

Mean field equation for vortex filament systems  
 -derivation, dual variational structure,  
 existence of the solution-

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1 Introduction

Mean field equations for the point vortex systems was first introduced by Onsager to investigate the large-scale long-lived vortex structures in two dimensional turbulence. The mean field equations are derived by applying the equilibrium statistical mechanics to the system, which is described by a Hamiltonian system, and then taking the mean field limit called high-energy scaling limit. Several different mean field equations are obtained according to the condition for the circulation of the vortices, though, these equations have a form of an elliptic partial differential equation with an exponential nonlinearity and a non-local term, mathematically.

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with a smooth boundary  $\partial\Omega$ ,  $x_i \in \Omega$  and  $\alpha_i \in \mathbb{R}$  be position and circulation of the each point vortex  $i = 1, \dots, P$ , respectively. Then the Hamiltonian  $H_N = H_N(x_1, \dots, x_N)$  which describes the motion of the system is defined by

$$H_N = \frac{1}{2} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \alpha_i \alpha_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^N (\alpha_i)^2 R(x_i).$$

In the mono circulation case in which every vortex has the same circulation, mean field equations is given by

$$\begin{cases} -\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v dx} & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega, \end{cases} \quad (1)$$

see [1]. Here,  $v = v(x)$  and  $\lambda \geq 0$  are related to the stream function and the inverse temperature, respectively. It is known that (1) has a variational functional

$$J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \int_{\Omega} e^v dx \quad v \in H_0^1(\Omega),$$

and the classical solution to (1) exists for  $\lambda \in [0, 8\pi)$  from the Trudinger-Moser inequality. In addition, a classification of the singular limits using the Green function [6] and the unique existence of the solution when  $\Omega$  is simply connected and  $\lambda \in [0, 8\pi)$  [11] are also known. Moreover, this system has a dual variational structure [13]. In more precisely, for the above mentioned variational structure in  $X = H_0^1(\Omega)$ , there is another variational

structure in the dual space  $X^* = H^{-1}(\Omega)$ . The variational functional on  $X^*$  is defined by

$$J^*(u) = -\frac{1}{2}\langle(-\Delta_D)^{-1}u, u\rangle + \int_{\Omega} u \log u dx - \lambda \log \lambda,$$

and the mean field equation for  $u \in X^*$  is given by

$$\begin{cases} (-\Delta_D)^{-1}u = \log u + \text{constant} & \text{in } \Omega \\ u \geq 0, \quad \int_{\Omega} u dx = \lambda \end{cases}$$

as the Euler-Lagrange equation for  $J^*$ . These two variational structures are governed by the Lagrangian  $L(v, u)$  on  $X \times X^*$  defined by

$$L(v, u) = \int_{\Omega} u(\log u - 1) dx - \lambda(\log \lambda - 1) + \frac{1}{2} \|\nabla v\|_2^2 - \langle v, u \rangle + 1_{D(F^*)}$$

$v \in X, u \in X^*$

$$D(F^*) = \{u \in X \mid u \geq 0, \int_{\Omega} u = \lambda\},$$

and the Toland duality

$$\inf_{(v,u) \in X \times X^*} L(v, u) = \inf_{u \in X^*} J^*(u) = \inf_{v \in X} J(v)$$

holds.

To generalize the condition in the circulation of the vortex in the system, we consider the deterministic system, in which the number density of the vortices on the circulation is given by  $\mathcal{P}(d\alpha)$  on  $[-1, 1]$ :

$$\mathcal{P}(d\tilde{\alpha}) = \sum_{m=1}^M \tilde{n}^m \delta_{\tilde{\alpha}^m}(d\tilde{\alpha}), \quad M \in \mathbb{N} \tag{2}$$

with  $\sum_{m=1}^M \tilde{n}^m = 1, \quad \tilde{n}^m \in [0, 1].$

The mean field equation for this generalized system is given by

$$\begin{cases} -\Delta v = \lambda \int_{[-1,1]} \alpha \frac{e^{\alpha v}}{\int_{\Omega} e^{\alpha v} dx} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

and analyzed through the variational functional

$$J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \log \left( \int_{\Omega} e^{\alpha v} dx \right) \mathcal{P}(d\alpha),$$

using the modified Trudinger-Moser inequality, see [7]. (3) is the mean field equation for a neutral system which is composed of opposite signed circulation obtained by [2], [9].

In this note, we consider vortex filament systems as a three dimensional extension of the two dimensional point vortex system. Even though the motion of the vortex filaments is very complex [14], the motion of the vortex filament system which is composed of nearly parallel vortex filaments is described by a Hamiltonian using the asymptotic theory [3].

The mean field equation for the vortex filament system with  $\mathcal{P}(d\tilde{\alpha})$  is given by

$$\begin{cases} -\Delta v_i = \lambda \int_{[-1,1]} \alpha \frac{K_i e^{\alpha v_i}}{\int_{\Omega} K_i e^{\alpha v_i} dx_i} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega \end{cases} \quad i = 1, \dots, P, \quad (4)$$

in the framework of the  $P$ -layer broken path model, where

$$\begin{aligned} K_i &= K_i(x_i, \lambda) \\ &= \int_{\Omega^{P-1}} e^{\gamma \sum_{j=1}^P |x_{j+1} - x_j|^2} e^{\alpha \sum_{j=1, j \neq i}^P v_j(x_j)} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_P \\ x_{P+1} &= x_1, \quad \gamma = \frac{\lambda S}{8\pi}, \quad S : \text{structure parameter.} \end{aligned}$$

This equation is regarded as the generalization of the mean field equation for the vortex filament system in  $\mathbb{R}^3$  with  $\mathcal{P}(d\tilde{\alpha}) = \delta_{+1}(d\tilde{\alpha})$  studied in [4, 5]. We analyze the mean field equation for this system, and obtain the following results.

