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Kinetic theory of Onsager's vortices in two-dimensional hydrodynamics: Virial theorem and BBGKY hierarchy

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We review our recent work on the kinetic theory of point vortices in two-dimensional hydrodynamics. We first consider statistical equilibrium states and derive the virial theorem of point vortices. It provides a relation between the angular velocity, the angular momentum and the temperature. Then, using the BBGKY hierarchy, we derive a kinetic equation for the point vortex gas, taking two-body correlations and collective effects into account. This equation is valid at the order $1/N$ where $N \gg 1$ is the number of point vortices in the system (we assume that their individual circulation scales like $\gamma \sim 1/N$). It gives the first correction, due to graininess and correlations, to the 2D Euler equation that is obtained for $N \to +\infty$. We discuss the implications of the kinetic theory regarding the relaxation (or not) of the point vortex gas towards the Boltzmann distribution. Finally, we consider the relaxation of a test vortex in a bath. It is described by a Fokker-Planck equation involving a diffusion and a drift.

I. INTRODUCTION

In 1949, Onsager [1] published a seminal paper in which he laid down the foundations of the statistical mechanics of vortices in two-dimensional hydrodynamics. He considered the point vortex gas as an idealization of more realistic vorticity fields and discovered that negative temperature states are possible for this system. At negative temperatures, corresponding to high energies, like-sign vortices have the tendency to cluster into "supervortices" similar to the large-scale vortices (e.g., Jupiter's great red spot) observed in the atmosphere of giant planets. If the point vortices all have the same sign, one gets a monopole. If they have different signs, one gets a dipole made of two clusters of opposite sign, or possibly a tripole made of a central vortex of a given sign surrounded by two vortices of opposite sign. Therefore, the statistical arguments of Onsager explain the ubiquity of large-scale vortices observed in geophysical and astrophysical flows.

The qualitative arguments of Onsager were developed more quantitatively in a mean field approximation by Joyce & Montgomery [2, 3], Kida [4] and Pointin & Lundgren [5, 6], and by Onsager himself in unpublished notes [7]. The statistical theory predicts that the point vortex gas should relax towards an equilibrium state described by the Boltzmann distribution. Specifically, the equilibrium stream function is solution of a Boltzmann-Poisson equation. Many mathematical works [8–12] have shown how a proper thermodynamic limit could be rigorously defined for the point vortex gas (in Onsager's picture). It is shown that the mean field approximation becomes exact in the limit $N \to +\infty$ with $\gamma \sim 1/N$ (where N is the number of point vortices and γ the circulation of a point vortex).

The statistical mechanics of continuous vorticity fields was developed later by Miller [13] and Robert & Sommeria [14]. Kuzmin [15] also developed a similar theory but his contribution is less well-known. The Miller-Robert-Sommeria (MRS) statistical theory is based on the 2D Euler equation which describes incompressible and inviscid flows. It predicts that the 2D Euler equation can reach a statistical equilibrium state (or metaequilibrium state) on a coarse-grained scale as a result of a mixing process. Recently, the MRS theory has been applied to geophysical and astrophysical flows, notably to oceanic circulation [16], jovian vortices (Jupiter's great red spot) [17–19], Fofonoff flows [20–22] and oceanic rings and jets [23]. The MRS theory shares many analogies with the theory of violent relaxation developed by Lynden-Bell [24] for collisionless stellar systems described by the Vlasov equation. The analogy between 2D vortices and stellar systems, first pointed out by Onsager in a letter to Lin [7], has been systematically developed in [25].

Another development of Onsager's seminal paper concerns the kinetic theory of point vortices. We have started this project in 1998 [26] and regularly worked on this topic since then [27–30]. We have derived a kinetic equation for the evolution of the smooth vorticity field, taking two-body correlations into account. This equation is valid at the order $1/N$ and provides therefore the first order correction to the 2D Euler equation obtained for $N \to +\infty$. This kinetic equation was derived by various methods such as the projection operator formalism, the quasilinear theory and the BBGKY hierarchy. In these works, we focused on (distant) two-body collisions and neglected collective effects. This leads to a kinetic equation similar to the Landau [31] equation in plasma physics. A kinetic theory of the point vortex gas had been previously developed by Dubin & O'Neil [32] in the context of non-neutral plasmas under a strong magnetic field. They used the Klimontovich formalism and took collective effects into account. This leads to a kinetic equation similar to the Lenard-Balescu [33, 34] equation in plasma physics. Their work was pursued in [35–38] in different directions. In a recent paper [39], we used the Klimontovich formalism to derive a Fokker-Planck equation describing the evolution of a test vortex in a bath of field vortices. This Fokker-Planck equation involves a diffusion and a drift. In a second recent paper [40], we derived the Lenard-Balescu-type kinetic equation of point vortices from the BBGKY hierarchy. These works properly take collective effects into account, unlike our previous works on the subject. Finally, in a third recent paper [41], we derived the virial theorem of point vortices and showed that it corresponds to the relation empirically obtained by Williamson [42] on the basis of numerical simulations (see also [6, 43] for alternative derivations). In these Proceedings, we shall provide a summary of our work on 2D point vortices, presenting in particular the main results obtained in our last two papers. The present paper should be self-contained although we shall not give all the details of the derivation which can be found in [40, 41].

Finally, we would like to quote the works of [44–48] who also developed kinetic theories of point vortices in different situations.

II. THE TWO-DIMENSIONAL POINT VORTEX GAS

We consider a multi-species system of point vortices in two-dimensional hydrodynamics described by the Kirchhoff-Hamilton equations [49, 50]:

$$\gamma_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \gamma_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \tag{1}$$

where γ_i is the circulation of point vortex i. The coordinates x and y of the point vortices are canonically conjugate. In an unbounded domain, the Hamiltonian can be written

$$H = \sum_{i<j} \gamma_i \gamma_j u(|\mathbf{r}_i - \mathbf{r}_j|), \tag{2}$$

where $u(|\mathbf{r}-\mathbf{r}'|) = -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|$ is the potential of interaction that is solution of the Poisson equation $\Delta u = -\delta$. For an isolated system, the energy $E = H$, the angular momentum $L = \sum_{i=1}^N \gamma_i r_i^2$ and the impulse $\mathbf{P} = -\mathbf{z}\times\sum_{i=1}^N \gamma_i \mathbf{r}$ are conserved ($\mathbf{z}$ is a unit vector normal to the plane of the flow). Of course, the total number N_a of point vortices of each species is also conserved (the total circulation of species a is $\Gamma_a = N_a\gamma_a$). The velocity of a point vortex located in $\mathbf{r}_i$ is given by

$$\mathbf{V}_i = -\mathbf{z}\times\frac{\partial\psi}{\partial\mathbf{r}_i}(\mathbf{r}_i) = -\mathbf{z}\times\sum_{j\neq i}\gamma_j\frac{\partial u}{\partial \mathbf{r}_i}(|\mathbf{r}_i-\mathbf{r}_j|) = \frac{1}{2\pi}\mathbf{z}\times\sum_{j\neq i}\gamma_j\frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}, \tag{3}$$

where $\psi(\mathbf{r}) = \sum_i \gamma_i u(|\mathbf{r}-\mathbf{r}_i|)$ is the stream function. The stream function is related to the vorticity $\omega(\mathbf{r}) = \sum_i \gamma_i \delta(\mathbf{r}-\mathbf{r}_i)$ by the Poisson equation $\Delta\psi = -\omega$.

Let us introduce the N-body distribution $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ of the system. We define the one- and two-body distribution functions by

$$P_1^{(a)}(\mathbf{r}_1, t) = \int P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)\, d\mathbf{r}_2...d\mathbf{r}_N, \tag{4}$$

$$P_2^{(ab)}(\mathbf{r}_1, \mathbf{r}_2, t) = \int P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)\, d\mathbf{r}_3...d\mathbf{r}_N. \tag{5}$$

The average energy, angular momentum and impulse can be written

$$E = \frac{1}{2}\sum_{a,b} N_a(N_b - \delta_{ab})\int P_2^{(ab)}(\mathbf{r},\mathbf{r}',t)\gamma_a\gamma_b u(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}d\mathbf{r}', \tag{6}$$

$$L = \sum_a \int P_1^{(a)}(\mathbf{r},t)\gamma_a r^2\, d\mathbf{r}, \qquad \mathbf{P} = -\mathbf{z}\times\sum_a\int P_1^{(a)}(\mathbf{r},t)\gamma_a \mathbf{r}\, d\mathbf{r}, \tag{7}$$

Finally, the N-body Boltzmann entropy is

$$S = -k_B \int P_N \ln P_N \prod_i d\mathbf{r}_i. \tag{8}$$

Remark: We note that the angular momentum $L = \sum_i \gamma_i r_i^2$ is similar to the moment of inertia for material particles. On the other hand, the conservation of linear impulse $\mathbf{P} = -\mathbf{z}\times\sum_i\gamma_i\mathbf{r}_i$ is equivalent to the conservation of $\mathbf{R} = \sum_i\gamma_i\mathbf{r}_i$, which is similar to the center of mass for material particles. The conservation of $\mathbf{R}$ for point vortices implies that the "center of vorticity" is fixed while, for material particles, the center of mass has a rectilinear motion with constant velocity (galilean invariance).

III. STATISTICAL EQUILIBRIUM STATE

A. The YBG-like hierarchy in the microcanonical ensemble

We consider the statistical equilibrium state of a system of point vortices in an unbounded domain. In that case, the circulations of the point vortices must have the same sign otherwise they would form dipoles $(+,-)$ and ballistically escape to infinity so that no equilibrium state would be possible. The microcanonical ensemble is the proper description of an isolated Hamiltonian system for which the energy E, the angular momentum L, and the center of vorticity $\mathbf{R}$ are conserved. The microcanonical N-body distribution is given by

$$P_N(\mathbf{r}_1,...,\mathbf{r}_N) = \frac{1}{g(E,L)}\delta(E - H(\mathbf{r}_1,...,\mathbf{r}_N))\delta\left(L - \sum_i \gamma_i r_i^2\right)\delta\left(\mathbf{R} - \sum_i \gamma_i \mathbf{r}_i\right). \tag{9}$$

Since $\int P_N\, d\mathbf{r}_1...d\mathbf{r}_N = 1$, the density of states is given by $g(E,L) = \int \delta(E-H)\delta(L-\sum_i \gamma_i r_i^2)\delta(\mathbf{R}-\sum_i \gamma_i \mathbf{r}_i)\, d\mathbf{r}_1...d\mathbf{r}_N$. It does not depend on $\mathbf{R}$ since this constraint may be absorbed by a shift of origin in the integral [6]. In the following, we take the origin of the coordinates at the center of vorticity so that $\mathbf{R} = \mathbf{0}$. The entropy is defined by $S(E,L) = k_B \ln g(E,L)$. Then, the temperature and the angular velocity are given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}, \qquad \frac{\Omega}{2T} = \frac{\partial S}{\partial L}. \tag{10}$$

As first realized by Onsager [1], the temperature of the point vortex gas can be positive or negative. At negative temperatures $\beta(E) < 0$, corresponding to high energy states, point vortices of the same sign group themselves in "supervortices" similar to large-scale vortices in planetary atmospheres.

Differentiating Eq. (4) with respect to $\mathbf{r}_i$ and using the microcanonical distribution (9), we obtain

$$\begin{aligned}\frac{\partial P_1^{(a)}}{\partial \mathbf{r}}(\mathbf{r}) = &-\sum_b (N_b - \delta_{ab})\int \frac{1}{g(E,L)}\frac{\partial}{\partial E}\left(g(E,L)P_2^{(ab)}(\mathbf{r},\mathbf{r}')\right)\gamma_a\gamma_b\frac{\partial u}{\partial \mathbf{r}}(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}' \\ &-\frac{1}{g(E,L)}\frac{\partial}{\partial L}\left(g(E,L)P_1^{(a)}(\mathbf{r})\right)2\gamma_a\mathbf{r} - \frac{1}{g(E,L)}\frac{\partial}{\partial \mathbf{R}}\left(g(E,L)P_1^{(a)}(\mathbf{r})\right)\gamma_a.\end{aligned} \tag{11}$$

This is the first equation of the Yvon-Born-Green (YBG) hierarchy in the microcanonical ensemble [6]. This equation involves the quantities

$$\frac{1}{g}\frac{\partial}{\partial E}\left(gP_2^{(ab)}\right) = \beta P_2^{(ab)} + \frac{\partial P_2^{(ab)}}{\partial E}, \tag{12}$$

$$\frac{1}{g}\frac{\partial}{\partial L}\left(gP_1^{(a)}\right) = \frac{1}{2}\beta\Omega P_1^{(a)} + \frac{\partial P_1^{(a)}}{\partial L}, \qquad \frac{1}{g}\frac{\partial}{\partial \mathbf{R}}\left(gP_1^{(a)}\right) = \frac{\partial P_1^{(a)}}{\partial \mathbf{R}}. \tag{13}$$

The last terms in these expressions are difficult to evaluate. If we neglect these terms, Eq. (11) reduces to

$$\frac{\partial P_1^{(a)}}{\partial \mathbf{r}}(\mathbf{r}) = -\beta\sum_b (N_b - \delta_{ab})\int P_2^{(ab)}(\mathbf{r},\mathbf{r}')\gamma_a\gamma_b\frac{\partial u}{\partial \mathbf{r}}(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}' - \beta\Omega\gamma_a P_1^{(a)}(\mathbf{r})\mathbf{r}. \tag{14}$$

This is the equation that we would have obtained if we had started from the canonical distribution [6, 41]. Equations (11) and (14) are different but they coincide at the thermodynamic limit $N \to +\infty$ in which the last terms in Eqs. (12)-(13) can be neglected.

B. Mean field approximation: The Boltzmann-Poisson equation

We consider the thermodynamic limit $N \to +\infty$ with $\gamma_i \sim 1/N$ [8–12]. This implies $E \sim 1$, $\beta \sim N$, $L \sim 1$ and $\Omega \sim 1$. The total circulation $\Gamma \sim N\gamma \sim 1$ and the dynamical time $t_D \sim 1/\omega \sim 1$ are of order unity. In this thermodynamic limit, it can be shown that the reduced correlation functions scale like $P_j' \sim (1/N)^{j-1}$ [6, 29].

