3^1(1)) \in A, \\ n'(g) &:= \exp(t(\mathcal{G}_1(\psi_1(\sigma_1 a'(g)g^{-1}) + 2\mathcal{G}_2(\psi_2(\sigma_1 a'(g)g^{-1}))) \in N^+, \\ m'(g) &:= \psi_3(\sigma_1 a'(g)g^{-1})^{-1}. \end{aligned}$$

Then

(i) $(gE_1|P^-) < 0$, *and* $a'(g)$, $n'(g)$, $m'(g)$ *are well-defined,*

(ii) $m'(g) \in M$ *and*

$$g = m'(g)a'(g)n'(g)\sigma_1 \in MAN^+\sigma_1 = MAN^+\sigma_1 N^-.$$

(3) *The following equations hold.*

$$\begin{aligned} MAN^+N^- =& \{g \in F_{4(-20)} \mid (gP^+|P^-) \neq 0\} \\ =& \{g \in F_{4(-20)} \mid (gP^+|P^-) < 0\} \neq \emptyset, \\ MAN^+\sigma_1 =& MAN^+\sigma_1 N^- \\ =& \{g \in F_{4(-20)} \mid (gP^+|P^-) = 0\} \neq \emptyset. \end{aligned}$$

Especially,

$$\begin{aligned} F_{4(-20)} &= MAN^+N^- \coprod MAN^+\sigma_1 N^- \qquad \text{(Bruhat decomposition)} \\ &= MAN^+N^- \coprod MAN^+\sigma_1 \end{aligned}$$

(4) MAN^+N^- *is open dense in* $F_{4(-20)}$. *Especially*

$$F_{4(-20)} = \overline{MAN^+N^-} \qquad \text{(Gauss decomposition).}$$

8. The Iwasawa decomposition of $F_{4(-20)}$.

We have

$$\mathcal{H}(\mathbf{O}) = Orb_{F_{4(-20)}}(E_1) \simeq F_{4(-20)}/(F_{4(-20)})_{E_1} = F_{4(-20)}/K.$$

So considering AN^+-orbits on $\mathcal{H}(\mathbf{O})$, we obtain:

Main Theorem 4. (The Iwasaws decomposition of $F_{4(-20)}$[26, Main Theorem 3])

For any $g \in F_{4(-20)}$, *let*

$$H(g) := \frac{1}{2}\log(-(gP^-|E_1))\tilde{A}_3^1(1) \in \mathfrak{a},$$

$$n(g) := \exp(\mathcal{G}_1\left(\frac{(g^{-1}E_1)_{F_1^1} - \overline{(g^{-1}E_1)_{F_2^1}}}{(gP^-|E_1)}\right) + \mathcal{G}_2\left(\frac{\mathrm{Im}((g^{-1}E_1)_{F_3^1})}{(gP^-|E_1)}\right))$$

$$\in N^+$$

$$k(g) := gn(g)^{-1}\exp(-H(g)).$$

Then

(1) $(gP^-|E_1) < 0$. *Especially* $H(g)$, $n(g)$ *and* $k(g)$ *is well-defined.*

(2) $k(g) \in K$ *and*

$$g = k(g)(\exp H(g))n(g) \in KAN^+.$$

9. The Iwasawa decomposition with respect to K_ϵ.

For $G = F_{4(-20)}$, let ϵ be a signature of root defined by

$$\epsilon(\alpha) = \epsilon(-\alpha) := -1,\ \epsilon(2\alpha) = \epsilon(-2\alpha) := 1.$$

Denote the $(\tilde{\sigma}_1, \epsilon)$−involution by $(\tilde{\sigma}_1)_\epsilon$, and use same notations $\mathfrak{k}_\epsilon$, $(K_\epsilon)_0$, K_ϵ, M^*, M_ϵ^*, W and W_ϵ corresponding to notations of general G respectively.

Proposition 9.1. ([26, Lemma 6.2])

(1) $(\tilde{\sigma}_1)_\epsilon = \tilde{\sigma}_2$.

(2) $K_\epsilon = (F_{4(-20)})_{E_2}$.

(3) $M^* = M\coprod \sigma_1 M$. *Especially* $W = \{M, \sigma_1 M\} \cong \mathbb{Z}_2$.

(4) $M_\epsilon^* = M\coprod \sigma_1 M$. *Especially* $W_\epsilon = \{M, \sigma_1 M\}$ *and* $[W : W_\epsilon] = 1$.

We have

$$\mathcal{H}'(\mathbf{O}) = Orb_{F_{4(-20)}}(E_2) \simeq F_{4(-20)}/(F_{4(-20)})_{E_2} = F_{4(-20)}/K_\epsilon.$$

So considering AN^+-orbits on $\mathcal{H}'(\mathbf{O})$, we obtain:

Main Theorem 5. (The Iwasawa decomposition with respect to K_ϵ[26, Main Theorem 4])

Let $\mathcal{D}$ *be the domain of* $F_{4(-20)}$ *defined by*

$$\mathcal{D} := \{g \in F_{4(-20)} \mid (gP^-|E_2) > 0\}.$$

For any $g \in \mathcal{D}$, *let*

$$H_\epsilon(g) := \frac{1}{2}\log((gP^-|E_2))\tilde{A}_3^1(1) \in \mathfrak{a},$$

$$n_\epsilon(g) := \exp(\mathcal{G}_1\left(\frac{(g^{-1}E_2)_{F_1^1} - \overline{(g^{-1}E_2)_{F_2^1}}}{(gP^-|E_2)}\right)$$

$$+ \mathcal{G}_2\left(\frac{\mathrm{Im}((g^{-1}E_2)_{F_3^1})}{(gP^-|E_2)}\right)) \in N^+$$

$$k_\epsilon(g) := gn_\epsilon(g)^{-1}\exp(-H_\epsilon(g)).$$

Then

(1) $k_\epsilon(g) \in K_\epsilon$ *and*

$$g = k_\epsilon(g)(\exp H_\epsilon(g))n_\epsilon(g) \in K_\epsilon AN^+.$$

(2) $\mathcal{D} = K_\epsilon AN^+ = \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\}$. *Furthermore,* $\mathcal{D}$ *is open dense in* $F_{4(-20)}$.

Moreover we have:

Theorem 9.2. ([26, Theorem 9.6])

(1) *The following equations hold.*

$$K_\epsilon MAN^+ = \{g \in F_{4(-20)} |\ (gP^-|E_2) \neq 0\}$$
$$= \{g \in F_{4(-20)} |\ (gP^-|E_2) > 0\} = K_\epsilon AN^+.$$

(2) $K_\epsilon \exp\left(-\tilde{A}_1^1(\frac{\pi}{2})\right) MAN^+ = \{g \in F_{4(-20)} |\ (gP^-|E_2) = 0\}$.

(3) $F_{4(-20)} = K_\epsilon MAN^+ \coprod K_\epsilon \exp\left(-\tilde{A}_1^1(\frac{\pi}{2})\right) MAN^+$.

Remark 9.3. Theorem 9.2(3) is a special case of [21, Theorems 3], so the decomposition in Theorem 9.2(3) is called a *Matsuki decomposition of* $F_{4(-20)}$.

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