**Conclusion 1** *The vortex filament system has the same dual variational structure as in the case of the point vortex system.*

**Conclusion 2** *There exist a global minimizer for the variational functional of (4) and a classical solution to (4), if  $\lambda \in (0, 8\pi)$  and  $\mathcal{P}(d\tilde{\alpha}) = \delta_{+1}(d\tilde{\alpha})$ .*

This note is organized as follows. In Section 2, we show the derivation of the mean field equation for the vortex filament system, applying the heuristic methods used in the derivation of the mean field equation for the point vortex system in [1]. In Section 3, we discuss on the dual variational structure of the vortex filament system and the existence of the solution to the mean field equations.

## 2 Derivation of the mean field equation

We consider the system where vortex filaments nearly parallel to the  $x_3$ -axis are included in a columnar region with cross section  $\Omega$ . Assume that the number density of the vortex filaments on the circulation is subject to a probability measure  $\mathcal{P}(d\tilde{\alpha})$  and set a periodic boundary condition (period  $L$ ) on  $x_3$ -axis.

We approximate the each vortex filament by a broken line with  $P$  nodes, and denote  $i$ -th vortex filament  $X_i^{(2)}$  by two dimensional coordinates  $x_i^\sigma \in \Omega$  at each layer  $\sigma$ :

$$X_i^{(2)} = (x_i^1, \dots, x_i^P) \in \Omega^P$$

The model in this framework is called the  $P$ -broken path model.

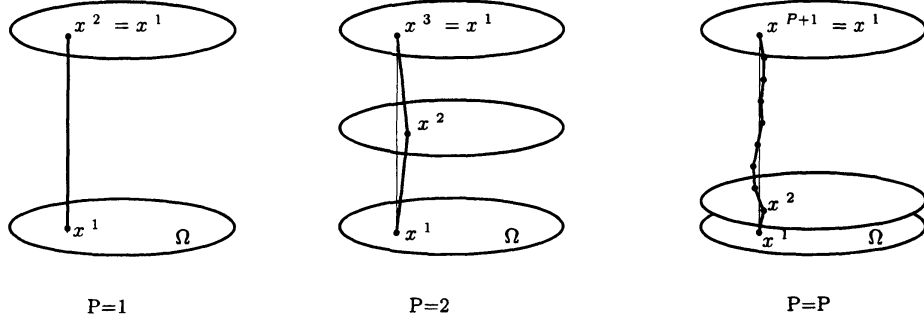


Figure 1: Sketch of the broken path model

The motion of the vortex filament system is described by a Hamiltonian given by

$$\begin{aligned} \mathcal{H}_{N,P} &= \frac{S}{8\pi} \sum_{\sigma=1}^P \sum_{i=1}^N (\alpha_i)^2 \frac{1}{l} |x_i^{\sigma+1} - x_i^\sigma|^2 \\ &\quad + \frac{1}{2} \sum_{\sigma=1}^P \sum_{i=1}^N \sum_{j \neq i}^N \alpha_i \alpha_j l G(x_i^\sigma, x_j^\sigma) \\ &\quad + \frac{1}{2} \sum_{\sigma=1}^P \sum_{i=1}^N (\alpha_i)^2 l R(x_i^\sigma), \end{aligned}$$

where,  $l = L/P$  and  $S$  is a structure parameter for the vortex filament.  $S$  describes the strength of the connection with upper and lower layers, and the filament becomes a straight line when  $S$  goes to infinity. Compared with the Hamiltonian  $H_N$  of the point vortex system, we can see that a self interaction term between layers is included in  $\mathcal{H}_{N,P}$ .

Using the Hamiltonian  $\mathcal{H}_{N,P}$ , Gibbs measure for the inverse temperature  $\beta_N$  is defined by

$$\begin{aligned} \mu_{N,P}^{\beta_N} &= \mu_{N,P}^{\beta_N}(dX_1^{(2)}, \dots, dX_N^{(2)}) \\ &= \frac{1}{Z(N, \beta_N)} \exp[-\beta_N \mathcal{H}_{N,P}] dX_1^{(1)} \dots dX_N^{(2)}, \\ Z(N, \beta_N) &= \int_{\Omega^{PN}} \exp[-\beta_N \mathcal{H}_{N,P}] dX_1^{(2)} \dots dX_N^{(2)}. \end{aligned}$$

Then, reduced probability function (pdf) for  $k$ -filaments with the circulation  $\tilde{\alpha}^m$  is defined by

$$\begin{aligned} \rho_{N,P,k}^m dX_1^{(2)} \dots dX_k^{(2)} &= \rho_{N,P,k}^m(X_1^{(2)}, \dots, X_k^{(2)}) dX_1^{(2)} \dots dX_k^{(2)} \\ &= \frac{1}{Z(N, \beta_N)} \int \exp[-\beta_N \mathcal{H}_{N,P}(X_1^{(2)}, \dots, X_N^{(2)})] dX_{k+1}^{(2)} \dots dX_N^{(2)}, \end{aligned} \quad (5)$$

and, similarly, reduced pdf for the position  $x_i^\sigma$  of  $i$ -th vortex filament with  $\tilde{\alpha}^m$  is defined

by

$$\begin{aligned}\eta_{P,\sigma}^m dx_i^\sigma &= \eta_{P,\sigma}^m(x_i^\sigma) dx_i^\sigma \\ &= \int_{\Omega^{P-1}} \rho_{N,P,1}^m(X_i^{(2)}) dx_i^1 \cdots dx_i^{\sigma-1} dx_i^{\sigma+1} \cdots dx_i^P.\end{aligned}\quad (6)$$