Therefore, when $N \to +\infty$, the mean field approximation becomes exact and the N-body distribution function factorizes in a product of N one-body distribution functions

$$P_N(\mathbf{r}_1, ..., \mathbf{r}_N) = \prod_i P_1(\mathbf{r}_i). \tag{15}$$

In particular, we have

$$P_2^{(ab)}(\mathbf{r}, \mathbf{r}') = P_1^{(a)}(\mathbf{r}) P_1^{(b)}(\mathbf{r}'). \tag{16}$$

In the thermodynamic limit, the last terms in Eqs. (12)-(13) can be neglected. Furthermore, we can make the approximation $N_a - 1 \simeq N_a$. Therefore, the first equation of the YBG hierarchy in microcanonical and canonical ensembles can be written [6]:

$$\nabla P_1^{(a)}(\mathbf{r}) = -\beta\gamma_a P_1^{(a)}(\mathbf{r}) \nabla\psi(\mathbf{r}) - \beta\gamma_a \Omega P_1^{(a)}(\mathbf{r})\mathbf{r}, \tag{17}$$

where $\psi(\mathbf{r}) = \langle \sum_i \gamma_i u(|\mathbf{r} - \mathbf{r}_i|) \rangle = \int \sum_b N_b \gamma_b P_1^{(b)}(\mathbf{r}') u(|\mathbf{r} - \mathbf{r}'|)\, d\mathbf{r}'$ is the average stream function. It is related to the average vorticity $\omega(\mathbf{r}) = \langle \sum_i \gamma_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle = \sum_b N_b \gamma_b P_1^{(b)}(\mathbf{r})$ by $\psi(\mathbf{r}) = \int u(|\mathbf{r} - \mathbf{r}'|)\omega(\mathbf{r}')\, d\mathbf{r}$. This is the solution of the Poisson equation

$$\Delta\psi = -\omega, \tag{18}$$

with the Gauge condition $\psi + \frac{\Gamma}{2\pi}\ln r \to 0$ for $r \to +\infty$. The mean velocity of a point vortex is $\mathbf{u} = -\mathbf{z} \times \nabla\psi$.

We introduce the number density $n_a(\mathbf{r}) = N_a P_1^{(a)}(\mathbf{r})$ of species a, and the total number density $n(\mathbf{r}) = \langle \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle = \sum_a n_a(\mathbf{r})$. Similarly, we introduce the vorticity $\omega_a(\mathbf{r}) = N_a \gamma_a P_1^{(a)}(\mathbf{r}) = \gamma_a n_a(\mathbf{r})$ of species a, and the total vorticity $\omega(\mathbf{r}) = \langle \sum_i \gamma_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle = \sum_a \omega_a(\mathbf{r})$. Equation (17) can be integrated to yield the Boltzmann distribution

$$\omega_a(\mathbf{r}) = N_a \gamma_a P_1^{(a)}(\mathbf{r}) = \Gamma_a \frac{e^{-\beta\gamma_a \psi_{eff}(\mathbf{r})}}{\int e^{-\beta\gamma_a \psi_{eff}(\mathbf{r})}\, d\mathbf{r}}, \tag{19}$$

where $\psi_{eff}(\mathbf{r}) = \psi(\mathbf{r}) + \frac{\Omega}{2} r^2$ is the relative stream function. The Boltzmann distribution (19) is steady in a frame rotating with angular velocity Ω. The vorticity profile decreases at large distances like $\omega_a(\mathbf{r}) \propto e^{-\beta\gamma_a \frac{\Omega}{2} r^2} r^{\frac{\beta\gamma_a \Gamma}{2\pi}}$. Because of the factor $e^{-\beta\gamma_a \frac{\Omega}{2} r^2}$, the distribution (19) can possibly be normalized only if

$$\mathrm{sgn}(\gamma)\beta\Omega \geq 0. \tag{20}$$

Summing Eq. (19) on the species and substituting the resulting equation in Eq. (18), we obtain the multi-species Boltzmann-Poisson equation

$$-\Delta\psi = \sum_a \Gamma_a \frac{e^{-\beta\gamma_a \psi_{eff}(\mathbf{r})}}{\int e^{-\beta\gamma_a \psi_{eff}(\mathbf{r})}\, d\mathbf{r}}. \tag{21}$$

When $\beta > 0$, the Boltzmann-Poisson equation (21) is similar to the one appearing in the theory of electrolytes and when $\beta < 0$, it is similar to the one appearing in the statistical mechanics of stellar systems [7, 27]. We emphasize that these results are valid both in microcanonical and in canonical ensembles. Furthermore, these equations remain valid in a bounded domain $\mathcal{D}$ in which case the Boltzmann-Poisson equation must be solved with the boundary condition $\psi = 0$ on $\partial\mathcal{D}$. In a disk, the angular momentum is conserved, as in an unbounded domain, and we get exactly the same equations (with different boundary conditions). In a bounded domain, there can be equilibrium states even if the signs of the circulations of the point vortices are different. Of course, if the domain does not possess the rotational invariance, the angular momentum is not conserved and $\Omega = 0$.

C. Maximum entropy state

In the mean field approximation, the entropy, energy, angular momentum and circulation of each species can be written

$$S = -k_B \sum_a \int \frac{\omega_a}{\gamma_a} \ln \frac{\omega_a}{\gamma_a}\, d\mathbf{r}, \quad E = \frac{1}{2}\int \omega\psi\, d\mathbf{r}, \quad L = \int \omega r^2\, d\mathbf{r}, \quad \Gamma_a = \int \omega_a\, d\mathbf{r}. \tag{22}$$

The entropy (22) can be obtained by substituting Eq. (15) in Eq. (8). It can also be obtained from a standard combinatorial analysis starting from the Boltzmann formula $S = k_B \ln W$, where W is the number of *microstates* (complexions) specified by the precise position $\{\mathbf{r}_1, ..., \mathbf{r}_N\}$ of each point vortex, corresponding to a given *macrostate* specified by the smooth vorticity field $\{\omega_a(\mathbf{r})\}$ giving the average number of point vortices of species a in macrocells of size $0 < \Delta \ll 1$. Using the Stirling formula for $N \gg 1$, we obtain the expression (22-a) of the Boltzmann entropy [2, 28]. The energy (22-b) is obtained by substituting Eq. (16) in Eq. (6) and making the approximation $N_a - 1 \simeq N_a$.

In the microcanonical ensemble, the statistical equilibrium state is obtained by maximizing the Boltzmann entropy S while conserving the energy E, the angular momentum L and the total circulation Γ_a of each species

$$S(E, \Gamma_a, L) = \max_{\omega_a} \quad \{S_B[\omega_a] \quad | \quad E[\omega] = E,\ \Gamma_a[\omega_a] = \Gamma_a,\ L[\omega] = L\}. \tag{23}$$

This maximum entropy principle amounts to determining the *most probable state*, i.e. the one that is the most represented at the microscopic level. Introducing Lagrange multipliers to take the constraints into account, and writing the variational problem in the form $\delta S/k_B - \beta\delta E - \beta\frac{\Omega}{2}\delta L - \sum_a \alpha_a \delta\Gamma_a = 0$, we find that the critical points of constrained entropy are given by the mean field Boltzmann distribution (19).

In the canonical ensemble, the statistical equilibrium state is obtained by maximizing the Boltzmann free energy $J = S/k_B - \beta E - \beta\frac{\Omega}{2}L$ (Massieu function) while conserving the total circulation Γ_a of each species

$$J(\beta, \Gamma_a, \Omega) = \max_{\omega_a} \quad \{J_B[\omega_a] \quad | \ \Gamma_a[\omega_a] = \Gamma_a, \}. \tag{24}$$

Writing the variational problem in the form $\delta J - \sum_a \alpha_a \delta\Gamma_a = 0$, we find that the critical points of constrained free energy are also given by the mean field Boltzmann distribution (19). Therefore, the series of equilibria are the same in MCE and CE but the nature of the solutions (maxima, minima, saddle points) may differ in each ensemble. When this happens, we speak of ensemble inequivalence [51].

D. The virial theorem

Taking the logarithmic derivative of Eq. (19), we obtain

$$\nabla\omega_a(\mathbf{r}) = -\beta\gamma_a\omega_a(\mathbf{r})\nabla\psi_{eff}(\mathbf{r}). \tag{25}$$

Dividing Eq. (25) by γ_a, summing on the species and introducing the local "pressure" [25]:

$$p(\mathbf{r}) = n(\mathbf{r})k_B T = k_B T \sum_a \frac{\omega_a(\mathbf{r})}{\gamma_a}, \tag{26}$$

we obtain

$$\nabla p + \omega\nabla\psi_{eff} = \mathbf{0}, \tag{27}$$

which is similar to the condition of hydrostatic equilibrium for material particles in a rotating frame. This equation is valid in MCE and CE. We note that the pressure is positive at positive temperatures and negative at negative temperatures. The equilibrium virial theorem can be derived as follows. Taking the scalar product of Eq. (27) with $\mathbf{r}$, integrating over the entire domain and integrating by parts the pressure term, we get

$$2\int p\, d\mathbf{r} - \mathcal{V} - \Omega L = 2PV, \tag{28}$$

where $\mathcal{V} = \int \omega \mathbf{r} \cdot \nabla\psi\, d\mathbf{r}$ is the "virial of the point vortices" and $P = \frac{1}{2V}\oint p\mathbf{r} \cdot d\mathbf{S}$ the average pressure on the boundary of the domain (if the pressure is uniform on the boundary, with value p_b, then $P = p_b$). Using the isothermal equation of state (26), we can rewrite the virial theorem as

$$2Nk_B T - \mathcal{V} - \Omega L = 2PV. \tag{29}$$

In an unbounded domain and for axisymmetric flows in a disk, we can easily show that $\mathcal{V} = -\Gamma^2/(4\pi)$ (see Appendix D). In that case, the virial theorem can be rewritten

$$PV = Nk_B(T - T_c) - \frac{1}{2}\Omega L, \tag{30}$$

where we have introduced the critical temperature

$$k_B T_c = -\frac{\Gamma^2}{8\pi N}. \tag{31}$$

For a single species system, we have $k_B T_c = -N\gamma^2/8\pi$. For a neutral system ($\Gamma = 0$), we find that $k_B T_c = 0$. The relation (30) is valid both in MCE and CE. If the vorticity field is a Dirac peak at $r = 0$ containing all the point vortices, then $L = P = 0$, implying $T = T_c$. Let us consider particular cases of the virial theorem.

(i) For an axisymmetric flow in a disk with $\Omega = 0$, we obtain the equation of state

$$PV = Nk_B(T - T_c), \tag{32}$$

where $P = p(R) = n(R)k_B T$. If $\Gamma = 0$, this equation reduces to $PV = Nk_B T$ like for a perfect gas. This relation is of course trivial if the flow is spatially homogeneous. On the other hand, recalling that $PV/Nk_B T \geq 0$, the equation of state (32) implies that $T \geq 0$ or $T \leq T_c$ (i.e. $\beta \geq \beta_c$). If $P = 0$ (e.g., if all the point vortices are located at $r = 0$), then $T = T_c$. For a single species system, the equation of state (32) can be directly derived from the analytical solution of the 2D Boltzmann-Poisson equation [41].

(ii) In an unbounded domain, $PV \to 0$ provided that the pressure decreases sufficiently rapidly with the distance. In that case, Eq. (30) reduces to

$$\frac{1}{2}\Omega L = Nk_B(T - T_c). \tag{33}$$

In the single species case, this relation was empirically established by Williamson [42] on the basis of numerical simulations and later explained by Kiessling [43] from the Boltzmann-Poisson equation. This relation was also derived by Pointin & Lundgren [5] from the density of states. Our approach clearly shows that this relation can be interpreted as "the virial theorem of point vortices" [41]. Let us discuss some consequences of the virial theorem (33) and of the inequality (20). We assume that all the vortices have a positive circulation. We first start by the microcanonical ensemble. (i) If $L = 0$, the vorticity distribution is a Dirac peak at $r = 0$ containing all the vortices: $\omega(\mathbf{r}) = \Gamma\delta(\mathbf{r})$. This solution exists only at $T = T_c$ (i.e. $\beta = \beta_c$). (ii) If $L \neq 0$, statistical equilibrium states can possibly exist only for $T > 0$ or $T < T_c$ (i.e. $\beta > \beta_c$). We now consider the canonical ensemble. (i) If $\Omega = 0$, statistical equilibrium states can possibly exist only at $T = T_c$ (i.e. $\beta = \beta_c$). (ii) If $\Omega > 0$, statistical equilibrium states can possibly exist only for $T > 0$ (i.e. $\beta > 0$). (iii) If $\Omega < 0$, statistical equilibrium states can possibly exist only for $T \leq T_c$ (i.e. $\beta_c \leq \beta < 0$). If $T = T_c$ (i.e. $\beta = \beta_c$), the virial theorem implies $L = 0$. In that case, the system forms a Dirac peak at $r = 0$ containing all the vortices: $\omega(\mathbf{r}) = \Gamma\delta(\mathbf{r})$.

Remark: In this paper, we have derived the virial theorem of point vortices in the mean field approximation. Interestingly, when this assumption is relaxed [41], we find that the exact virial theorem in CE keeps the same form as the mean field virial theorem (33), except that the mean field critical temperature is replaced by the exact critical temperature

$$k_B T_c = -\frac{1}{8\pi N}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}\gamma_\alpha\gamma_\beta = -\frac{1}{8\pi N}\left(\Gamma^2 - \sum_{\alpha=1}^{N}\gamma_\alpha^2\right) = -\frac{1}{8\pi N}\left(\Gamma^2 - \Gamma_2\right), \tag{34}$$

where $\Gamma = \sum_a N_a \gamma_a$ is the total circulation and $\Gamma_2 = \sum_a N_a \gamma_a^2$ the "enstrophy". In the single species case, we get $k_B T_c = -(N-1)\gamma^2/(8\pi)$. We see that the mean field critical temperature only differs from the exact critical temperature in CE by the replacement of $N - 1$ by N.

