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ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS

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ABSTRACT. Let $\mathcal{S}$ be the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic and univalent in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. The subclasses of $\mathcal{S}$ whose coefficients a_n belong to a quadratic field have been studied by Friedman [3] and Bernardi [1]. Linis [7] gave a short proof of Friedman's theorem which states that if all the a_n are "rational integers" then f is rational and has nine forms. In this paper, we consider what will happen if all the a_n are "half-integers"; that is, $2a_n\in\mathbb{Z}$.

1. PRELIMINARIES

1.1. **Notation and Definitions.** A *domain* is an open connected set in the complex plane $\mathbb{C}$. The *unit disk* $\mathbb{D}$ consists of all points $z\in\mathbb{C}$ of modulus $|z|<1$. A single-valued function f is said to be *univalent* in a domain $D\subset\mathbb{C}$ if it is injective; that is, if $f(z_1)\neq f(z_2)$ for all points z_1 and z_2 in D with $z_1\neq z_2$. The function f is said to be *locally univalent* at a point $z_0\in D$ if it is univalent in some neighborhood of z_0. For analytic functions f, the condition $f'(z_0)\neq 0$ is equivalent to local univalence at z_0.

We shall be concerned primarily with the class $\mathcal{S}$ of functions f analytic and univalent in $\mathbb{D}$, normalized by the conditions $f(0)=0$ and $f'(0)=1$. Thus each $f\in\mathcal{S}$ has a Taylor series expansion of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots,\quad |z|<1.$$

The important example of a function in the class $\mathcal{S}$ is the *Koebe function*

$$k(z)=\frac{z}{(1-z)^2}=z+2z^2+3z^3+\cdots.$$

1.2. **Bieberbach's Conjecture.** In 1916, Bieberbach estimated the second coefficient a_2 of a function in the class $\mathcal{S}$. (See [2, p. 30].)

Theorem 1. *If $f\in\mathcal{S}$ then $|a_2|\leq 2$. Equality occurs if and only if f is the Koebe function or one of its rotations.*

This suggests the general problem to find

$$A_n:=\sup_{f\in\mathcal{S}}|a_n|,\quad n=2,3,\ldots.$$

In a footnote, he wrote "Vielleicht ist überhaupt $A_n=n$ (Perhaps it is generally $A_n=n$)." Since the Koebe function plays the extremal role in so many problems for the class $\mathcal{S}$, it is natural to suspect that it maximizes $|a_n|$ for all n. This is the famous conjecture of Bieberbach, first proposed in 1916.

Many partial results were obtained in the intervening years, including results for special subclasses of $\mathcal{S}$ and for particular coefficients, as well as asymptotic estimates and estimates for general n. Finally, de Branges [4] gave a remarkable proof in 1985. (See [6].)

Theorem 2. *If $f \in \mathcal{S}$ then*

$$|a_n| \le n, \quad n = 2, 3, \ldots. \tag{1}$$

Equality occurs if and only if f is the Koebe function or one of its rotations.

1.3. **Prawitz' Inequality.** Let $f \in \mathcal{S}$. Set $F(z) = z/f(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$F(z) = 1 - a_2 z + (a_2^2 - a_3)z^2 + \cdots.$$

Hence, we have $b_0 = 1$, $b_1 = -a_2$, $b_2 = a_2^2 - a_3, \ldots$. The coefficient b_n $(n \ge 1)$ can be computed by the relation

$$b_n = (-1)^n \begin{vmatrix} a_2 & 1 & & \cdots & 0 \\ a_3 & a_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & a_n & & \cdots & a_2 \end{vmatrix}.$$

Prawitz [8] discovered an estimate for the coefficient b_n. It is a generalization of the Gronwall area theorem (see [2, p. 29]) and may be formulated as follows:

Theorem 3. *Let $f \in \mathcal{S}$ and $[z/f(z)]^{\alpha/2} = \sum_{n=0}^{\infty} \beta_n z^n$. Then*

$$\sum_{n=0}^{\infty} \frac{(2n-\alpha)}{\alpha} |\beta_n|^2 \le 1$$

for all real α.

In particular, for $\alpha = 2$ we have the following

Corollary 1. *Let $f \in \mathcal{S}$ and $z/f(z) = \sum_{n=0}^{\infty} b_n z^n$. Then*

$$\sum_{n=1}^{\infty} (n-1)|b_n|^2 \le 1. \tag{2}$$

This corollary is essentially equivalent to the Gronwall area theorem.

2. Motivation

2.1. **Friedman's Theorem.** Friedman [3] proved the following theorem which is a part of Salem's theorem on univalent functions [10]:

Theorem 4. *Let $f \in \mathcal{S}$. If all the coefficients a_n are rational integers then $f(z)$ is one of the following nine functions:*

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

Proof. Set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then the coefficients b_n are rational integers. Since $b_1 = -a_2$ and $|a_2| \le 2$, it follows that $|b_1| \le 2$. Applying the inequality (2), we have $|b_2| \le 1$ and $b_n = 0$ for $n \ge 3$. Therefore, the possible values for b_n are:

$$b_1 = 0, \pm 1, \pm 2; \quad b_2 = 0, \pm 1; \quad b_n = 0 \text{ for } n \ge 3.$$

From the combination of these values we obtain 15 functions. However, the following six functions must be rejected as having zeros in $\mathbb{D}$:

$$1 \pm 2z, \quad 1 \pm 2z - z^2, \quad 1 \pm z - z^2.$$

The remaining nine functions prove the theorem. □

2.2. **Extensions of Friedman's Theorem.** The method of the proof of Friedman's theorem in the previous section was given by Linis [7]. He also proved the following

Theorem 5. *Let $f \in \mathcal{S}$. If all the coefficients a_n are Gaussian integers then f has 15 forms. Here, a Gaussian integer is a complex number whose real and imaginary part are both rational integers.*

Royster [9] extended the method of the proof given by Linis to quadratic fields with negative discriminant as follows:

Theorem 6. *Let $f \in \mathcal{S}$. If all the coefficients a_n are algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{d})$ for some square-free rational negative integer d, then f has 36 forms.*

As mentioned above, they have obtained new results by replacing the condition "rational integers" with other conditions.

3. Main Result

3.1. **Subclass of $\mathcal{S}$ Having Half-integer Coefficients.** Now, we shall consider what will happen if all the coefficients a_n are half-integers. Here, a_n is said to be a *half-integer* if $2a_n$ is a rational integer.

In a similar way used in the proof of Friedman's theorem in the second chapter, we set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then the coefficients b_n are rational numbers. In the case when the a_n are rational integers, we could obtain all the possible values for the b_n. But, in this case we cannot obtain them immediately. However, using the inequalities (1) and (2), we can examine the possibilities of coefficients one by one, and obtain the following

Theorem 7. *Let $f \in \mathcal{S}$. If all the coefficients a_n are half-integers then $f(z)$ is one of the following 13 functions:*

$$z, \quad z \pm \frac{1}{2}z^2, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}, \quad \frac{z(2 \pm z)}{2(1 \pm z)^2}.$$

The detailed proof of this theorem is given in [5].

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