Hence,  $k$ -filament reduced pdf with  $\tilde{\alpha}^m$  is given by

$$\begin{aligned}&\rho_{N,P,k}^m(X_1^{(2)}, \dots, X_k^{(2)}) \\ &= \frac{1}{Z(N, \beta_N)} \exp[-\beta_N \mathcal{H}_{k,P}] \\ &\quad \int_{\Omega^{(N-k)P}} \exp[-\beta_N \mathcal{H}_{N-k,P}] \exp\left[-\beta_N \sum_{\sigma=1}^P \sum_{i=1}^k \sum_{j=k+1}^N \tilde{\alpha}^m \alpha_i \alpha_j l G_\Omega(x_i^\sigma, x_j^\sigma)\right] \\ &\quad \cdot \exp\left[\frac{N\beta_N}{N-k} \mathcal{H}_{N-k,P}\right] Z(N-k, \beta_N) \mu_{N-k,P}^{\beta_N}(dX_{N-k}^{(2)}, \dots, dX_N^{(2)}) \\ &= \frac{Z(N-k, \beta_N)}{Z(N, \beta_N)} \exp[-\beta_N \mathcal{H}_{k,P}] \int_{\Omega^{(N-k)P}} \exp\left[\frac{\beta_N k}{N-k} \mathcal{H}_{N-k,P}\right] \\ &\quad \cdot \exp\left[-\beta_N \sum_{\sigma=1}^P \sum_{i=1}^k \sum_{j=k+1}^N \tilde{\alpha}^m \alpha_i \alpha_j l G_\Omega(x_i^\sigma, x_j^\sigma)\right] \mu_{N-k,P}^{\beta_N}(dX_{N-k}^{(2)}, \dots, dX_N^{(2)}),\end{aligned}$$

using

$$dX_{k+1}^{(2)} \cdots dX_N^{(2)} = Z(N-k, \beta_N) e^{\frac{N\beta_N}{N-k} \mathcal{H}_{N-k,P}} \mu_{N-k,P}^{\beta_N} \quad (7)$$

which is derived from the decomposition of the Hamiltonian

$$\begin{aligned}\mathcal{H}_{N,P}(X_1^{(2)}, \dots, X_N^{(2)}) &= \mathcal{H}_{k,P}(X_1^{(2)}, \dots, X_k^{(2)}) + \mathcal{H}_{N-k,P}(X_{k+1}^{(2)}, \dots, X_N^{(2)}) \\ &\quad + \sum_{\sigma=1}^P \sum_{i=1}^k \sum_{j=k+1}^N \alpha_i \alpha_j l G_\Omega(x_i^\sigma, x_j^\sigma)\end{aligned}\quad (8)$$

and

$$\mu_{N-k,P}^{\beta_N} = \frac{1}{Z(N-k, \beta_N)} \exp\left[-\frac{N\beta_N}{(N-k)} \mathcal{H}_{N-k,P}\right] dX_{k+1}^{(2)} \cdots dX_N^{(2)}.$$

For  $k=1$ , we have

$$\begin{aligned}&\rho_{N,P,1}^m(X_1^{(2)}) \\ &= \frac{Z(N-1, \beta_N)}{Z(N, \beta_N)} \exp\left[-\frac{\alpha^2 \beta_N}{8\pi} (\tilde{\alpha}^m)^2 \sum_{\sigma=1}^P \frac{1}{l} |x_1^{\sigma+1} - x_1^\sigma|^2 - \frac{\alpha^2 \beta_N (\tilde{\alpha}^m)^2}{2} \sum_{\sigma=1}^P l \gamma_\Omega(x_1^\sigma)\right] \\ &\quad \int_{\Omega^{(N-1)P}} \exp\left[-\frac{\beta_N S}{8\pi(N-1)} \sum_{\sigma=1}^P (\alpha_i)^2 \frac{1}{l} |x_i^{\sigma+1} - x_i^\sigma|^2\right. \\ &\quad \left. + \frac{\beta_N}{2(N-1)} \sum_{\sigma=1}^P \sum_{i=2}^N \sum_{j=2, j \neq i}^N \alpha_i \alpha_j l G_\Omega(x_i^\sigma, x_j^\sigma) + \frac{\beta_N}{2(N-1)} \sum_{\sigma=1}^P \sum_{i=2}^N (\alpha_i)^2 l \gamma(x_i^\sigma)\right] \\ &\quad \exp\left[-\alpha \beta_N \tilde{\alpha}^m \sum_{\sigma=1}^P \sum_{j=2}^N \alpha_j l G_\Omega(x_1^\sigma, x_j^\sigma)\right] \mu_{N-1,P}^{\beta_N}(dX_2^{(2)}, \dots, dX_N^{(2)}).\end{aligned}$$

Here, we take the mean field limit:

$$N \rightarrow \infty, \quad \tilde{\beta} = \alpha^2 \beta_N N = \text{constant}, \quad \tilde{S} = S/N = \text{constant}, \quad (9)$$

under the following assumption

- the propagation of chaos  $(\rho_{P,1}^n(X_1^{(2)})) = \lim_{N \rightarrow \infty} \rho_{N,P,1}^n(X_1^{(2)})$
- the existence of the smooth function

$$\begin{aligned} \rho_{P,1}^m &= \lim_{N \rightarrow \infty} \rho_{N,P,1}^m, \quad \eta_{P,\sigma}^m = \lim_{N \rightarrow \infty} \eta_{N,P,\sigma}^m \\ Z &= \lim_{N \rightarrow \infty} Z(N, \beta_N) / Z(N-1, \beta_N). \end{aligned}$$