E. Summary

Regrouping the previous results, the equilibrium stream function is determined, in the mean field approximation, by the multi-species Boltzmann-Poisson equation (21). Then, the distribution of each species is determined by the Boltzmann distribution (19). These results are valid in the microcanonical and canonical ensembles. In the canonical ensemble, the temperature and the angular velocity are prescribed. In the microcanonical ensemble, they must be expressed in terms of the energy and angular momentum that are the relevant control parameters (conserved quantities) in that case. This is done by substituting the Boltzmann distribution (19) in the constraints (22). In an unbounded domain, we can use the virial theorem (33) to relate the angular velocity to the temperature:

$$\Omega = \frac{2N}{\beta L}\left(1 - \frac{\beta}{\beta_c}\right), \tag{35}$$

where $\beta_c = -8\pi N/\Gamma^2$. This equation replaces the constraint on the angular momentum. Then, the temperature is determined by the energy (22-b). These equations generalize the equations derived by Lundgren & Pointin [6] for the single species point vortex gas in an unbounded domain. The above procedure just determines critical points of entropy in MCE and critical points of free energy in CE. We must then study the second variations of these functionals to select *maxima* of entropy S and free energy J. In principle, minima or saddle points must be discarded. When several maxima exist for the same values of the constraints, we must distinguish stable (global) and metastable (local) states. Finally, phase transitions and ensemble inequivalence must be considered.

IV. TWO-DIMENSIONAL POINT VORTICES: EVOLUTION OF THE SYSTEM AS A WHOLE

A. BBGKY-like hierarchy and $1/N$ expansion

In order to establish whether the point vortex gas will reach the Boltzmann distribution (19) predicted by statistical mechanics and determine the timescale of the relaxation, in particular its scaling with N, we need to develop a kinetic theory of point vortices. Basically, the evolution of the N-body distribution $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ of the point vortex gas is governed by the Liouville equation

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \mathbf{V}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} = 0, \tag{36}$$

where

$$\mathbf{V}_i = \sum_{j\neq i} \mathbf{V}(j \to i) = \frac{\gamma}{2\pi}\mathbf{z} \times \sum_{j\neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{37}$$

is the velocity of point vortex i due to its interaction with the other vortices. Here, $\mathbf{V}(j \to i)$ denotes the velocity induced by point vortex j on point vortex i. For simplicity, we shall assume that the point vortices have the same circulation γ and that they move in the infinite plane. The Liouville equation (36), which is equivalent to the Hamiltonian system (1)-(2), contains too much information to be of practical use. In general, we are only interested in the one-body or two-body distributions [75]. From the Liouville equation (36) we can construct a complete BBGKY-like hierarchy for the reduced distributions $P_j(\mathbf{r}_1, ..., \mathbf{r}_j, t) = \int P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)\, d\mathbf{r}_{j+1}...d\mathbf{r}_N$. It reads [29]:

$$\frac{\partial P_j}{\partial t} + \sum_{i=1}^{j} \sum_{k=1,k\neq i}^{j} \mathbf{V}(k \to i) \cdot \frac{\partial P_j}{\partial \mathbf{r}_i} + (N-j)\sum_{i=1}^{j} \int \mathbf{V}(j+1 \to i) \cdot \frac{\partial P_{j+1}}{\partial \mathbf{r}_i}\, d\mathbf{r}_{j+1} = 0. \tag{38}$$

This hierarchy of equations is not closed since the equation for the one-body distribution $P_1(\mathbf{r}_1, t)$ involves the two-body distribution $P_2(\mathbf{r}_1, \mathbf{r}_2, t)$, the equation for the two-body distribution $P_2(\mathbf{r}_1, \mathbf{r}_2, t)$ involves the three-body distribution $P_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t)$, and so on... The idea is to close the hierarchy by using a systematic expansion of the solutions in powers of $1/N$ in the thermodynamic limit $N \to +\infty$.

The first two equations of the hierarchy are

$$\frac{\partial P_1}{\partial t}(\mathbf{r}_1, t) + (N-1)\int \mathbf{V}(2 \to 1) \cdot \frac{\partial P_2}{\partial \mathbf{r}_1}(\mathbf{r}_1, \mathbf{r}_2, t)\, d\mathbf{r}_2 = 0. \tag{39}$$

$$\frac{1}{2}\frac{\partial P_2}{\partial t}(\mathbf{r}_1, \mathbf{r}_2, t) + \mathbf{V}(2 \to 1) \cdot \frac{\partial P_2}{\partial \mathbf{r}_1}(\mathbf{r}_1, \mathbf{r}_2, t) + (N-2)\int \mathbf{V}(3 \to 1) \cdot \frac{\partial P_3}{\partial \mathbf{r}_1}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t)\, d\mathbf{r}_3 + (1 \leftrightarrow 2) = 0. \tag{40}$$

We decompose the two- and three-body distributions in the suggestive form

$$P_2(\mathbf{r}_1, \mathbf{r}_2, t) = P_1(\mathbf{r}_1, t)P_1(\mathbf{r}_2, t) + P_2'(\mathbf{r}_1, \mathbf{r}_2, t), \tag{41}$$

$$\begin{aligned} P_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t) &= P_1(\mathbf{r}_1, t)P_1(\mathbf{r}_2, t)P_1(\mathbf{r}_3, t) + P_2'(\mathbf{r}_1, \mathbf{r}_2, t)P_1(\mathbf{r}_3, t) \\ &+ P_2'(\mathbf{r}_1, \mathbf{r}_3, t)P_1(\mathbf{r}_2, t) + P_2'(\mathbf{r}_2, \mathbf{r}_3, t)P_1(\mathbf{r}_1, t) + P_3'(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t). \end{aligned} \tag{42}$$

This is similar to the Mayer [52] decomposition in plasma physics. Substituting Eqs. (41) and (42) in Eqs. (39) and (40) and simplifying some terms, we obtain

$$\frac{\partial P_1}{\partial t}(\mathbf{r}_1, t) + (N-1)\left[\int \mathbf{V}(2 \to 1)P_1(\mathbf{r}_2, t)\, d\mathbf{r}_2\right] \cdot \frac{\partial P_1}{\partial \mathbf{r}_1}(\mathbf{r}_1, t) = -(N-1)\frac{\partial}{\partial \mathbf{r}_1} \cdot \int \mathbf{V}(2 \to 1)P_2'(\mathbf{r}_1, \mathbf{r}_2, t)\, d\mathbf{r}_2, \tag{43}$$

$$\frac{1}{2}\frac{\partial P_2'}{\partial t}(\mathbf{r}_1,\mathbf{r}_2,t)+(N-2)\left[\int \mathbf{V}(3\to 1)P_1(\mathbf{r}_3,t)\,d\mathbf{r}_3\right]\cdot\frac{\partial P_2'}{\partial \mathbf{r}_1}(\mathbf{r}_1,\mathbf{r}_2,t)$$
$$+\left[\mathbf{V}(2\to 1)-\int \mathbf{V}(3\to 1)P_1(\mathbf{r}_3,t)\,d\mathbf{r}_3\right]\cdot\frac{\partial P_1}{\partial \mathbf{r}_1}(\mathbf{r}_1,t)P_1(\mathbf{r}_2,t)+\mathbf{V}(2\to 1)\cdot\frac{\partial P_2'}{\partial \mathbf{r}_1}(\mathbf{r}_1,\mathbf{r}_2,t)$$
$$+(N-2)\left[\int \mathbf{V}(3\to 1)P_2'(\mathbf{r}_2,\mathbf{r}_3,t)\,d\mathbf{r}_3\right]\cdot\frac{\partial P_1}{\partial \mathbf{r}_1}(\mathbf{r}_1,t)$$
$$-\frac{\partial}{\partial \mathbf{r}_1}\cdot\int \mathbf{V}(3\to 1)P_2'(\mathbf{r}_1,\mathbf{r}_3,t)P_1(\mathbf{r}_2,t)\,d\mathbf{r}_3+(N-2)\frac{\partial}{\partial \mathbf{r}_1}\cdot\int \mathbf{V}(3\to 1)P_3'(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,t)\,d\mathbf{r}_3+(1\leftrightarrow 2)=0. \quad (44)$$

Equations (43) and (44) are exact for all N but they are not closed. We shall close these equations by expanding the solutions in powers of $1/N$ for $N\gg 1$. In this limit, the correlation functions $P_j'(\mathbf{r}_1,...,\mathbf{r}_j,t)$ scale like $1/N^{j-1}$. In particular, $P_2'\sim 1/N$ and $P_3'\sim 1/N^2$. On the other hand, $P_1\sim 1$ and $|\mathbf{V}(i\to j)|\sim\gamma\sim 1/N$. We are aiming at obtaining a kinetic equation that is valid at the order $1/N$. Let us consider the terms in Eq. (44) one by one. The first and second terms are of order $1/N$. They represent the transport of the two-body correlation function by the mean flow. The third term represents the effect of "soft" binary collisions between vortices; it is of order $1/N$. If we consider only these first three terms as done in [29], we obtain a kinetic equation that is the counterpart of the Landau equation in plasma physics. The fourth term represents the effect of "hard" binary collisions between vortices. This is the term which, together with the previous ones, gives rise to the Boltzmann equation in the theory of gases. It is of order $1/N^2$ but it may become large at small scales so its effect is not entirely negligible. For example, in plasma physics, hard collisions must be taken into account in order to regularize the logarithmic divergence that appears at small scales in the Landau and Lenard-Balescu equations. In the case of point vortices, there is no divergence at small scales in the kinetic equation that we shall obtain. Therefore, in this paper, we shall ignore the contribution of this term (it will be studied specifically in another paper). The fifth term is of order $1/N$ and it corresponds to collective effects. In plasma physics, this term leads to the Lenard-Balescu equation. It accounts for dynamical Debye shielding and regularizes the divergence at large scales that appears in the Landau equation. We shall take this term into account in the kinetic theory of point vortices in order to obtain a Lenard-Balescu-type kinetic equation from the BBGKY hierarchy. The last two terms are of order $1/N^2$ and they will be neglected. In particular, at the order $1/N$, we can neglect the three-body correlation function. In this way, the hierarchy of equations is closed.

If we introduce the smooth vorticity field $\omega(\mathbf{r}_1,t)=N\gamma P_1(\mathbf{r}_1,t)$ and the two-body correlation function $g(\mathbf{r}_1,\mathbf{r}_2,t)=N^2P_2'(\mathbf{r}_1,\mathbf{r}_2,t)$, we get at the order $1/N$:

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=-\gamma\frac{\partial}{\partial\mathbf{r}_1}\cdot\int\mathbf{V}(2\to 1)g(\mathbf{r}_1,\mathbf{r}_2,t)\,d\mathbf{r}_2, \quad (45)$$

$$\frac{1}{2}\frac{\partial g}{\partial t}(\mathbf{r}_1,\mathbf{r}_2,t)+\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial g}{\partial\mathbf{r}_1}(\mathbf{r}_1,\mathbf{r}_2,t)+\left[\frac{1}{\gamma}\int\mathbf{V}(3\to 1)g(\mathbf{r}_2,\mathbf{r}_3,t)\,d\mathbf{r}_3\right]\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)$$
$$+\frac{1}{\gamma^2}\tilde{\mathbf{V}}(2\to 1)\cdot\frac{\partial}{\partial\mathbf{r}_1}\omega(\mathbf{r}_1,t)\omega(\mathbf{r}_2,t)+(1\leftrightarrow 2)=0. \quad (46)$$

We have introduced the mean velocity in $\mathbf{r}_1$ created by all the vortices

$$\langle\mathbf{V}\rangle(\mathbf{r}_1,t)=\frac{1}{\gamma}\int\mathbf{V}(2\to 1)\omega(\mathbf{r}_2,t)\,d\mathbf{r}_2=-\mathbf{z}\times\nabla\psi(\mathbf{r}_1,t), \quad (47)$$

and the fluctuating velocity created by point vortex 2 on point vortex 1:

$$\tilde{\mathbf{V}}(2\to 1)=\mathbf{V}(2\to 1)-\frac{1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t). \quad (48)$$

We also recall that the exact velocity created by point vortex 2 on point vortex 1 can be written

$$\mathbf{V}(2\to 1)=-\gamma\,\mathbf{z}\times\frac{\partial u_{12}}{\partial\mathbf{r}_1}, \quad (49)$$

where $u_{12}=u(|\mathbf{r}_1-\mathbf{r}_2|)$ is the binary potential of interaction between point vortices. Equations (45) and (46) are exact at the order $O(1/N)$. They form the right basis to develop a kinetic theory of point vortices at this order of approximation.

B. The limit $N \to +\infty$: The 2D Euler equation (collisionless regime)

In the limit $N \to +\infty$ for a fixed time t, the correlations between point vortices can be neglected and the N-body distribution factorizes in a product of N one-body distributions:

$$P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \prod_{i=1}^{N} P_1(\mathbf{r}_i, t). \tag{50}$$

Therefore, for long-range interactions, the mean field approximation is exact at the thermodynamic limit $N \to +\infty$. Substituting the factorization (50) in the Liouville equation (36), and integrating on $\mathbf{r}_2$, $\mathbf{r}_3$, ..., $\mathbf{r}_N$, we find that the smooth vorticity field $\omega(\mathbf{r}, t)$ of the point vortex gas is solution of the 2D Euler equation

$$\frac{\partial \omega}{\partial t}(\mathbf{r}_1, t) + \langle \mathbf{V} \rangle(\mathbf{r}_1, t) \cdot \frac{\partial \omega}{\partial \mathbf{r}_1}(\mathbf{r}_1, t) = 0. \tag{51}$$

This equation also results from Eq. (45) if we neglect the correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t)$ in the right-hand side. The 2D Euler equation describes the *collisionless* evolution of the point vortex gas for times smaller than Nt_D. In practice, N is large so the domain of validity of the 2D Euler equation is huge. The 2D Euler equation is the counterpart of the Vlasov equation in plasma physics and stellar dynamics. It can undergo a process of mixing and violent relaxation towards a quasistationary state (QSS) on a very short timescale, of the order of a few dynamical times t_D. This QSS has the form of a large-scale vortex. Miller [13] and Robert & Sommeria [14] have developed a statistical mechanics of the 2D Euler equation to predict these QSSs. The MRS theory is the counterpart of the Lynden-Bell theory for collisionless stellar systems [24]. The analogy between two-dimensional vortices and stellar systems is developed in [25].

C. The order $O(1/N)$: An exact kinetic equation (collisional regime)

If we want to describe the *collisional* evolution of the point vortex gas, we need to consider finite N effects. Equations (45) and (46) are valid at the order $1/N$ so they describe the evolution of the system on a timescale of order Nt_D. The equation for the evolution of the smooth vorticity field is of the form

$$\frac{\partial \omega}{\partial t}(\mathbf{r}_1, t) + \frac{N-1}{N} \langle \mathbf{V} \rangle(\mathbf{r}_1, t) \cdot \frac{\partial \omega}{\partial \mathbf{r}_1}(\mathbf{r}_1, t) = C\left[\omega(\mathbf{r}_1, t)\right], \tag{52}$$

where $C(\omega)$ is a "collision" term analogous to the one arising in the Boltzmann equation in the theory of gases. In the present context, there are no real collisions between point vortices. The term on the right-hand side of Eq. (45) is due to the development of correlations between vortices. It is induced by the two-body correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t)$ which is determined in terms of the vorticity by Eq. (46). Our aim is to derive a kinetic equation that is valid at the order $1/N$ and that gives the first correction to the 2D Euler equation.

The formal solution to Eq. (46) is

$$g(\mathbf{r}_1, \mathbf{r}_2, t) = -\frac{1}{\gamma^2} \int d\mathbf{r}'_1 \int d\mathbf{r}'_2 \int_0^t dt' U(\mathbf{r}_1, \mathbf{r}'_1, t - t') U(\mathbf{r}_2, \mathbf{r}'_2, t - t')$$
$$\times \left[\tilde{\mathbf{V}}(2' \to 1') \cdot \frac{\partial}{\partial \mathbf{r}'_1} + \tilde{\mathbf{V}}(1' \to 2') \cdot \frac{\partial}{\partial \mathbf{r}'_2}\right] \omega(\mathbf{r}'_1, t') \omega(\mathbf{r}'_2, t'), \tag{53}$$

where the propagator $U(\mathbf{r}_1, \mathbf{r}'_1, t - t')$ satisfies the equation

$$\frac{\partial U}{\partial t}(\mathbf{r}_1, \mathbf{r}'_1, t - t') + \langle \mathbf{V} \rangle(\mathbf{r}_1, t) \cdot \frac{\partial}{\partial \mathbf{r}_1} U(\mathbf{r}_1, \mathbf{r}'_1, t - t') + \left[\int \mathbf{V}(2 \to 1) U(\mathbf{r}_2, \mathbf{r}'_1, t - t')\, d\mathbf{r}_2\right] \frac{\partial \omega}{\partial \mathbf{r}_1}(\mathbf{r}_1, t) = 0, \tag{54}$$

with the initial condition $U(\mathbf{r}_1, \mathbf{r}'_1, 0) = \delta(\mathbf{r}_1 - \mathbf{r}'_1)$. Equation (54) can be viewed as a linearized version of the 2D Euler equation (see Appendix A). Indeed, if we make the replacement $\omega \to \omega + \delta\omega$ in Eq. (A1) and linearize it with respect to $\delta\omega$ [see Eq. (A2)], we obtain Eq. (54) in which U plays the role of $\delta\omega$. Consequently, the propagator U obeys the linearized Euler equation.

Substituting Eq. (53) in Eq. (45), we obtain a kinetic equation

$$\frac{\partial \omega}{\partial t}(\mathbf{r}_1, t) + \frac{N-1}{N} \langle \mathbf{V} \rangle(\mathbf{r}_1, t) \cdot \frac{\partial \omega}{\partial \mathbf{r}_1}(\mathbf{r}_1, t) = \frac{1}{\gamma} \frac{\partial}{\partial \mathbf{r}_1} \cdot \int d\mathbf{r}_2 \int d\mathbf{r}'_1 \int d\mathbf{r}'_2 \int_0^t d\tau U(\mathbf{r}_1, \mathbf{r}'_1, \tau) U(\mathbf{r}_2, \mathbf{r}'_2, \tau)$$
$$\times \mathbf{V}(2 \to 1) \left[\tilde{\mathbf{V}}(2' \to 1') \cdot \frac{\partial}{\partial \mathbf{r}'_1} + \tilde{\mathbf{V}}(1' \to 2') \cdot \frac{\partial}{\partial \mathbf{r}'_2}\right] \omega(\mathbf{r}'_1, t - \tau) \omega(\mathbf{r}'_2, t - \tau), \tag{55}$$

that is exact at the order $1/N$. If we neglect collective effects, we recover the generalized Landau equation (C6) that was derived in our previous articles [27, 29, 30]. Equation (55) is a complicated non-Markovian integrodifferential equation. It is furthermore coupled to Eq. (54) which determines the evolution of the propagator. In order to resolve this coupling, it is necessary to consider the timescales involved in the dynamics. We shall argue that, for a given vorticity profile ω, the two-body correlation function relaxes to its asymptotic form on a timescale short compared with that on which ω changes appreciably. This is the equivalent of the Bogoliubov ansatz in plasma physics. It is expected to be a very good approximation for $N \gg 1$ since the two-body correlation function relaxes on a few dynamical times t_D while the vorticity field changes on a collisional relaxation time of the order Nt_D or larger. Therefore, it is possible to neglect the time variation of $\omega(\mathbf{r}, t-\tau)$ in the calculation of the collision term and extend the time integration to $+\infty$. This amounts to replacing the two-body correlation function in Eq. (45) by its asymptotic value $g(\mathbf{r}_1, \mathbf{r}_2, +\infty)$ for a given vorticity profile ω. After the correlation function has been obtained as a functional of ω, the time dependence of ω can be reinserted. With this Bogoliubov ansatz (or adiabatic hypothesis), the kinetic equation (55) can be rewritten

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=\frac{1}{\gamma}\frac{\partial}{\partial\mathbf{r}_1}\cdot\int d\mathbf{r}_2\int d\mathbf{r}_1'\int d\mathbf{r}_2'\int_0^{+\infty}d\tau U(\mathbf{r}_1,\mathbf{r}_1',\tau)U(\mathbf{r}_2,\mathbf{r}_2',\tau)$$
$$\times\mathbf{V}(2\to1)\left[\tilde{\mathbf{V}}(2'\to1')\cdot\frac{\partial}{\partial\mathbf{r}_1'}+\tilde{\mathbf{V}}(1'\to2')\cdot\frac{\partial}{\partial\mathbf{r}_2'}\right]\omega(\mathbf{r}_1',t)\omega(\mathbf{r}_2',t). \qquad (56)$$

Similarly, the equation for the propagator takes the form

$$\frac{\partial U}{\partial\tau}(\mathbf{r}_1,\mathbf{r}_1',\tau)+\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial}{\partial\mathbf{r}_1}U(\mathbf{r}_1,\mathbf{r}_1',\tau)+\left[\int\mathbf{V}(2\to1)U(\mathbf{r}_2,\mathbf{r}_1',\tau)\,d\mathbf{r}_2\right]\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=0, \qquad (57)$$

with the initial condition $U(\mathbf{r}_1,\mathbf{r}_1',0)=\delta(\mathbf{r}_1-\mathbf{r}_1')$. The two equations (56) and (57) are now completely decoupled. For a *given* vorticity profile $\omega(\mathbf{r},t)$ at time t, one can solve Eq. (57) to obtain $U(\mathbf{r}_1,\mathbf{r}_1',\tau)$ and determine the collision term in the right-hand side of Eq. (56). Then, the vorticity profile $\omega(\mathbf{r},t)$ evolves with time on a slow timescale according to Eq. (56).

The kinetic equation (56) is valid at the order $1/N$ and, for $N\to+\infty$, it reduces to the (smooth) 2D Euler equation (51) which describes the collisionless evolution of the point vortex gas. Interestingly, the structure of the collision term in Eq. (56) bears a clear physical meaning in terms of generalized Kubo relations [29]. The kinetic equation (56) is, of course, equivalent to the pair of equations

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=-\gamma\frac{\partial}{\partial\mathbf{r}_1}\cdot\int\mathbf{V}(2\to1)g(\mathbf{r}_1,\mathbf{r}_2,+\infty)\,d\mathbf{r}_2, \qquad (58)$$

$$g(\mathbf{r}_1,\mathbf{r}_2,+\infty)=-\frac{1}{\gamma^2}\int d\mathbf{r}_1'\int d\mathbf{r}_2'\int_0^{+\infty}d\tau U(\mathbf{r}_1,\mathbf{r}_1',\tau)U(\mathbf{r}_2,\mathbf{r}_2',\tau)$$
$$\times\left[\mathbf{V}(2'\to1')\cdot\frac{\partial}{\partial\mathbf{r}_1'}+\mathbf{V}(1'\to2')\cdot\frac{\partial}{\partial\mathbf{r}_2'}\right]\omega(\mathbf{r}_1',t)\omega(\mathbf{r}_2',t), \qquad (59)$$

which correspond to the first two equations of the BBGKY-like hierarchy at the order $1/N$ within the Bogoliubov ansatz. These equations, supplemented by Eq. (57) for the propagator, provide the formal solution of the problem in the general case. In order to obtain more explicit expressions, we have to consider particular types of flows.

V. EXPLICIT KINETIC EQUATION FOR AXISYMMETRIC FLOWS

A. Laplace-Fourier transforms

We consider an axisymmetric distribution of point vortices that is a stable steady state of the 2D Euler equation. Therefore, the vorticity field evolves in time only because of the development of correlations between point vortices due to finite N effects (graininess). In that case, an explicit form of the kinetic equation can be derived.

For an axisymmetric flow, introducing a system of polar coordinates, the vorticity field and the two-body correlation function can be written as $\omega(\mathbf{r}_1,t)=\omega(r_1,t)$ and $g(\mathbf{r}_1,\mathbf{r}_2,t)=g(r_1,r_2,\theta_1-\theta_2,t)$, and the mean velocity as $\langle\mathbf{V}\rangle(\mathbf{r}_1,t)=\langle V\rangle_\theta(r_1,t)\mathbf{e}_\theta$. On the other hand, according to Eq. (49), the radial velocity (in the direction of $\mathbf{r}_1$) created by point vortex 2 on point vortex 1, is

$$V_{r_1}(2\to1)=\frac{\gamma}{r_1}\frac{\partial u_{12}}{\partial\theta_1}=-\frac{r_2}{r_1}V_{r_2}(1\to2), \qquad (60)$$

where $u_{12} = u(r_1, r_2, \theta_1 - \theta_2)$ is symmetric in r_1 and r_2 and even in $\phi = \theta_1 - \theta_2$ (see Appendix B). In that case, Eqs. (58) and (59) take the form

$$\frac{\partial\omega}{\partial t}(r_1,t) = -\gamma^2 \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2\, dr_2 \int_0^{2\pi} d\theta_2 \frac{\partial u}{\partial\theta_1}(r_1, r_2, \theta_1-\theta_2) g(r_1, r_2, \theta_1-\theta_2, +\infty), \tag{61}$$

$$g(r_1, r_2, \theta_1-\theta_2, +\infty) = -\frac{1}{\gamma}\int r_1' dr_1' d\theta_1' \int r_2' dr_2' d\theta_2' \int_0^{+\infty} d\tau \frac{\partial u}{\partial\theta_1}(r_1', r_2', \theta_1'-\theta_2')\left[\left(\frac{1}{r_1'}\frac{\partial}{\partial r_1'} - \frac{1}{r_2'}\frac{\partial}{\partial r_2'}\right)\omega(r_1')\omega(r_2')\right]$$
$$\times U(r_1, r_1', \theta_1-\theta_1', \tau) U(r_2, r_2', \theta_2-\theta_2', \tau). \tag{62}$$

For convenience, we have not written the time t in the vorticity field $\omega(r,t)$ appearing in the correlation function. As we have previously explained, the vorticity profile is assumed "frozen" on the short timescale that we consider to compute the asymptotic expression of the correlation function and the collision term (Bogoliubov ansatz). The time t will be restored at the end in the kinetic equation.