Then we obtain

$$\begin{aligned} &\rho_{P,1}^m(X_1^{(2)}) \\ &= \frac{1}{Z} \exp \left[ -\frac{\tilde{\beta}(\tilde{\alpha}^m)^2 \tilde{S}}{8\pi} \sum_{\sigma=1}^P \frac{1}{l} |x_1^{\sigma+1} - x_1^\sigma|^2 \right] \\ &\quad \exp \left[ \frac{\tilde{\beta}}{2} \sum_{\sigma=1}^P l \int_{\Omega^2} G_\Omega(x, y) \left( \sum_{n=1}^M \tilde{\alpha}^n \tilde{n}^n \eta_{P,\sigma}^n(x) \right) \left( \sum_{n'=1}^M \tilde{\alpha}^{n'} \tilde{n}^{n'} \eta_{P,\sigma}^{n'}(y) \right) dx dy \right] \\ &\quad \exp \left[ -\tilde{\beta} \tilde{\alpha}^m \sum_{\sigma=1}^P l \int_{\Omega} G_\Omega(x_1^\sigma, y) \left( \sum_{n=1}^M \tilde{n}^n \tilde{\alpha}^n \eta_{P,\sigma}^n(y) \right) dy \right]. \end{aligned}$$

By the condition  $\int_{\Omega} \rho_{P,1}^m dX_1^{(2)} = 1$  and  $\mathcal{P}(d\alpha)$ , it follows that

$$\begin{aligned} \rho_{P,1}^m(X_1^{(2)}) &= \frac{1}{Z} \exp \left[ -\frac{\tilde{\beta}(\tilde{\alpha}^m)^2 \tilde{S}}{8\pi} \sum_{\sigma=1}^P \frac{1}{l} |x_1^{\sigma+1} - x_1^\sigma|^2 \right] \\ &\quad \exp \left[ -\tilde{\beta} \tilde{\alpha}^m \sum_{\sigma=1}^P l \int_{\Omega} G_\Omega(x_1^\sigma, y) \left( \sum_{n=1}^M \tilde{n}^n \tilde{\alpha}^n \eta_{P,\sigma}^n(y) \right) dy \right]. \end{aligned}$$

From the pdfs  $\rho_{P,1}^m$  and  $\eta_{P,\sigma}^m$ , limiting functions associated to the vorticity and the stream function are described by

$$\begin{aligned} \omega_{P,\sigma}(x) &= \int_{[-1,1]} \tilde{\alpha} \tilde{\eta}_{P,\sigma}^{\tilde{\alpha}}(y) \mathcal{P}(d\tilde{\alpha}) \\ \psi_{P,\sigma}(x) &= \int_{\Omega} G_\Omega(x, y) \left[ \int_{[-1,1]} \tilde{\alpha} \tilde{\eta}_{P,\sigma}^{\tilde{\alpha}}(y) \mathcal{P}(d\tilde{\alpha}) \right] dy. \end{aligned}$$

Therefore, using the relation between  $\psi_{P,\sigma}$  and  $\omega_{P,\sigma}$  on each cross section  $\Omega$ , and the following translations

$$\begin{aligned} v_\sigma &= -\tilde{\beta} \psi_{P,\sigma}, \quad x_\sigma = x^\sigma, \\ l &= 1, \quad \tilde{\beta} = -\lambda, \quad \tilde{\alpha} = \alpha, \quad \tilde{S} = S, \\ \gamma &= \frac{\lambda \alpha^2 S}{8\pi}, \end{aligned}$$

we obtain the mean field equation

$$\begin{cases} -\Delta_i v_i = \lambda \int_{[-1,1]} \alpha \frac{K_i e^{\alpha v_i}}{\int_{\Omega} K_i e^{\alpha v_i} dx_i} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \dots, P \quad (10)$$

where

$$\begin{aligned} K_i &= K_i(x_i, \lambda) \\ &= \int_{\Omega^{P-1}} e^{\gamma \sum_{j=1}^P |x_{j+1} - x_j|^2} e^{\alpha \sum_{j=1, j \neq i}^P v_j(x_j)} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_P. \end{aligned}$$

In the case that  $P$  goes to infinity, the vortex filaments are described by using a conditional Wiener measure, and the corresponding 1-filament pdf is determined through the Green function of a partial differential equation indicated by the Feynman-Kac formula [4, 5].

### 3 Analysis of the vortex filament system

#### 3.1 Dual variational structure

In this section, we consider a dual variational structure for the vortex filament system, using the same approach in [13]. Our starting point is the mean field equation (10). This mean field equation has a variation structure, and the variational functional for  $v_i \in H_0^1(\Omega)$   $i = 1, \dots, P$  is given by

$$J(v_1, \dots, v_P) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \log \left( \int_{\Omega^P} K \exp \left[ \alpha \sum_{i=1}^P v_i \right] dx_1 \cdots dx_P \right) \mathcal{P}(d\alpha), \quad (11)$$

where

$$K = K(x_1, \dots, x_P; \gamma) = \exp \left[ \gamma \sum_{i=1}^P |x_{i+1} - x_i|^2 \right].$$

The mean field equation (10) is the Euler-Lagrange equation for this variational functional (11), in fact, we have

$$\begin{aligned} \langle \varphi_i, \delta_i J(v_1, \dots, v_P) \rangle &= \left. \frac{d}{ds} J(v_1, \dots, v_i + s\varphi_i, \dots, v_P) \right|_{s=0} \\ &= \int_{\Omega} \varphi_i \left( (-\Delta_i) v_i - \lambda \int_{[-1,1]} \alpha \frac{K_i e^{\alpha v_i}}{\int_{\Omega} K_i e^{\alpha v_i}} \mathcal{P}(d\alpha) \right) dx_i = 0 \end{aligned}$$

for  $\forall \varphi_i \in H_0^1(\Omega)$ ,  $i = 1, \dots, P$ .