We now expand the potential of interaction in Fourier series

$$u(r, r', \theta-\theta') = \sum_n e^{in(\theta-\theta')}\hat{u}_n(r, r'), \tag{63}$$

and perform similar expansions for $g(r_1, r_2, \theta_1-\theta_2)$ and $U(r_1, r_1', \theta_1-\theta_1', t)$. In terms of these Fourier transforms, Eqs. (61) and (62) can be rewritten

$$\frac{\partial\omega}{\partial t}(r_1,t) = 2i\pi\gamma^2 \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2\, dr_2 \sum_n n\,\hat{u}_n(r_1, r_2)\hat{g}_n(r_1, r_2, +\infty), \tag{64}$$

$$\hat{g}_n(r_1, r_2, +\infty) = -i\frac{(2\pi)^2}{\gamma}\int_0^{+\infty} r_1' dr_1' \int_0^{+\infty} r_2' dr_2' \int_0^{+\infty} d\tau\, n\,\hat{u}_n(r_1', r_2')\left[\left(\frac{1}{r_1'}\frac{\partial}{\partial r_1'} - \frac{1}{r_2'}\frac{\partial}{\partial r_2'}\right)\omega(r_1')\omega(r_2')\right]$$
$$\times U_n(r_1, r_1', \tau) U_{-n}(r_2, r_2', \tau). \tag{65}$$

Introducing the Laplace transform of $U_n(r_1, r_1', \tau)$ (see Appendix A for the definition of Laplace transforms) and integrating on time τ, we get

$$\hat{g}_n(r_1, r_2, +\infty) = -\frac{1}{\gamma}\int_0^{+\infty} r_1' dr_1' \int_0^{+\infty} r_2' dr_2' \int_C d\sigma \int_C d\sigma' \frac{1}{\sigma+\sigma'} n\hat{u}_n(r_1', r_2')\left[\left(\frac{1}{r_1'}\frac{\partial}{\partial r_1'} - \frac{1}{r_2'}\frac{\partial}{\partial r_2'}\right)\omega(r_1')\omega(r_2')\right]$$
$$\times U_n(r_1, r_1', \sigma) U_{-n}(r_2, r_2', \sigma'), \tag{66}$$

where C is the Laplace contour in the complex σ-plane. The integration over σ' can be performed by closing the contour by an infinite semicircle in the upper half-plane. Since $U_{-n}(r_2, r_2', \sigma')$ vanishes for $|\sigma'| \to +\infty$, the only contribution of the integral comes from the pole at $\sigma' = -\sigma$. Using the residue theorem, we obtain

$$\hat{g}_n(r_1, r_2, +\infty) = -\frac{2\pi i}{\gamma}\int_0^{+\infty} r_1' dr_1' \int_0^{+\infty} r_2' dr_2' \int_{-\infty}^{+\infty} d\sigma\, n\hat{u}_n(r_1', r_2')\left[\left(\frac{1}{r_1'}\frac{\partial}{\partial r_1'} - \frac{1}{r_2'}\frac{\partial}{\partial r_2'}\right)\omega(r_1')\omega(r_2')\right]$$
$$\times U_n(r_1, r_1', \sigma) U_{-n}(r_2, r_2', -\sigma). \tag{67}$$

Finally, substituting Eq. (67) in Eq. (64), the kinetic equation takes the form

$$\frac{\partial\omega}{\partial t}(r_1,t) = (2\pi)^2\gamma \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2\, dr_2 \sum_n n\hat{u}_n(r_1, r_2)\int_0^{+\infty} r_1' dr_1' \int_0^{+\infty} r_2' dr_2' \int_{-\infty}^{+\infty} d\sigma\, n\hat{u}_n(r_1', r_2')$$
$$\times \left[\left(\frac{1}{r_1'}\frac{\partial}{\partial r_1'} - \frac{1}{r_2'}\frac{\partial}{\partial r_2'}\right)\omega(r_1')\omega(r_2')\right] U_n(r_1, r_1', \sigma) U_{-n}(r_2, r_2', -\sigma). \tag{68}$$

On the other hand, for an axisymmetric vorticity distribution $\omega(r)$, the Laplace-Fourier transform of the propagator is explicitly given by (see Appendix A):

$$U_n(r, r', \sigma) = \frac{i}{\sigma - n\Omega(r)}\frac{\delta(r-r')}{2\pi r} + i\frac{G(n, r, r', \sigma)}{(\sigma - n\Omega(r))(\sigma - n\Omega(r'))} n\frac{1}{r}\frac{\partial\omega}{\partial r}(r). \tag{69}$$

The first term on the right-hand side represents the advection by the mean flow, i.e. it corresponds to a pure rotation with angular velocity $\Omega(r)$. The second term takes collective effects into account.

Before going further, some technical details must be given. Since $U_n(r_1, r_1', \sigma)$ is obtained as the Laplace transform of the propagator $U_n(r_1, r_1', \tau)$, this function is analytic in the upper half of the complex σ plane. It is then continued analytically into the lower half-plane where it generally has singularities. The contour of σ integration in Eq. (68) sees all singularities of $U_n(r_1, r_1', \sigma)$ from above. On the other hand, the function $U_{-n}(r_1, r_1', -\sigma)$ which is the complex conjugate to $U_n(r_1, r_1', \sigma)$ is an analytic function in the lower half-plane and is continued analytically into the upper half-plane where it generally has singularities. The integration contour in Eq. (68) sees all the singularities of $U_{-n}(r_1, r_1', -\sigma)$ from below. In order to take these boundary conditions into account, we shall write $\sigma + i0^+$ in place of σ for the functions which are well defined and analytic in the upper half-plane and we use $\sigma - i0^+$ for the functions well defined and analytic in the lower half-plane.

B. Without collective effects: Landau-type equation

Before solving the general case, we first derive a Landau-type equation obtained by neglecting collective effects. In that case, the propagator (69) reduces to

$$U_n(r, r', \sigma) = \frac{i}{\sigma - n\Omega(r)} \frac{\delta(r - r')}{2\pi r}, \tag{70}$$

and the kinetic equation (68) becomes

$$\frac{\partial \omega}{\partial t}(r_1, t) = \gamma \frac{1}{r_1} \frac{\partial}{\partial r_1} \int_0^{+\infty} r_2\, dr_2 \int_{-\infty}^{+\infty} d\sigma \sum_n n^2 \hat{u}_n(r_1, r_2)^2 \frac{1}{(\sigma - n\Omega(r_1) + i0^+)(\sigma - n\Omega(r_2) - i0^+)} \times \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_2} \frac{\partial}{\partial r_2} \right) \omega(r_1)\omega(r_2). \tag{71}$$

The integration on σ may be carried out by closing the contour with an infinite semicircle in the lower half-plane. Only the pole at $\sigma = n\Omega(r_1)$ contributes. Using the residue theorem, we obtain

$$\frac{\partial \omega}{\partial t}(r_1, t) = -2\pi i \gamma \frac{1}{r_1} \frac{\partial}{\partial r_1} \int_0^{+\infty} r_2\, dr_2 \sum_n n^2 \hat{u}_n(r_1, r_2)^2 \frac{1}{n\Omega(r_1) - n\Omega(r_2) - i0^+} \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_2} \frac{\partial}{\partial r_2} \right) \omega(r_1)\omega(r_2). \tag{72}$$

Then, with the aid of the Plemelj formula

$$\frac{1}{x \pm i0^+} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{73}$$

the foregoing expression can be rewritten

$$\frac{\partial \omega}{\partial t}(r_1, t) = 2\pi^2 \gamma \frac{1}{r_1} \frac{\partial}{\partial r_1} \int_0^{+\infty} r_2\, dr_2 \sum_n n^2 \hat{u}_n(r_1, r_2)^2 \delta[n(\Omega(r_1) - \Omega(r_2))] \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_2} \frac{\partial}{\partial r_2} \right) \omega(r_1)\omega(r_2). \tag{74}$$

Finally, using the identity $\delta(\lambda x) = \frac{1}{|\lambda|}\delta(x)$ and putting back the (slow) time dependence in the kinetic equation, we get

$$\frac{\partial \omega}{\partial t}(r_1, t) = 2\pi^2 \gamma \frac{1}{r_1} \frac{\partial}{\partial r_1} \int_0^{+\infty} r_2\, dr_2 \sum_n |n| \hat{u}_n(r_1, r_2)^2 \delta[\Omega(r_1, t) - \Omega(r_2, t)] \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_2} \frac{\partial}{\partial r_2} \right) \omega(r_1, t)\omega(r_2, t). \tag{75}$$

Using Eq. (B4), the series can be explicitly calculated and we obtain the alternative form

$$\frac{\partial \omega}{\partial t}(r_1, t) = 2\pi^2 \gamma \frac{1}{r_1} \frac{\partial}{\partial r_1} \int_0^{+\infty} r_2 dr_2\, \chi(r_1, r_2) \delta\left[\Omega(r_1, t) - \Omega(r_2, t)\right] \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} - \frac{1}{r_2} \frac{\partial}{\partial r_2} \right) \omega(r_1, t)\omega(r_2, t), \tag{76}$$

with

$$\chi(r_1, r_2) = \sum_n |n| \hat{u}_n(r_1, r_2)^2 = \frac{1}{8\pi^2} \sum_{m=1}^{+\infty} \frac{1}{m} \left(\frac{r_<}{r_>} \right)^{2m} = -\frac{1}{8\pi^2} \ln \left[1 - \left(\frac{r_<}{r_>} \right)^2 \right], \tag{77}$$

where $r_> = \max(r_1, r_2)$ and $r_< = \min(r_1, r_2)$. This kinetic equation, which neglects collective effects and takes only two-body encounters into account, is the counterpart of the Landau [31] equation in plasma physics. It was derived in [27, 29, 30] from different formalisms. In plasma physics, the Landau equation presents a logarithmic divergence at large scales. Therefore, collective effects which lead to Debye shielding are crucial because they regularize the logarithmic divergence at large scales. This is essentially what the works of Lenard [33] and Balescu [34] have demonstrated. Since the kinetic equation (75) does not present any divergence, the neglect of collective effects may not be crucial for point vortices.

C. With collective effects: Lenard-Balescu-type equation

If we take collective effects into account, the kinetic equation is obtained by substituting the expression (69) of the propagator in Eq. (68) and carrying out the integrations. The calculations are similar to those developed by Ichimaru [53] in his derivation of the Lenard-Balescu equation from the BBGKY hierarchy. They are detailed in [40]. The final result is

$$\frac{\partial \omega}{\partial t}(r_1, t) = 2\pi^2 \gamma \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2\, dr_2 \sum_n |n| |G(n, r_1, r_2, n\Omega(r_1,t))|^2 \delta[\Omega(r_1,t) - \Omega(r_2,t)] \times \left(\frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{1}{r_2}\frac{\partial}{\partial r_2}\right)\omega(r_1,t)\omega(r_2,t). \tag{78}$$

This can be rewritten

$$\frac{\partial \omega}{\partial t}(r_1, t) = 2\pi^2 \gamma \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2 dr_2\, \chi(r_1, r_2, \Omega(r_1,t))\delta\left[\Omega(r_1,t) - \Omega(r_2,t)\right]\left(\frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{1}{r_2}\frac{\partial}{\partial r_2}\right)\omega(r_1,t)\omega(r_2,t), \tag{79}$$

with

$$\chi(r_1, r_2) = \sum_n |n| |G(n, r_1, r_2, n\Omega(r_1, t))|^2. \tag{80}$$

This kinetic equation, which properly takes collective effects into account, is the counterpart of the Lenard-Balescu equation in plasma physics. It was derived in [32, 39] from the Klimontovich formalism. We have here provided an alternative derivation of this equation from the BBGKY hierarchy. Note that the Lenard-Balescu-type equation (78) differs from the Landau-type equation (75) only by the replacement of the "bare" potential of interaction $\hat{u}_n(r_1, r_2)$ by the "dressed" potential of interaction $G(n, r_1, r_2, n\Omega(r_1, t))$ [defined by Eq. (A13)] taking the contribution of the polarization cloud into account. This is similar to the case of plasma physics.

D. Numerical resolution of the kinetic equation: No relaxation towards the Boltzmann distribution

The kinetic equation (79) is valid at the order $1/N$ so it describes the "collisional" evolution of the point vortex gas on a timescale of order Nt_D. This kinetic equation conserves the circulation Γ, the energy E and the angular momentum L. It also monotonically increases the Boltzmann entropy in the sense that $\dot{S}_B \geq 0$ (H-theorem). These properties are proven in [28, 32]. The change of the vorticity distribution in r_1 is due to a condition of resonance (encapsulated in the δ-function) between vortices located in r_1 and vortices located in $r_2 \neq r_1$ which rotate with the *same* angular velocity $\Omega(r_2, t) = \Omega(r_1, t)$ (the self-interaction at $r_2 = r_1$ does not produce transport since the term in parenthesis vanishes identically). Of course, this condition can be satisfied only when the profile of angular velocity is non-monotonic. The collisional evolution of the point vortices is thus truly due to long-range interactions since the current in r_1 is caused by "distant collisions" with vortices located in $r_2 \neq r_1$ that can be far away. This is different from the case of plasma physics and stellar dynamics where the collisions are assumed to be *local* in space. The mean field Boltzmann distribution (19) is a particular steady state of Eq. (79) but it is not the only one: The kinetic equation (79) admits an infinite number of steady states. Indeed, all the vorticity profiles $\omega(r)$ associated with a monotonic profile of angular velocity $\Omega(r)$ are steady states of the kinetic equation (79) since the δ-function is zero for these profiles. Therefore, the collisional evolution of the point vortex gas described by Eq. (79) stops when the profile of angular velocity becomes monotonic (so there is no resonance) even if the system has not reached the Boltzmann distribution. In that case, the system settles on a QSS that is *not* the most mixed state predicted by statistical mechanics (see Fig. 1) [28]. On the timescale Nt_D on which the kinetic theory is valid, the collisions tend to create a monotonic profile of angular velocity. Since the entropy increases monotonically, the vorticity profile *tends*

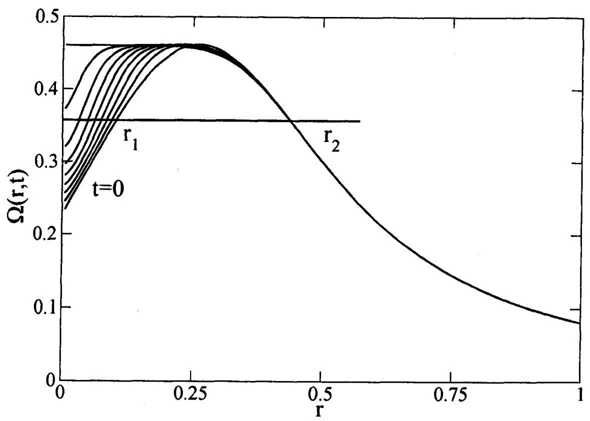

FIG. 1: Evolution of the profile of angular velocity obtained by solving numerically the kinetic equation (76) [28]. Time increases from bottom to top. The horizontal line shows the resonance between point vortices in r_1 and r_2 which rotate with the same angular velocity. The evolution stops when the profile of angular velocity becomes monotonic so there is no resonance anymore. The final profile does *not* correspond to the Boltzmann distribution (19). The kinetic equation (76) is valid on a timescale Nt_D. Therefore, on this timescale the vortex gas does not reach statistical equilibrium but remains "blocked" in a QSS with a monotonic profile of angular velocity. The Boltzmann distribution (19) may be reached on a longer timescale (due to higher order correlations) but this timescale of relaxation is not known. It is not even clear whether the point vortex gas will ever relax towards the statistical equilibrium distribution (19) in the axisymmetric case.