We see that  $v_i$  is connected with each other by  $K$  which originates from the self interaction term. In order to describe the behavior of the vortex filament system, a suitable function space  $X$  on  $x_1, \dots, x_P$  is needed. We try to construct a dual variational structure on  $X$  and its dual space  $X^*$ , considering the two candidates for the space  $X$ . In both cases, we set  $x = (x_1, \dots, x_P)$ ,  $x_i \in \Omega$   $i = 1, \dots, P$  and  $dx = dx_1 \cdots dx_P$ .

**Case 1** Here, we set

$$X = H^1(\Omega^P), \quad X^* = (H^1(\Omega^P))^*.$$

In order to express the variational functional (11), we use a subspace

$$D = \left\{ v = \bigoplus_{i=1}^P v_i(x_i) \mid v_i \in H_0^1(\Omega), \quad i = 1, \dots, P \right\}.$$

Then, for  $v \in D$ , we have

$$\begin{aligned} \nabla v &= \begin{pmatrix} \frac{\partial v}{\partial x_{11}} \\ \frac{\partial v}{\partial x_{12}} \\ \vdots \\ \frac{\partial v}{\partial x_{P1}} \\ \frac{\partial v}{\partial x_{P2}} \end{pmatrix} = \begin{pmatrix} \nabla_1 v_1 \\ \vdots \\ \nabla_P v_P \end{pmatrix}, \\ \|\nabla v\|_2^2 &= \int_{\Omega^P} |\nabla v|_2^2 dx \\ &= |\Omega|^{P-1} \left( \|\nabla_1 v_1\|_2^2 + \dots + \|\nabla_P v_P\|_2^2 \right), \end{aligned}$$

$$\begin{aligned} \int_{\Omega^P} K e^v dx_1 \cdots dx_P &= \int_{\Omega} K_i e^{v_i(x_i)} dx_i \\ &= \int_{\Omega^P} e^{\gamma \sum_{i=1}^P |x_{i+1} - x_i|^2} e^{\sum_{i=1}^P v_i(x_i)} dx_1 \cdots dx_P. \end{aligned}$$

The variational functional corresponding to (11) is given by

$$J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda |\Omega|^{P-1} \int_{[-1,1]} \log \left( \int_{\Omega^P} K e^{\alpha v} \right) \mathcal{P}(d\alpha) + 1_D(v)$$

on  $X$ . It is confirmed that the mean field equation (10) is the Euler-Lagrange equation of  $J(v)$  on  $X$ , since we have

$$\delta J(v) = 0 \Leftrightarrow \begin{cases} v = \sum_{i=1}^P v_i(x_i), \quad v_i \in H_0^1(\Omega) \\ -\Delta_i v_i = \lambda \int_{[-1,1]} \frac{K_i e^{v_i}}{\int_{\Omega} K_i e^{v_i}} \mathcal{P}(d\alpha) \quad i = 1, \dots, P. \end{cases}$$

This variational functional is represented by

$$J(v) = G(v) - F(v),$$

using

$$\begin{aligned} G(v) &= \frac{1}{2} \|\nabla v\|_2^2 + 1_D(v) \\ F(v) &= \lambda |\Omega|^{P-1} \int_{[-1,1]} \log \left( \int_{\Omega^P} K e^{\alpha v} \right) \mathcal{P}(d\alpha). \end{aligned}$$



Since  $G(v)$  is proper, convex, lower semi continuous (l.s.c), its conjugate function  $G^*(u)$  on  $X^*$  (Legendre transform of  $G(v)$ ) is defined by

$$\begin{aligned} G^*(u) &= \sup\{\langle v, u \rangle - G(v)\} = \langle v, u \rangle - G(v) \Big|_{u=\delta G(v)} \\ &= \frac{1}{2}|\Omega|^{P-1} \sum_{i=1}^P \langle (-\Delta)^{-1}u_i, u_i \rangle + 1_H(u), \end{aligned}$$

where

$$H = \left\{ u = \bigoplus_{i=1}^P u_i(x_i) \mid u_i \in H^{-1}(\Omega), \quad i = 1, \dots, P \right\}.$$

In this derivation,

$$u = \delta G(v) \Rightarrow u_i = (-\Delta_i)v_i \quad i = 1, \dots, P$$

is used.

Similarly, since  $F(v)$  is also proper, convex, l.s.c., its conjugate function  $F^*(u)$  on  $X^*$  is defined by

$$\begin{aligned} F^*(u) &= \sup\{\langle v, u \rangle - F(v)\} = \langle v, u \rangle - F(v) \Big|_{u=\delta F(v)} \\ &= \int_{[-1,1]} \int_{\Omega^P} u_\alpha \left( \log u_\alpha - \log K - 1 \right) dx \mathcal{P}(d\alpha) \\ &\quad - \lambda |\Omega|^{P-1} \left( \log \lambda |\Omega|^{P-1} - 1 \right) + 1_{D(F^*)}(u), \end{aligned}$$

where

$$D(F^*) = \left\{ u = \int_{[-1,1]} \alpha u_\alpha \mathcal{P}(d\alpha) \mid u_\alpha \in X^*, \quad u_\alpha \geq 0, \quad \int_{\Omega^P} u_\alpha dx = \lambda |\Omega|^{P-1} \right\}.$$

In this derivation,

$$u = \delta F(v) \Rightarrow u_\alpha = \lambda |\Omega|^{P-1} \frac{K e^{\alpha v}}{\int_{\Omega^P} K e^{\alpha v} dx} \geq 0$$

is used.

Considering the subspaces  $H$  and  $D(F^*)$ , we set

$$\begin{aligned} E &= \left\{ u = \sum_{i=1}^P \int_{[-1,1]} \alpha u_i^\alpha(x_i) \mathcal{P}(d\alpha) \mid \right. \\ &\quad \left. u_i^\alpha \in H^{-1}(\Omega), \quad \sum_{i=1}^P u_i^\alpha \geq 0, \quad \int_{\Omega^P} \sum_{i=1}^P u_i^\alpha dx = \lambda |\Omega|^{P-1} \quad i = 1, \dots, P \right\}, \end{aligned}$$

in order to define the variational functional  $J^*(u)$  on  $X^*$ :

$$\begin{aligned} J^*(u) &= F^*(u) - G^*(u) \\ &= \int_{[-1,1]} \int_{\Omega^P} u_\alpha \left( \log u_\alpha - \log K - 1 \right) dx \mathcal{P}(d\alpha) \\ &\quad - \frac{1}{2} |\Omega|^{P-1} \sum_{i=1}^P \langle (-\Delta)^{-1}u_i, u_i \rangle - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1) + 1_E(u). \end{aligned}$$

For  $u_i^\alpha \in H^{-1}(\Omega)$  satisfying

$$\sum_{i=1}^P u_i^\alpha \geq 0, \quad \int_{\Omega^P} \sum_{i=1}^P u_i^\alpha dx = \lambda |\Omega|^{P-1},$$

we have

$$\begin{aligned} J^*(\oplus u_i^\alpha) &= \int_{[-1,1]} \int_{\Omega^P} \left( \sum_{i=1}^P u_i^\alpha \right) \left( \log \left( \sum_{i=1}^P u_i^\alpha \right) - \log K - 1 \right) dx \mathcal{P}(d\alpha) \\ &\quad - \frac{1}{2} |\Omega|^{P-1} \int_{[-1,1]} \int_{[-1,1]} \alpha \alpha' \sum_{i=1}^P \langle (-\Delta)^{-1} u_i^\alpha, u_i^{\alpha'} \rangle \mathcal{P}(d\alpha) \mathcal{P}(d\alpha') \\ &\quad - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1). \end{aligned}$$

The Euler-Lagrange equation for this variational functional  $J^*(u)$  is given by

$$\begin{cases} \alpha (-\Delta)^{-1} u_i = \log \sum_{i=1}^P u_i^\alpha - \log K + \text{constant} \\ \sum_{i=1}^P u_i \geq 0, \quad \int_{\Omega^P} \sum_{i=1}^P u_i^\alpha dx = \lambda |\Omega|^{P-1}, \quad u_i = \int_{[-1,1]} \alpha u_i^\alpha \mathcal{P}(d\alpha) \end{cases} \quad i = 1, \dots, P.$$

This is the mean field equation for  $u \in X^*$  related to the vorticity.

These two variational structures on  $X$  and  $X^*$  are governed by the Lagrangian  $L(v, u)$  on  $X \times X^*$  defined by

$$\begin{aligned} L(v, u) &= F^*(u) + G(v) - \langle v, u \rangle \\ &= \int_{[-1,1]} \int_{\Omega^P} u_\alpha (\log u_\alpha - \log K - 1) dx \mathcal{P}(d\alpha) - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1) \\ &\quad + \frac{1}{2} \|\nabla v\|_2^2 - \langle v, u \rangle + 1_{D(F^*)}(u) + 1_D(v), \end{aligned}$$

and it holds that

$$\begin{aligned} \inf_{v \in X} L(v, u) &= \inf_{v \in X} \{ F^*(u) + G(v) - \langle v, u \rangle \} \\ &= F^*(u) - \sup_{v \in X} \{ \langle v, u \rangle - G(v) \} \\ &= F^*(u) - G^*(u) = J^*(u) \\ \inf_{u \in X^*} L(v, u) &= J(v) \\ \inf_{(v,u) \in X \times X^*} L(v, u) &= \inf_{u \in X^*} J^*(u) = \inf_{v \in X} J(v). \end{aligned}$$

Moreover, we have

$$\begin{aligned} L(v, u)|_{u \in \partial F(v)} &= J(v) \\ L(v, u)|_{v \in \partial G^*(u)} &= J^*(u) \\ L(v, u) &\geq \max\{J(v), J^*(u)\}. \end{aligned}$$

**Case 2** Let

$$\begin{aligned} X &= \{v = \oplus_{i=1}^P v_i(x_i) \mid v_i \in H_0^1(\Omega), \quad i = 1, \dots, P\} \\ X^* &= \{u = \oplus_{i=1}^P u_i(x_i) \mid u_i \in H^{-1}(\Omega) \quad i = 1, \dots, P\}. \end{aligned}$$

The norm and the duality pairing are defined by

$$\begin{aligned} \|\nabla v\|_2^2 &= \sum_{i=1}^P \|\nabla_i v_i\|_2^2 \\ \langle v, u \rangle_{X, X^*} &= \sum_{i=1}^P \langle v_i, u_i \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}. \end{aligned}$$

On this space  $X$ , the variational functional is given by

$$J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left( \int_{\Omega^P} K e^v dx_1 \cdots dx_P \right)$$

and represented by

$$J(v) = G(v) - F(v)$$

using

$$\begin{aligned} G(v) &= \frac{1}{2} \|\nabla v\|_2^2 \\ F(v) &= \lambda \log \left( \int_{\Omega^P} K e^v dx_1 \cdots dx_P \right). \end{aligned}$$