to approach the Boltzmann distribution (the system becomes "more mixed") but does not attain it in general because of the absence of resonance. This is particularly obvious if we start from an initial condition with a monotonic profile of angular velocity that is non-Boltzmannian. In that case, the collision term vanishes, so that $\partial\omega/\partial t = 0$ meaning that there is no evolution on the timescale Nt_D. The Boltzmann distribution may be reached on longer timescales, larger than Nt_D. To describe this regime, we need to determine terms of order N^{-2}, or smaller, in the expansion of the solutions of the BBGKY hierarchy for $N \to +\infty$. This implies in particular the determination of the three-body, or higher, correlation functions, which is a formidable task. At the moment, we can only conclude from the kinetic theory that, for an axisymmetric distribution of point vortices, the relaxation time satisfies

$$t_R > Nt_D. \tag{81}$$

The collisional relaxation towards the Boltzmann distribution (19) is therefore a very slow process. In fact, up to now, there is no rigorous proof coming from the kinetic theory that the point vortex gas will ever relax towards the Boltzmann distribution predicted by statistical mechanics. Indeed, the point vortices may not mix well enough through the action of "collisions". Therefore, the relaxation (or not) of the vorticity profile towards the Boltzmann distribution (19) for $t \to +\infty$ still remains an open problem. This is at variance with the Landau [31] and Lenard-Balescu [33, 34] equations of plasma physics which always converge towards the Boltzmann distribution (it is the unique steady state of these equations). Therefore, in plasma physics, it is sufficient to develop the kinetic theory at the order $1/N$. The kinetic theory of point vortices is more complicated since it requires to go to higher order in the expansion in power of $1/N$. This is similar to the case of spatially homogeneous one-dimensional systems with long-range interactions, such as one-dimensional plasmas [54, 55] and the HMF model [56, 57], for which the Lenard-Balescu collision term also vanishes at the order $1/N$. Therefore, $t_R > Nt_D$ for these systems. For spatially homogeneous one-dimensional plasmas, the relaxation time is found numerically to scale like $t_R \sim N^2 t_D$ [58, 59] which is the next order term in the expansion of the BBGKY hierarchy in powers of $1/N$ (here N represents the number of charges in the Debye sphere). For the spatially homogeneous HMF model, the scaling of the relaxation time with N is still controversial and different scalings such as $t_R \sim N^2 t_D$ [60] or $t_R \sim e^N t_D$ [61] have been reported. An interesting problem would be to determine numerically the scaling with N of the relaxation time of an axisymmetric distribution of point vortices. If the collision term does not vanish at the next order of the $1/N$ expansion, this would imply a timescale scaling like $N^2 t_D$, but this scaling has to be ascertained (this project is currently under way).

Note, finally, that the above results are only valid for axisymmetric distributions of point vortices. Non-axisymmetric distributions are described by a more complex kinetic equation (56) which may present new resonances allowing the system to reach the Boltzmann distribution on a timescale of the order $t_R \sim Nt_D$ (the natural first order of the kinetic theory). A linear Nt_D scaling is indeed observed numerically for the relaxation of a non-axisymmetric distribution of point vortices [62]. However, very little is known concerning the properties of Eq. (56) and its convergence (or not) towards the Boltzmann distribution. It could approach the Boltzmann distribution (since entropy increases) without reaching it exactly. New resonances also appear for spatially inhomogeneous one dimensional systems with long-range interactions [30, 63]. This may explain the linear Nt_D scaling of the relaxation time observed numerically for spatially inhomogeneous one dimensional stellar systems [64–67] and for the spatially inhomogeneous HMF model [68]. On the other hand, for the HMF model, Yamaguchi *et al.* [69] find a relaxation time scaling like $N^{\delta}t_D$ with $\delta \simeq 1.7$. In their simulation, the initial distribution function is spatially homogeneous but the collisional evolution makes it Vlasov unstable so that it becomes spatially inhomogeneous. This corresponds to a dynamical phase transition from a non-magnetized to a magnetized state as theoretically studied in [70]. In that case, the relaxation time could be *intermediate* between $N^2 t_D$ (permanently homogeneous) and Nt_D (permanently inhomogeneous). This argument (leading to $1 < \delta < 2$) may provide a first step towards the explanation of the anomalous exponent $\delta \simeq 1.7$ reported in [69]. The same phenomenon (loss of Euler stability due to "collisions" and dynamical phase transition from an axisymmetric distribution to a non-axisymmetric distribution) could happen for the point vortex system.

VI. RELAXATION OF A TEST VORTEX IN A BATH: THE FOKKER-PLANCK EQUATION

In the previous sections, we have studied the evolution of the system "as a whole". In that approach, all the vortices are treated on the same footing. We now consider the relaxation of a "test" vortex (tagged particle) evolving in a steady distribution of "field" vortices. If the field vortices are at statistical equilibrium, described by the Boltzmann distribution (19), their density profile does not evolve at all. In that case, they form a thermal bath. However, we shall also consider the case of an out-of-equilibrium (i.e. non-Boltzmannian) bath corresponding to a vorticity distribution $\omega(r)$ that is a stable steady state of the 2D Euler equation with a monotonic profile of angular velocity. As we have previously seen, this distribution does not change on a timescale of order Nt_D. Since the relaxation time of a test vortex in a bath is of order $(N/\ln N)t_D$ (see below), we can consider that the distribution of the field vortices is "frozen" on this timescale. They form therefore an out-of-equilibrium bath.

Let us call $P(\mathbf{r},t)$ the probability density of finding the test vortex at position $\mathbf{r}$ at time t. For simplicity, we consider axisymmetric distributions (some results valid in the general case can be found in [29]). The evolution of $P(r,t)$ can be obtained from the kinetic equation (79) by considering that the distribution of the field vortices, described by the vorticity profile $\omega(r_2,t)$, is *fixed* [76]. In the BBGKY hierarchy, this amounts to specializing a particular point vortex in the system (the test vortex described by the variable 1) and assuming that the other vortices (the field vortices described by the running variable 2) are in a steady state. If we replace $\omega(r_1,t)$ by $P(r,t)$ and $\omega(r_2,t)$ by $\omega(r')$, we get

$$\frac{\partial P}{\partial t}(r,t) = 2\pi^2\gamma\frac{1}{r}\frac{\partial}{\partial r}\int_0^{+\infty} r'dr'\chi(r,r',\Omega(r))\,\delta(\Omega(r)-\Omega(r'))\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r'}\frac{d}{dr'}\right)P(r,t)\omega(r'). \tag{82}$$

This procedure transforms the integrodifferential equation (79) describing the evolution of the system "as a whole" into a differential equation (82) describing the evolution of a test vortex in a bath of field vortices. Equation (82) can be written in the form of a Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(D\frac{\partial P}{\partial r} - PV_r^{pol}\right)\right], \tag{83}$$

involving a diffusion coefficient

$$D(r) = \frac{2\pi^2\gamma}{r^2}\int_0^{+\infty} r'dr'\chi(r,r',\Omega(r))\,\delta(\Omega(r)-\Omega(r'))\omega(r'), \tag{84}$$

and a drift term due to the polarization

$$V_r^{pol}(r) = \frac{2\pi^2\gamma}{r}\int_0^{+\infty} dr'\chi(r,r',\Omega(r))\,\delta(\Omega(r)-\Omega(r'))\frac{d\omega}{dr}(r'). \tag{85}$$

Physically, the diffusion coefficient is due to the fluctuations of the velocity field produced by a discrete number of point vortices; it can be directly derived from the Kubo formula [27]. On the other hand, the drift arises from the retroaction of the perturbation on the field vortices induced by the test vortex, just like in a polarization process; it can be directly derived from a linear response theory [26]. In the present case, the coefficients of diffusion and drift depend on the position r of the test vortex. Hence, it is more appropriate to write Eq. (82) in a form that is fully consistent with the general Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\frac{1}{2r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{\langle(\Delta r)^2\rangle}{\Delta t}P\right)\right]-\frac{1}{r}\frac{\partial}{\partial r}\left(rP\frac{\langle\Delta r\rangle}{\Delta t}\right), \tag{86}$$

with

$$\frac{\langle(\Delta r)^2\rangle}{2\Delta t}=D,\qquad \frac{\langle\Delta r\rangle}{\Delta t}\equiv V_r^{drift}=\frac{dD}{dr}+V_r^{pol}. \tag{87}$$

Substituting Eqs. (84) and (85) in Eq. (87), and using an integration by parts, we find that the total drift term is given by

$$V_r^{drift}(r)=2\pi^2\gamma\int_0^{+\infty}rr'dr'\omega(r')\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r'}\frac{d}{dr'}\right)\chi(r,r',\Omega(r))\,\delta(\Omega(r)-\Omega(r'))\frac{1}{r^2}. \tag{88}$$

The two expressions (83) and (86) of the Fokker-Planck equation have their own interest. The expression (86) where the diffusion coefficient is placed after the second derivative $\partial^2(DP)$ involves the total drift $\mathbf{V}_{drift}$ and the expression (83) where the diffusion coefficient is placed between the derivatives $\partial D\partial P$ isolates the part of the drift $\mathbf{V}_{pol}$ due to the polarization. This expression is directly connected to the form of the Lenard-Balescu-type equation (79). It has therefore a clear physical interpretation. The Fokker-Planck equation (82) and the expressions (84), (85) and (88) of the diffusion coefficient and drift term can also be obtained directly by calculating the first and second moments of the increment of radial position of the test vortex using the Klimontovich approach [39].

If the profile of angular velocity of the field vortices $\Omega(r)$ is monotonic, we can use the identity $\delta(\Omega(r)-\Omega(r'))=\delta(r-r')/|\Omega'(r)|$ and we find that

$$D(r)=2\pi^2\gamma\frac{\chi(r,r,\Omega(r))}{|\Sigma(r)|}\omega(r), \tag{89}$$

and

$$V_r^{pol}(r)=2\pi^2\gamma\frac{\chi(r,r,\Omega(r))}{|\Sigma(r)|}\frac{d\omega}{dr}(r), \tag{90}$$

where $\Sigma(r)=r\Omega'(r)$ is the local shear. If we neglect collective effects, we can replace $\chi(r,r,\Omega(r))$ by

$$\chi(r,r)=\sum_n |n|\hat{u}_n^2(r,r)=\frac{1}{8\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n}\equiv\frac{1}{8\pi^2}\ln\Lambda, \tag{91}$$

where $\ln\Lambda\equiv\sum_{n=1}^{+\infty}\frac{1}{n}$ is a Coulomb factor [77] that has to be regularized with appropriate cut-offs as discussed in [26, 28, 38]. It is then found that $\ln\Lambda\sim\frac{1}{2}\ln N$ in the thermodynamic limit $N\to+\infty$. We note that the diffusion coefficient and the drift due to the polarization are inversely proportional to the shear. Furthermore, the diffusion coefficient is proportional the vorticity profile of the field vortices while the drift is proportional to its gradient. Comparing Eqs. (89) and (90), we find that the drift velocity is related to the diffusion coefficient by the relation [27]:

$$V_r^{pol}(r)=D(r)\frac{d\ln\omega}{dr}(r). \tag{92}$$

This can be viewed as a generalized form of Einstein relation for an out-of-equilibrium distribution of field vortices. Combining the previous results, we find that the Fokker-Planck equation (83) can be written

$$\frac{\partial P}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\left(\frac{\partial P}{\partial r}-P\frac{d\ln\omega}{dr}\right)\right], \tag{93}$$

with a diffusion coefficient given by Eq. (89).

If the field vortices are at statistical equilibrium (thermal bath), their vorticity profile is the Boltzmann distribution

$$\omega(r)=A\gamma e^{-\beta\gamma\psi_{eff}(r)}, \tag{94}$$

where $\psi_{eff}(r) = \psi(r) + \frac{\Omega_L}{2}r^2$ is the relative stream function (see Sec. III B). We have denoted by Ω_L the Lagrange multiplier associated with the conservation of angular momentum in order to distinguish it from the angular velocity $\Omega(r)$. We first note the identity

$$\frac{d\omega}{dr}(r') = -\beta\gamma\omega(r')\frac{d\psi_{eff}}{dr}(r') = \beta\gamma\omega(r')(\Omega(r') - \Omega_L)r', \tag{95}$$

where we have used $\Omega(r) = -(1/r)d\psi/dr$. Substituting this relation in Eq. (85), using the δ-function to replace $\Omega(r')$ by $\Omega(r)$, using $\Omega(r) - \Omega_L = -(1/r)d\psi_{eff}/dr$ and comparing the resulting expression with Eq. (84), we finally obtain

$$V_r^{pol}(r) = -D(r)\beta\gamma\frac{d\psi_{eff}}{dr}(r). \tag{96}$$

The drift velocity $\mathbf{V}_{pol} = -D\beta\gamma\nabla\psi_{eff}$ is perpendicular to the relative mean field velocity $\langle \mathbf{V}_{eff}\rangle = -\mathbf{z}\times\nabla\psi_{eff}$. Furthermore, the drift coefficient (or mobility) satisfies an Einstein relation $\xi(r) = D(r)\beta\gamma$. We recall that the drift coefficient and the diffusion coefficient depend on the position r of the test vortex and that the temperature is negative in cases of physical interest. We also stress that the Einstein relation is valid for the drift $\mathbf{V}_{pol}$ due to the polarization, not for the total drift $\mathbf{V}_{drift}$ which has a more complicated expression. We do not have this subtlety for the usual Brownian motion where the diffusion coefficient is constant. For a thermal bath, using Eq. (96), the Fokker-Planck equation (83) can be written

$$\frac{\partial P}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\left(\frac{\partial P}{\partial r} + \beta\gamma P\frac{d\psi_{eff}}{dr}\right)\right], \tag{97}$$

where $D(r)$ is given by Eq. (84) with Eq. (94). Of course, if the profile of angular velocity of the Boltzmann distribution is monotonic, we find that Eq. (93) with Eq. (94) returns Eq. (97) with a diffusion coefficient given by Eq. (89) with Eq. (94). Finally, we note that the systematic drift $\mathbf{V}_{pol} = -D\beta\gamma\nabla\psi_{eff}$ of a point vortex [26] is the counterpart of the dynamical friction $\mathbf{F}_{pol} = -D_{\|}\beta m\mathbf{v}$ of a star [71]. Similarly, the Smoluchowski-type form of the Fokker-Planck equation (97) describing the relaxation a point vortex in a "sea" of field vortices [26] is the counterpart of the Kramers-type form of the Fokker-Planck equation describing the relaxation of a star in a globular cluster [71]. This is an aspect of the numerous analogies that exist between two-dimensional vortices and stellar systems [25].