The conjugate function  $G^*(u)$  on  $X^*$  for  $G(v)$  is defined by

$$\begin{aligned} G^*(u) &= \sup_{v \in X} \{ \langle v, u \rangle - G(v) \} \\ &= \frac{1}{2} \sum_{i=1}^P \langle (-\Delta_i)^{-1} u_i, v_i \rangle \end{aligned}$$

using the relation

$$u = \delta G(v) \Rightarrow u_i = (-\Delta_i) v_i \quad i = 1, \dots, P.$$

Similarly, the conjugate function  $F^*(u)$  on  $X^*$  for  $F(v)$  is defined by

$$\begin{aligned} F^*(u) &= \sup_{v \in X} \{ \langle v, u \rangle - F(v) \} \\ &= \sum_{i=1}^P \int_{\Omega} u_i (\log u_i - \log K_i - 1) dx_i - P\lambda (\log \lambda - 1) \\ &\quad - \lambda(P-1) \log \int_{\Omega^P} K \exp \left[ \sum_{i=1}^P v_i \right] dx + 1_{D(F^*)}(u), \end{aligned}$$

where

$$D(F^*) = \left\{ u = \oplus_{i=1}^P u_i \in X^* \mid u_i \geq 0, \int_{\Omega} u_i dx_i = \lambda \quad i = 1, \dots, P \right\}.$$

In this derivation,

$$u = \delta F(v) \Rightarrow u_i = \lambda \frac{K_i e^{v_i}}{\int_{\Omega^P} K e^{\sum v_i} dx} \geq 0 \quad i = 1, \dots, P$$

is used. But we fail to express  $K_i$  by  $u_i$ , so  $F^*(u)$  is not closed in  $X^*$ . It is concluded that Case 2 is not a valid space to construct the dual variational structure of the vortex filament system.

### 3.2 Existence of the solution

We consider the solution of the mean field equations for the vortex filament system with  $\mathcal{P}(d\alpha) = \delta_{+1}(d\alpha)$  on the space  $X = H^1(\Omega^P)$  and its dual space  $X^*$ .

First, we show the existence of the solution to the mean field equation on  $X$

$$\begin{cases} -\Delta v_i = \frac{K_i e^{v_i}}{\int_{\Omega} K_i e^{v_i} dx_i} & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega \end{cases} \quad i = 1, \dots, P, \quad (12)$$

where  $v_i \in H_0^1(\Omega)$   $i = 1, \dots, P$ . Since  $\Omega \subset \mathbb{R}^2$  is bounded, there exists a constant  $K_\Omega$  satisfying

$$1 \leq K = \exp \left[ \gamma \sum_{i=1}^P |x_{i+1} - x_i|^2 \right] \leq K_\Omega,$$

and it holds that

$$J(v) \geq \frac{1}{2} |\Omega|^{P-1} \sum_{i=1}^P \|\nabla v_i\|_2^2 - \lambda |\Omega|^{P-1} \log \left[ K_\Omega \prod_{i=1}^P \int_{\Omega} e^{v_i} dx_i \right]$$

for  $v \in D$ . Then, from the Trudinger-Moser inequality

$$\int_{\Omega} e^{v_i} dx_i \leq C_i |\Omega| e^{\frac{1}{16\pi} \|\nabla v_i\|_2^2},$$

we have

$$\begin{aligned} J(v) &\geq \frac{1}{2} |\Omega|^{P-1} \sum_{i=1}^P \|\nabla v_i\|_2^2 - \lambda |\Omega|^{P-1} \log \left[ K_\Omega \prod_{i=1}^P C_i |\Omega| e^{\frac{1}{16\pi} \|\nabla v_i\|_2^2} \right] \\ &= |\Omega|^{P-1} \sum_{i=1}^P \left( \frac{1}{2} - \frac{\lambda}{16\pi} \right) \|\nabla v_i\|_2^2 - \lambda |\Omega|^{P-1} \log \left[ K_\Omega |\Omega|^P \prod_{i=1}^P C_i \right]. \end{aligned}$$

Thus, we have the boundedness from below:

$$\lambda \leq 8\pi \Rightarrow \inf_{v \in X} J(v) > -\exists C. \quad (13)$$

This indicates the existence of the solution to (12) for  $\lambda \in (0, 8\pi)$ .

Next, we consider the existence of the solution for the mean field equation on  $X^*$ .

$$\begin{cases} (-\Delta)^{-1}u_i = \log\left(\sum_{i=1}^P u_i\right) - \log K + \text{constant} & i = 1, \dots, P \\ \sum_{i=1}^P u_i \geq 0, \quad \int_{\Omega^P} \sum_{i=1}^P u_i dx = \lambda |\Omega|^{P-1}, \end{cases} \quad (14)$$

where  $u_i \in H^{-1}(\Omega)$   $i = 1, \dots, P$ . The associated variational functional is given by

$$\begin{aligned} J^*(\oplus u_i) &= \int_{\Omega^P} \left(\sum_{i=1}^P u_i(x_i)\right) \left(\log\left(\sum_{i=1}^P u_i(x_i)\right) - \log K - 1\right) dx_1 \cdots dx_P \\ &\quad - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1) - \frac{1}{2} |\Omega|^{P-1} \sum_{i=1}^P \langle (-\Delta_i)^{-1} u_i, u_i \rangle \end{aligned}$$

for  $u_i$  with the property  $\sum_{i=1}^P u_i \geq 0$ ,  $\int_{\Omega^P} \sum_{i=1}^P u_i dx = \lambda |\Omega|^{P-1}$ . The form of this functional is different from the one discussed in the Theorem 5 of [10]:

$$\begin{aligned} \Psi(\tilde{u}) &= \sum_i \int_{\Omega} \tilde{u}_i(x) \log \tilde{u}_i(x) dx - \frac{1}{2} \sum_{i,j} a_{i,j} \int_{\Omega} \int_{\Omega} \tilde{u}_i(x) G(x,y) \tilde{u}_j(y) dx dy \\ &\quad \text{on } \{\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_I) \mid \tilde{u}_i \geq 0, \int_{\Omega} \tilde{u}_i \log \tilde{u}_i < \infty, \int_{\Omega} \tilde{u}_i = \lambda_i\} \\ A = (a_{i,j}) &: \text{symmetric, } a_{i,j} \geq 0, \end{aligned}$$

and the lower boundedness conjectured from the dual variational principle is not confirmed yet. The difficulty of this problem is caused by the absence of the positivity of each  $u_i$ .