The Fokker-Planck equations (93) and (97) have been studied for different types of bath distribution in [28]. The distribution of the test vortex $P(\mathbf{r}, t)$ relaxes towards the distribution of the bath $\omega(\mathbf{r})/\Gamma$ on a typical timescale

$$t_R^{bath} \sim \frac{N}{\ln N}t_D, \tag{98}$$

where the logarithmic correction comes from the scaling with N of the Coulombian factor $\ln\Lambda$. However, the relaxation process towards the steady state is very peculiar and differs from the usual exponential relaxation of Brownian particles. In particular, the evolution of the front profile in the tail of the distribution is very slow, scaling like $(\ln t)^{1/2}$, and the temporal correlation function $\langle r(0)r(t)\rangle$ decreases algebraically, like $\ln t/t$ (for a thermal bath), instead of exponentially. This is due to the rapid decay of the diffusion coefficient $D(r)$ for large r. Similar results had been found earlier for the HMF model [56, 72].

Finally, we stress that the evolution of the system "as a whole" is very different from the evolution of a test vortex in a bath. We have seen in Sec. V D that the relaxation time of the system as a whole is strictly larger than Nt_D (for axisymmetric distributions) while the relaxation time of a test vortex in a bath is of order $(N/\ln N)t_D$. In particular, a steady state of the 2D Euler equation with a monotonic profile of angular velocity does not change on this timescale. This justifies our procedure of developing a bath approximation for out-of-equilibrium (i.e. non-Boltzmannian) distributions of the field vortices.

VII. CONCLUSION

We have developed a kinetic theory of 2D point vortices, taking two-body correlations and collective effects into account. The theory is valid at the order $1/N$. An important conclusion of our study is that the relevance of the Boltzmann distribution is not established by present-day kinetic theories. More precisely, we have shown that, if the point vortex gas ever relaxes towards the Boltzmann statistical equilibrium state, this takes place on a very long time, larger than Nt_D (for axisymmetric distributions). The collisional relaxation of point vortices is therefore a very slow process and requires to take into account three-body, four-body... correlations, of order $1/N^2$, $1/N^3$..., that are ignored so far in the kinetic theory. It is also possible that mixing by "collisions" is not sufficient to drive the system towards the Boltzmann distribution. We emphasize, however, that the point vortex gas can rapidly reach a QSS as a result of a "collisionless" violent relaxation. However, this QSS is described by the Miller-Robert-Sommeria, or Lynden-Bell, distribution, not by the Boltzmann distribution.

To make the presentation simpler, we have developed the kinetic theory of point vortices in the case where the point vortices have the same circulation γ. However, it is straightforward to generalize the formalism of the kinetic theory to a multi-species system. The implications of the kinetic theory for a multi-species system are relatively interesting. This is discussed in detail in [28, 38, 44].

APPENDIX A: THE LINEARIZED 2D EULER EQUATION

The 2D Euler equation can be written

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \tag{A1}$$

where $\mathbf{u} = -\mathbf{z} \times \nabla \psi$ is the velocity field ($\mathbf{z}$ is a unit vector normal to the plane of the flow). The stream function $\psi(\mathbf{r}, t)$ is related to the vorticity $\omega(\mathbf{r}, t)$ by the Poisson equation $\Delta \psi = -\omega$. The potential of interaction $u(\mathbf{r}, \mathbf{r}')$, which is the Green function of the Laplacian operator Δ, is defined by $\Delta u(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$. In an unbounded domain, $u = u(|\mathbf{r}-\mathbf{r}'|)$ only depends on the absolute distance between two points and is given by $u(|\mathbf{r}-\mathbf{r}'|) = -(1/2\pi) \ln |\mathbf{r}-\mathbf{r}'|$. This corresponds to a Newtonian (or Coulombian) interaction in two dimensions. Therefore, the stream function is related to the vorticity field by an expression of the form $\psi(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|)\omega(\mathbf{r}', t)\, d\mathbf{r}$. This can be written as a product of convolution $\psi = u * \omega$.

Let us consider a small perturbation $\delta\omega(\mathbf{r}, t)$ around a steady state $\omega(\mathbf{r})$ of the 2D Euler equation. We write $\omega(\mathbf{r}, t) = \omega(\mathbf{r}) + \delta\omega(\mathbf{r}, t)$, $\psi(\mathbf{r}, t) = \psi(\mathbf{r}) + \delta\psi(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}) + \delta\mathbf{u}(\mathbf{r}, t)$. Substituting these decompositions in Eq. (A1) and neglecting the quadratic terms, we obtain the linearized Euler equation

$$\frac{\partial \delta\omega}{\partial t} + \mathbf{u} \cdot \nabla \delta\omega + \delta\mathbf{u} \cdot \nabla \omega = 0. \tag{A2}$$

If we restrict ourselves to axisymmetric mean flows, then $\mathbf{u}(\mathbf{r}) = u(r)\mathbf{e}_\theta$ with $u(r) = -\frac{\partial \psi}{\partial r}(r) = \Omega(r) r$, where $\Omega(r) = \frac{1}{r^2} \int_0^r \omega(r') r'\, dr'$ is the angular velocity (see Sec. 6.1 of [28]). In that case, Eq. (A2) becomes

$$\frac{\partial \delta\omega}{\partial t} + \Omega(r) \frac{\partial \delta\omega}{\partial \theta} + \frac{1}{r} \frac{\partial \delta\psi}{\partial \theta} \frac{\partial \omega}{\partial r} = 0. \tag{A3}$$

To solve the initial value problem, it is convenient to introduce Fourier-Laplace transforms. The Fourier-Laplace transform of the perturbation of the vorticity field $\delta\omega$ is defined by

$$\delta\tilde{\omega}(n, r, \sigma) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{+\infty} dt\, e^{-i(n\theta - \sigma t)} \delta\omega(\theta, r, t). \tag{A4}$$

This expression for the Laplace transform is valid for $\mathrm{Im}(\sigma)$ sufficiently large. For the remaining part of the complex σ plane, it is defined by an analytic continuation. The inverse transform is

$$\delta\omega(\theta, r, t) = \sum_{n=-\infty}^{+\infty} \int_C \frac{d\sigma}{2\pi}\, e^{i(n\theta - \sigma t)} \delta\tilde{\omega}(n, r, \sigma), \tag{A5}$$

where the Laplace contour C in the complex σ plane must pass above all poles of the integrand. Similar expressions hold for the perturbation of the stream function $\delta\psi$. If we take the Fourier-Laplace transform of the linearized Euler equation (A3), we obtain

$$-\delta\hat{\omega}(n, r, 0) - i\sigma\delta\tilde{\omega}(n, r, \sigma) + in\Omega(r)\delta\tilde{\omega}(n, r, \sigma) + in\frac{1}{r}\frac{\partial \omega}{\partial r}\delta\tilde{\psi}(n, r, \sigma) = 0, \tag{A6}$$

where the first term is the spatial Fourier transform of the initial value

$$\delta\hat{\omega}(n, r, 0) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-in\theta} \delta\omega(\theta, r, 0). \tag{A7}$$

The foregoing equation can be rewritten

$$\delta\tilde{\omega}(n,r,\sigma)=\frac{n\frac{1}{r}\frac{\partial\omega}{\partial r}}{\sigma-n\Omega(r)}\delta\tilde{\psi}(n,r,\sigma)-\frac{\delta\hat{\omega}(n,r,0)}{i(\sigma-n\Omega(r))},\tag{A8}$$

where the first term on the right hand side corresponds to "collective effects" and the second term is related to the initial condition. The perturbation of the stream function is related to the perturbation of the vorticity by the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\delta\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\delta\psi}{\partial\theta^2}=-\delta\omega.\tag{A9}$$

Taking the Fourier-Laplace transform of this equation, we obtain

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right]\delta\tilde{\psi}(n,r,\sigma)=-\delta\tilde{\omega}(n,r,\sigma).\tag{A10}$$

Substituting Eq. (A8) in Eq. (A10), we find that

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{n^2}{r^2}+\frac{n\frac{1}{r}\frac{\partial\omega}{\partial r}}{\sigma-n\Omega(r)}\right]\delta\tilde{\psi}(n,r,\sigma)=\frac{\delta\hat{\omega}(n,r,0)}{i(\sigma-n\Omega(r))}.\tag{A11}$$

Therefore, the Fourier-Laplace transform of the perturbation of the stream function is related to the initial condition by

$$\delta\tilde{\psi}(n,r,\sigma)=-\int_0^{+\infty}2\pi r'\,dr'\,G(n,r,r',\sigma)\frac{\delta\hat{\omega}(n,r',0)}{i(\sigma-n\Omega(r'))},\tag{A12}$$

where the Green function $G(n,r,r',\sigma)$ is defined by

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{n^2}{r^2}+\frac{n\frac{1}{r}\frac{\partial\omega}{\partial r}}{\sigma-n\Omega(r)}\right]G(n,r,r',\sigma)=-\frac{\delta(r-r')}{2\pi r}.\tag{A13}$$

If we neglect collective effects in the foregoing equation, we obtain

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right]G_{bare}(n,r,r')=-\frac{\delta(r-r')}{2\pi r}.\tag{A14}$$

Therefore, $G_{bare}(n,r,r')=\hat{u}_n(r,r')$ is the Fourier transform of the "bare" potential of interaction u that is solution of the Poisson equation $\Delta u=-\delta$, while $G(n,r,r',\sigma)$ is the Laplace-Fourier transform of the potential of interaction "dressed" by its polarization cloud, i.e. taking collective effects into account.

Using the previous formulae, we can now relate the Laplace-Fourier transform $\delta\tilde{\omega}(n,r,\sigma)$ of the perturbation of the vorticity to the initial condition $\delta\hat{\omega}(n,r,0)$. Substituting Eq. (A12) in Eq. (A8), we obtain

$$\delta\tilde{\omega}(n,r,\sigma)=-\frac{n\frac{1}{r}\frac{\partial\omega}{\partial r}}{\sigma-n\Omega(r)}\int_0^{+\infty}2\pi r''\,dr''\,G(n,r,r'',\sigma)\frac{\delta\hat{\omega}(n,r'',0)}{i(\sigma-n\Omega(r''))}-\frac{\delta\hat{\omega}(n,r,0)}{i(\sigma-n\Omega(r))}.\tag{A15}$$

If we consider an initial condition of the form $\delta\omega(r,\theta,0)=\gamma\delta(\theta-\theta')\delta(r-r')/r$, implying

$$\delta\hat{\omega}(n,r,0)=\frac{\gamma}{2\pi r}e^{-in\theta'}\delta(r-r'),\tag{A16}$$

we find that

$$\delta\tilde{\omega}(n,r,\sigma)=-\frac{n\frac{1}{r}\frac{\partial\omega}{\partial r}}{\sigma-n\Omega(r)}\gamma e^{-in\theta'}\frac{G(n,r,r',\sigma)}{i(\sigma-n\Omega(r'))}-\gamma\frac{e^{-in\theta'}}{2\pi r}\frac{\delta(r-r')}{i(\sigma-n\Omega(r))}.\tag{A17}$$

As shown in Sec. V, this equation gives the expression of the propagator $U_n(r,r',\sigma)$ [see Eq. (69)]. This operator is also called the resolvent operator as it connects $\delta\tilde{\omega}(n,r,\sigma)$ to its initial value:

$$\delta\tilde{\omega}(n,r,\sigma)=\int_0^{+\infty}U_n(r,r'',\sigma)\delta\hat{\omega}(n,r'',0)\,2\pi r''\,dr''.\tag{A18}$$

APPENDIX B: AN INTEGRAL EQUATION

The perturbation of the stream function is related to the perturbation of the vorticity by an expression of the form

$$\delta\psi(\mathbf{r},t)=\int u(|\mathbf{r}-\mathbf{r}'|)\delta\omega(\mathbf{r}',t)\,d\mathbf{r}. \tag{B1}$$

The potential of interaction can be written

$$u(|\mathbf{r}-\mathbf{r}'|)=u\left(\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\right)=u(r,r',\phi), \tag{B2}$$

where $\phi=\theta-\theta'$. We note that $u(r,r'\phi)=u(r',r,\phi)$ and $u(r,r',-\phi)=u(r,r',\phi)$. Due to its ϕ-periodicity, the potential of interaction can be decomposed in Fourier series as

$$u(r,r',\phi)=\sum_n e^{in\phi}\hat{u}_n(r,r'),\qquad \hat{u}_n(r,r')=\int_0^{2\pi}\frac{d\phi}{2\pi}u(r,r',\phi)\cos(n\phi). \tag{B3}$$

For the Coulombian potential $u(|\mathbf{r}-\mathbf{r}'|)=-(1/2\pi)\ln|\mathbf{r}-\mathbf{r}'|$, satisfying $u(r,r',\phi)=-(1/4\pi)\ln(r^2+r'^2-2rr'\cos\phi)$, the integrals in Eq. (B3) can be performed analytically [30]. We find that

$$\hat{u}_n(r,r')=\frac{1}{4\pi|n|}\left(\frac{r_<}{r_>}\right)^{|n|},\qquad \hat{u}_0(r,r')=-\frac{1}{2\pi}\ln r_>. \tag{B4}$$

Therefore, the potential of interaction can be written

$$u(r,r',\phi)=-\frac{1}{2\pi}\ln r_>+\frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{|n|}\left(\frac{r_<}{r_>}\right)^{|n|}e^{in\phi}, \tag{B5}$$

which is just the Fourier decomposition of the logarithm in two dimensions. Taking the Fourier-Laplace transform of Eq. (B1) and using the fact that the integral is a product of convolution, we obtain

$$\delta\tilde{\psi}(n,r,\sigma)=2\pi\int_0^{+\infty}r'dr'\,\hat{u}_n(r,r')\delta\tilde{\omega}(n,r',\sigma). \tag{B6}$$

If we substitute Eq. (A8) in Eq. (B6), we obtain the equation

$$\delta\tilde{\psi}(n,r,\sigma)-2\pi\int_0^{+\infty}r'dr'\,\hat{u}_n(r,r')\frac{n\frac{1}{r'}\frac{\partial\omega}{\partial r'}(r')}{\sigma-n\Omega(r')}\delta\tilde{\psi}(n,r',\sigma)=-2\pi\int_0^{+\infty}r'dr'\,\hat{u}_n(r,r')\frac{\delta\hat{\omega}(n,r',0)}{i(\sigma-n\Omega(r'))}, \tag{B7}$$

which is equivalent to Eq. (A12). This implies that the Green function $G(n,r,r',\sigma)$ satisfies an integral equation of the form

$$G(n,r,r',\sigma)-2\pi\int_0^{+\infty}r''dr''\,\hat{u}_n(r,r'')\frac{n\frac{1}{r''}\frac{\partial\omega}{\partial r''}(r'')}{\sigma-n\Omega(r'')}G(n,r'',r',\sigma)=\hat{u}_n(r,r'), \tag{B8}$$

which is equivalent to Eq. (A13). If we neglect collective effects, Eq. (B7) reduces to

$$\delta\tilde{\psi}(n,r,\sigma)=-2\pi\int_0^{+\infty}r'dr'\,\hat{u}_n(r,r')\frac{\delta\hat{\omega}(n,r,0)}{i(\sigma-n\Omega)}. \tag{B9}$$

Comparing Eq. (B9) with Eq. (A12), we see that the bare Green function $G_{bare}(n,r,r')=\hat{u}_n(r,r')$ is the Fourier transform of the potential of interaction u.

APPENDIX C: GENERAL KINETIC EQUATION WITHOUT COLLECTIVE EFFECTS

If we neglect collective effects, the first two equations (45) and (46) of the BBGKY-like hierarchy reduce to

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=-\gamma\frac{\partial}{\partial\mathbf{r}_1}\cdot\int\mathbf{V}(2\to 1)g(\mathbf{r}_1,\mathbf{r}_2,t)\,d\mathbf{r}_2, \tag{C1}$$

$$\frac{\partial g}{\partial t}(\mathbf{r}_1,\mathbf{r}_2,t)+\left[\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial}{\partial\mathbf{r}_1}+\langle\mathbf{V}\rangle(\mathbf{r}_2,t)\cdot\frac{\partial}{\partial\mathbf{r}_2}\right]g(\mathbf{r}_1,\mathbf{r}_2,t)$$
$$=-\frac{1}{\gamma^2}\left[\tilde{\mathbf{V}}(2\to 1)\cdot\frac{\partial}{\partial\mathbf{r}_1}+\tilde{\mathbf{V}}(1\to 2)\cdot\frac{\partial}{\partial\mathbf{r}_2}\right]\omega(\mathbf{r}_1,t)\omega(\mathbf{r}_2,t). \tag{C2}$$

Equation (C2) is just a first order differential equation in time. It can be written symbolically as

$$\frac{\partial g}{\partial t}+\mathcal{L}g=S[\omega], \tag{C3}$$

where $\mathcal{L}$ is an advection operator and S is a source term: The correlation function is transported by the mean flow (l.h.s.) and modified by two-body collisions (r.h.s.). This equation can be solved by the method of characteristics. Introducing the Green function

$$G(t,t')=\exp\left\{-\int_t^{t'}\mathcal{L}(\tau)\,d\tau\right\}, \tag{C4}$$

constructed with the smooth velocity field $\langle\mathbf{V}\rangle$, we obtain

$$g(\mathbf{r}_1,\mathbf{r}_2,t)=-\frac{1}{\gamma^2}\int_0^t d\tau\, G(t,t-\tau)\left[\tilde{\mathbf{V}}(2\to 1)\cdot\frac{\partial}{\partial\mathbf{r}_1}+\tilde{\mathbf{V}}(1\to 2)\cdot\frac{\partial}{\partial\mathbf{r}_2}\right]\omega(\mathbf{r}_1,t-\tau)\omega(\mathbf{r}_2,t-\tau), \tag{C5}$$

where we have assumed that no correlation is present initially so that $g(\mathbf{r}_1,\mathbf{r}_2,t=0)=0$ (if correlations are present initially, it can be shown that they are rapidly washed out). Substituting this result in Eq. (C1), we obtain

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=\frac{\partial}{\partial r_1^\mu}\int_0^t d\tau\int d\mathbf{r}_2 V^\mu(2\to 1)G(t,t-\tau)$$
$$\times\left[\tilde{V}^\nu(2\to 1)\frac{\partial}{\partial r_1^\nu}+\tilde{V}^\nu(1\to 2)\frac{\partial}{\partial r_2^\nu}\right]\omega(\mathbf{r}_1,t-\tau)\frac{\omega}{\gamma}(\mathbf{r}_2,t-\tau). \tag{C6}$$

In writing this equation, we have adopted a Lagrangian point of view: The coordinates $\mathbf{r}_i$ following the Greenian must be viewed as $\mathbf{r}_i(t-\tau)=\mathbf{r}_i(t)-\int_0^\tau ds\,\langle\mathbf{V}\rangle(\mathbf{r}_i(t-s),t-s)\,ds$. The generalized Landau equation (C6) which is valid for possibly non-axisymmetric flows and which takes non-Markovian effects into account has been derived in our previous papers from the projection operator formalism [27], the BBGKY hierarchy [29] and the quasilinear theory based on the Klimontovich equation [29]. The generalized Lenard-Balescu equation (55) can be viewed as an extension of this equation taking collective effects into account. Note that the Markovian approximation may not be justified in every situation since it has been found numerically that point vortices can exhibit long jumps (Lévy flights) and strong correlations [73, 74].

If we implement the Bogoliubov ansatz and extend the integration on time τ to infinity, which amounts to making a Markovian approximation, we get

$$\frac{\partial\omega}{\partial t}(\mathbf{r}_1,t)+\frac{N-1}{N}\langle\mathbf{V}\rangle(\mathbf{r}_1,t)\cdot\frac{\partial\omega}{\partial\mathbf{r}_1}(\mathbf{r}_1,t)=\frac{\partial}{\partial r_1^\mu}\int_0^{+\infty} d\tau\int d\mathbf{r}_2 V^\mu(2\to 1,t)G(t,t-\tau)$$
$$\times\left[\tilde{V}^\nu(2\to 1)\frac{\partial}{\partial r_1^\nu}+\tilde{V}^\nu(1\to 2)\frac{\partial}{\partial r_2^\nu}\right]\omega(\mathbf{r}_1,t)\frac{\omega}{\gamma}(\mathbf{r}_2,t), \tag{C7}$$

where, now, $\mathbf{r}_i(t-\tau)=\mathbf{r}_i(t)-\int_0^\tau ds\,\langle\mathbf{V}\rangle(\mathbf{r}_i(t-s),t)\,ds$. If we consider axisymmetric flows, this equation can be simplified [27, 29, 30] and we obtain the Landau-type equation (75). If we take collective effects into account, Eq. (C7) is replaced by Eq. (56) which reduces to the Lenard-Balescu-type kinetic equation (78) for axisymmetric flows.

If we assume right from the beginning that the mean flow is axisymmetric, Eqs. (C1) and (C2) can be rewritten as

$$\frac{\partial\omega}{\partial t}(r_1,t)=-\gamma^2\frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty}r_2\,dr_2\int_0^{2\pi}d\theta_2\,\frac{\partial u}{\partial\theta_1}(r_1,r_2,\theta_1-\theta_2)g(r_1,r_2,\theta_1-\theta_2,t), \tag{C8}$$

$$\frac{\partial g}{\partial t}(r_1,r_2,\theta_1-\theta_2,t)+[\Omega(r_1,t)-\Omega(r_2,t)]\,\frac{\partial g}{\partial\theta_1}(r_1,r_2,\theta_1-\theta_2,t)$$
$$=-\frac{\partial u}{\partial\theta_1}(r_1,r_2,\theta_1-\theta_2)\left(\frac{1}{r_1}\frac{\partial}{\partial r_1}-\frac{1}{r_2}\frac{\partial}{\partial r_2}\right)\omega(r_1,t)\omega(r_2,t). \tag{C9}$$

Introducing the Fourier transforms of the potential of interaction and of the correlation function, we obtain

$$\frac{\partial \omega}{\partial t}(r_1,t) = 2i\pi\gamma^2 \frac{1}{r_1}\frac{\partial}{\partial r_1}\int_0^{+\infty} r_2\,dr_2 \sum_n n\,\hat{u}_n(r_1,r_2)\hat{g}_n(r_1,r_2,t), \tag{C10}$$

$$\frac{d\hat{g}_n}{dt}(r_1,r_2,t) + in\left[\Omega(r_1,t)-\Omega(r_2,t)\right]\hat{g}_n = -\frac{i}{\gamma} n\,\hat{u}_n(r_1,r_2)\left(\frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{1}{r_2}\frac{\partial}{\partial r_2}\right)\omega(r_1,t)\omega(r_2,t). \tag{C11}$$

With the Bogoliubov ansatz, Eq. (C10) becomes Eq. (64). On the other hand, the asymptotic value of the correlation function can be obtained by taking the Laplace transform of Eq. (C11) and considering the limit $t \to +\infty$. Equivalently, it is obtained by making the substitution

$$\frac{d\hat{g}_n}{dt}(r_1,r_2,t) \to \lim_{\epsilon\to 0^+} \epsilon\hat{g}_n(r_1,r_2,+\infty), \tag{C12}$$

in Eq. (C11). This yields

$$\hat{g}_n(r_1,r_2,+\infty) = -\frac{1}{\gamma}\frac{n\,\hat{u}_n(r_1,r_2)}{n\left[\Omega(r_1)-\Omega(r_2)\right]-i0^+}\left(\frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{1}{r_2}\frac{\partial}{\partial r_2}\right)\omega(r_1)\omega(r_2). \tag{C13}$$

Substituting Eq. (C13) in Eq. (64), we obtain Eq. (72) which finally leads to the Landau-type equation (75).

APPENDIX D: THE VIRIAL OF POINT VORTICES

The virial of point vortices is $\mathcal{V} = \int \omega\mathbf{r}\cdot\nabla\psi\,d\mathbf{r}$. In an unbounded domain, the stream function can be written $\psi(\mathbf{r}) = -\frac{1}{2\pi}\int \ln|\mathbf{r}-\mathbf{r}'|\omega(\mathbf{r}')\,d\mathbf{r}'$. Therefore, $\mathcal{V} = -\frac{1}{2\pi}\int \omega(\mathbf{r})\omega(\mathbf{r}')\mathbf{r}\cdot(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^2\,d\mathbf{r}d\mathbf{r}'$. Interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$ and adding the resulting equation to the initial one, we get $\mathcal{V} = -\Gamma^2/(4\pi)$. On the other hand, for an axisymmetric flow in a disk, the Gauss theorem yields $d\psi/dr = -\Gamma(r)/2\pi r$ where $\Gamma(r) = \int_0^r \omega(r')2\pi r'\,dr'$ is the circulation within the disk of radius r. In that case, $\mathcal{V} = -\frac{1}{2\pi}\int_0^R \Gamma(r)\Gamma'(r)\,dr = -\Gamma^2/(4\pi)$ like in an unbounded domain.

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[75] Nevertheless, a conclusion of our study will be that higher order distributions should also be considered.
[76] Indeed, we can interprete the kinetic equation (79) as describing the "collisions" between a test vortex described by the variable 1 and field vortices described by the running variable 2. In Eq. (79) all the vortices are equivalent so that the distribution of the field vortices $\omega(r_2,t)$ changes with time exactly like the distribution of the test vortex $\omega(r_1,t)$. In the bath approximation, we assume that the distribution of the field vortices $\omega(r_2)$ is prescribed.
[77] For point vortices, the kinetic equation (79) governing the evolution of the system as a whole does *not* present any divergence contrary to the Lenard-Balescu equation in plasma physics that presents a divergence at small scales. However, a logarithmic divergence at small scales occurs in the Fokker-Planck equation (82) when we make the bath approximation and consider that the dominant contribution to the δ-function comes from the interaction at $r' = r$. This divergence comes from the assumption made in the kinetic theory that the point vortices essentially follow the streamlines produced by the mean flow (modified by collective effects). This neglects the contribution of "hard" collisions with small impact parameter

that lead to more complex trajectories and more complex interactions. As explained in Sec. IV A, these hard collisions could be taken into account by keeping the contribution of the fourth term in Eq. (44). Alternatively, we can proceed heuristically and regularize the logarithmic divergence by introducing a cut-off at the scale at which hard collisions come into play [26, 28, 38]. With this regularization, it can be shown that $\ln\Lambda \sim \frac{1}{2}\ln N$, leading to a $\ln N$ factor in the Fokker-Planck equation (82). Since only *distant* collisions $r_2 \neq r_1$ occur in the kinetic equation (79) describing the evolution of the system as a whole, there is no divergence at close separation, and therefore no $\ln N$ factor, in that equation.