## Table of formulas

Dual variational structure for the point vortex and the vortex filament systems:

- Point vortex system with mono circulation

$$\begin{aligned} J(v) &= \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log\left(\int_{\Omega} e^v\right) \\ &\quad \begin{cases} -\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \\ J^*(u) &= \int_{\Omega} u(\log u - 1) dx - \lambda(\log \lambda - 1) - \frac{1}{2} \langle (-\Delta)^{-1} u, u \rangle + 1_E(u) \\ &\quad \begin{cases} (-\Delta)^{-1} u = \log u + \text{constant} & \text{in } \Omega \\ u \geq 0, \quad \int_{\Omega} u = \lambda \end{cases} \end{aligned}$$

- Point vortex system with generalized circulation  $\mathcal{P}(d\alpha)$

$$\begin{aligned}
J(v) &= \frac{1}{2} \|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \log \left( \int_{\Omega} e^{\alpha v} \right) \mathcal{P}(d\alpha) \\
&\begin{cases} -\Delta v = \lambda \int_{[-1,1]} \alpha \frac{e^{\alpha v}}{\int_{\Omega} e^{\alpha v}} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \\
J^*(\oplus u_{\alpha}) &= \int_{[-1,1]} \int_{\Omega} u_{\alpha} (\log(u_{\alpha} - 1)) dx \mathcal{P}(d\alpha) \\
&\quad - \lambda (\log \lambda - 1) - \frac{1}{2} \int_{[-1,1]} \int_{[-1,1]} \alpha \alpha' \langle (-\Delta)^{-1} u_{\alpha}, u_{\alpha'} \rangle \mathcal{P}(d\alpha) \mathcal{P}(d\alpha') \\
&\begin{cases} \alpha (-\Delta)^{-1} u = \log u_{\alpha} + \text{constant in } \Omega \\ u_{\alpha} \geq 0, \quad \int_{\Omega} u_{\alpha} = \lambda, \quad u = \int_{[-1,1]} \alpha u_{\alpha} \mathcal{P}(d\alpha) \end{cases}
\end{aligned}$$

- Vortex filament system with mono circulation

$$\begin{aligned}
J(v) &= \frac{1}{2} \|\nabla v\|_2^2 - \lambda |\Omega|^{P-1} \log \left( \int_{\Omega^P} K e^v \right) + 1_D(v) \\
&\begin{cases} -\Delta v_i = \lambda \frac{K_i e^{v_i}}{\int_{\Omega} K_i e^{v_i}} & \text{in } \Omega \quad i = 1, \dots, P \\ v_i = 0 & \text{on } \partial\Omega, \end{cases} \\
J^*(\oplus u_i) &= \int_{\Omega^P} \left( \sum_{i=1}^P u_i(x_i) \right) (\log \left( \sum_{i=1}^P u_i(x_i) \right) - \log K - 1) dx_1 \cdots dx_P \\
&\quad - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1) - \frac{1}{2} |\Omega|^{P-1} \sum_{i=1}^P \langle (-\Delta_i)^{-1} u_i, u_i \rangle \\
&\begin{cases} (-\Delta)^{-1} u_i = \log \left( \sum_{i=1}^P u_i \right) - \log K + \text{constant} & i = 1, \dots, P \\ \sum_{i=1}^P u_i \geq 0, \quad \int_{\Omega^P} \sum_{i=1}^P u_i dx = \lambda |\Omega|^{P-1} \end{cases}
\end{aligned}$$

- Vortex filament system with generalized circulation  $\mathcal{P}(d\alpha)$

$$\begin{aligned}
J(v) &= \frac{1}{2} \|\nabla v\|_2^2 - \lambda |\Omega|^{P-1} \int_{[-1,1]} \log \left( \int_{\Omega^P} K e^{\alpha v} \right) \mathcal{P}(d\alpha) + 1_D(v) \\
&\begin{cases} -\Delta v_i = \lambda \int_{[-1,1]} \alpha \frac{K_i e^{\alpha v_i}}{\int_{\Omega} K_i e^{\alpha v_i} dx_i} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \dots, P \\
J^*(\oplus u_i^\alpha) &= \int_{[-1,1]} \int_{\Omega^P} \left( \sum_{i=1}^P u_i^\alpha \right) \left( \log \left( \sum_{i=1}^P u_i^\alpha \right) - \log K - 1 \right) dx_1 \cdots dx_P \mathcal{P}(d\alpha) \\
&\quad - \lambda |\Omega|^{P-1} (\log \lambda |\Omega|^{P-1} - 1) \\
&\quad - \frac{1}{2} |\Omega|^{P-1} \int_{[-1,1]} \int_{[-1,1]} \alpha \alpha' \sum_{i=1}^P \langle (-\Delta)^{-1} u_i^\alpha, u_i^{\alpha'} \rangle \mathcal{P}(d\alpha) \mathcal{P}(d\alpha') \\
&\begin{cases} \alpha (-\Delta)^{-1} u_i = \log \left( \sum_{i=1}^P u_i^\alpha \right) - \log K + \text{constant} & i = 1, \dots, P \\ \sum_{i=1}^P u_i^\alpha \geq 0, \int_{\Omega^P} \sum_{i=1}^P u_i^\alpha dx = \lambda |\Omega|^{P-1}, u_i = \int_{[-1,1]} \alpha u_i^\alpha \mathcal{P}(d\alpha) \end{cases}
\end{aligned}$$

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